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ON THE OSCILLATION OF SECOND ORDER NONLINEAR NEUTRAL DYNAMIC EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS ON TIME-SCALES

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. In this paper, we investigate some new oscillation criteria and give sufficient conditions to ensure that all solutions of second order nonlinear neutral dynamic equations with distributed deviating arguments are oscillatory on a time-scale $\mathbb{T}$, via comparison with second order nonlinear dynamic equations whose oscillatory character are known and extensively studied in the literature.

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1. INTRODUCTION

The purpose of this paper is to establish some oscillation criteria for a second order nonlinear neutral dynamic equation

$$\left(a\left(t\right)\left[\left(x\left(t\right)+\int_{a_1}^{b_1}F_1\left(t,\tau_1,x\left(\theta_1\left(t,\tau_1\right)\right)\right)\triangle\tau_1\right)^{\triangle}\right]^{\alpha}\right)^{\triangle}+\int_{a_2}^{b_2}F_2\left(t,\tau_2,x\left(\theta_2\left(t,\tau_2\right)\right)\right)\triangle\tau_2=0 \tag{1.1}$$

on a time scale $\mathbb{T}$ which is unbounded above.

Throughout this paper we assume that

(i) $\alpha\geq 1$ is a quotient of positive odd integers, $0<a_i<b_i$, $i=1,2$;
(ii) $a:\mathbb{T}\rightarrow(0,\infty)$ is rd-continuous and

$$\int^{\infty}a^{-1/\alpha}\left(s\right)\triangle s=\infty; \tag{1.2}$$

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(iii) $\theta_i(t,\tau_i) : \mathbb{T}\times[a_i,b_i] \to \mathbb{T}$, $\theta_i(t,\tau_i) \le t$ for $\tau_i \in [a_i,b_i]$, $\theta_i(t,\tau_i) \to \infty$ as $t\to\infty$, $i=1,2$ and $\theta_2(t,\tau_2)$ is decreasing with respect τ_2;

(iv) $F_i(t,\tau_i,x) : \mathbb{T}\times[a_i,b_i]\times\mathbb{R} \to \mathbb{R}$, $i=1,2$ are continuous, assume that there exist a constant λ which is the ratio of positive odd integers and the functions $P_i : \mathbb{T}\times[a_i,b_i] \to (0,\infty)$, $i=1,2$ are rd-continuous such that

$$\begin{cases} F_1(t,\tau_1,x) \le P_1(t,\tau_1)\,x & \text{for } x>0, \quad t\in\mathbb{T},\ \tau_1\in[a_1,b_1] \\ F_1(t,\tau_1,x) \ge P_1(t,\tau_1)\,x & \text{for } x<0, \quad t\in\mathbb{T},\ \tau_1\in[a_1,b_1] \end{cases} \tag{1.3}$$

and

$$\begin{cases} F_2(t,\tau_2,x) \ge P_2(t,\tau_2)\,x^{\lambda} & \text{for } x>0, \quad t\in\mathbb{T},\ \tau_2\in[a_2,b_2] \\ F_2(t,\tau_2,x) \le P_2(t,\tau_2)\,x^{\lambda} & \text{for } x<0, \quad t\in\mathbb{T},\ \tau_2\in[a_2,b_2]. \end{cases} \tag{1.4}$$

We set

$$y(t) := x(t) + \int_{a_1}^{b_1} F_1(t,\tau_1,x(\theta_1(t,\tau_1)))\,\triangle\tau_1. \tag{1.5}$$

A nontrivial function $x(t)$ is said to be a solution of equation (1.1) if $y(t)\in C^1_{rd}([t_x,\infty),\mathbb{R})$ and $a\left(y^{\triangle}\right)^{\alpha}(t)\in C^1_{rd}([t_x,\infty),\mathbb{R})$ for $t_x\ge t_0$. A solution of equation (1.1) is called oscillatory if it has no last zero. Otherwise, a solution is called nonoscillatory.

The study of dynamic equations on time-scales which goes back to its founder Hilger [8] as an area of mathematics that has received a lot of attention. It has been created in order to unify the study of differential and discrete equations.

Recently, there has been an increasing interest in studying the oscillatory behavior of all orders of dynamic equations on time-scales, see [3–6]. With respect to dynamic equations on time-scales it is fairly new topic for general basic ideas and background, we refer to [2].

It appears that very little is known regarding the oscillation of second order nonlinear neutral dynamic equations with distributed deviating arguments on time-scales, see [3]. Our aim here is to establish some new criteria for the oscillation of equation (1.1) via comparison with second order nonlinear dynamic equations whose oscillatory character are known. We also provide some sufficient conditions guaranteeing the oscillatory behavior of all solutions of equation (1.1). The results are new for the special cases when $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$.

2. **Preliminaries**

We shall employ the following lemmas.

Consider the inequality

$$\left(a\left(x^{\triangle}\right)^{\alpha}\right)^{\triangle}(t) + Q(t)\,x^{\lambda}(g(t)) \le 0 \tag{2.1}$$

where a, Q are positive real valued, rd-continuous function on $\mathbb{T}$ and a satisfies condition (1.2), $g : \mathbb{T} \to \mathbb{T}$ is a rd-continuous function, $g(t) \leq t$ and $g(t) \to \infty$ as $t \to \infty$, α and λ are ratios of positive odd integers, $\alpha \geq 1$.

Now we present the following lemma.

Lemma 2.1. *If inequality* (2.1) *has an eventually positive solution, then the equation*

$$\left(a\left(x^{\triangle}\right)^{\alpha}\right)^{\triangle}(t) + Q(t)\, x^{\lambda}(g(t)) = 0 \tag{2.2}$$

also has an eventually positive solution.

Proof. Let $x(t)$ be an eventually positive solution of inequality (2.1). It is easy to see that $x^{\triangle}(t) > 0$ eventually, see [5]. Let t_0 be sufficiently large so that $x(t) > 0$, $x(g(t)) > 0$ and $y(t) := a(t)\left(x^{\triangle}(t)\right)^{\alpha}$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Then in view of

$$x(t) = x(t_0) + \int_{t_0}^{t} \left(\frac{y(s)}{a(s)}\right)^{1/\alpha} \triangle s$$

inequality (2.1) becomes

$$y^{\triangle}(t) + Q(t)\left(x(t_0) + \int_{t_0}^{g(t)} \left(\frac{y(s)}{a(s)}\right)^{1/\alpha} \triangle s\right)^{\lambda} \leq 0. \tag{2.3}$$

Integrating (2.3 from t to $u \geq t \geq t_0$ and letting $u \to \infty$, we have

$$y(t) \geq G(t, y(t)) \text{ for } t \in [t_0, \infty)_{\mathbb{T}},$$

where

$$G(t, y) := \int_{t}^{\infty} Q(\nu)\left(x(t_0) + \int_{t_0}^{g(\nu)} \left(\frac{y(s)}{a(s)}\right)^{1/\alpha} \triangle s\right)^{\lambda} \triangle \nu,$$

Now, we define a sequence of successive approximations $\{w_j(t)\}$ as follows:

$$w_0(t) = y(t)$$
$$w_{j+1}(t) = G(t, w_j(t)), \quad j = 0, 1, 2, \ldots.$$

It is easy to show that

$$0 < w_j(t) \leq y(t) \text{ and } w_{j+1}(t) \leq w_j(t), \quad j = 0, 1, 2, \ldots.$$

Then, the sequence $\{w_j(t)\}$ is non-increasing and bounded for each $t \geq t_0$. This means, we may define $w(t) := \lim_{j\to\infty} w_j(t) \geq 0$. Since,

$$0 \leq w(t) \leq w_j(t) \leq y(t) \text{ for all } j \geq 0, \text{ we find that}$$

$$\int_{t_0}^{t} w_j(s) \triangle s \leq \int_{t_0}^{t} y(s) \triangle s.$$

By the Lebesgue dominated convergence theorem on time-scale, one can easily find

$$w(t) = G(t, w(t)).$$

Therefore,

$$w^{\triangle}(t) = -Q(t)\left(x(t_0) + \int_{t_0}^{g(t)} \left(\frac{w(s)}{a(s)}\right)^{1/\alpha} \triangle s\right)^{\lambda} := -Q(t)\, m^{\lambda}(g(t)),$$

where

$$m(t) = x(t_0) + \int_{t_0}^{t} \left(\frac{w(s)}{a(s)}\right)^{1/\alpha} \triangle s.$$

Thus

$$m(t) > 0 \text{ and } a(t)\left(m^{\triangle}(t)\right)^{\alpha} = w(t) \ \text{ for } t \geq t_0.$$

Equation (2.4) then gives

$$\left(a(t)\left(m^{\triangle}(t)\right)^{\alpha}\right)^{\triangle} + Q(t)\, m^{\lambda}(g(t)) = 0.$$

Hence equation (2.2) has a positive solution $m(t)$. This completes the proof. □

Lemma 2.2 ([6])**.** *Suppose that* $|x|^{\triangle} > 0$ *on* $[t_0, \infty)_{\mathbb{T}}$, $\lambda > 0$ *and* $\lambda \neq 1$. *Then*

$$\frac{|x|^{\triangle}}{(|x^{\sigma}|)^{\lambda}} \leq \frac{\left(|x|^{1-\lambda}\right)^{\triangle}}{1-\lambda} \leq \frac{|x|^{\triangle}}{(|x|)^{\lambda}} \ \text{ on } \ [t_0, \infty)_{\mathbb{T}}.$$

Lemma 2.3 ([7])**.** *If* X *and* Y *are nonnegative and* $\beta > 1$, *then*

$$X^{\beta} - \lambda X Y^{\beta-1} + (\lambda - 1) Y^{\beta} \geq 0,$$

where equality holds if and only if $X = Y$.

Lemma 2.4 ([2])**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuously differentiable and suppose* $g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable. Then* $f \circ g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable and the formula*

$$(f \circ g)^{\triangle}(t) = \left\{\int_0^1 \left(f'(g(t)) + h\mu(t)\, g^{\triangle}(t)\right) dh\right\} g^{\triangle}(t)$$

holds.

Lemma 2.5 ([1])**.** *Assume* $x \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$.

If $x(t) > 0, x^{\triangle}(t) \geq 0$ *on* $[t_0, \infty)_{\mathbb{T}}$ *and* $\lambda > 1$, *then*

$$\int_t^{\infty} \frac{x^{\triangle}(s)}{(x^{\sigma}(s))^{\lambda}} \triangle s < \infty, \ t \in [t_0, \infty)_{\mathbb{T}}.$$

3. **Main results**

We will employ the following notation

$$Q(t) := \int_{a_2}^{b_2} P_2(t, \tau_2)\left(1 - \int_{a_1}^{b_1} P_1(\theta_2(t, \tau_2), \tau_1) \triangle \tau_1\right)^{\lambda} \triangle \tau_2 \tag{3.1}$$

and

$$g(t) := \theta_2(t, b_2). \tag{3.2}$$

Now we present our first result.

Theorem 3.1. *Let conditions* (i)–(iv) *hold and*

$$0 \leq \int_{a_1}^{b_1} P_1(t,\tau_1)\triangle\tau_1 < 1. \tag{3.3}$$

If the equation

$$\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\triangle} + Q(t)\,y^{\lambda}(g(t)) = 0 \tag{3.4}$$

where Q and g are as in (3.1) *and* (3.2), *is oscillatory, then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$ for $t \geq t_0$. Then $x(\theta_i(t,\tau_i)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$, $b_i \geq \tau_i \geq a_i$, $i = 1,2$. In the case when $x(t)$ is negative the proof is similar. In view of (1.1) and (1.3)–(1.5), we see that

$$\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\triangle} + \int_{a_2}^{b_2} P_2(t,\tau_2)\left(x(\theta_2(t,\tau_2))\right)^{\lambda}\triangle\tau_2 \leq 0 \text{ for } t \geq t_1. \tag{3.5}$$

It is easy to see that $y(t) > 0$ for $t \geq t_1$. Now we claim that $y^{\triangle}(t) > 0$ eventually. If not, then there exists a $t_2 \geq t_1$ such that $a\left(y^{\triangle}\right)^{\alpha}(t_2) = c < 0$. Thus,

$$a\left(y^{\triangle}\right)^{\alpha}(t) \leq c \text{ for } t \geq t_2,$$

or

$$y^{\triangle}(t) \leq \left(\frac{c}{a(t)}\right)^{1/\alpha} \quad \text{for } t \geq t_2.$$

Integrating this inequality from t_2 to t and using condition (1.2) we obtain a contradiction to the fact that $y(t) > 0$ for $t \geq t_2$. Therefore, one can easily have

$$y(t) > 0 \text{ and } y^{\triangle}(t) > 0 \text{ for } t \geq t_2. \tag{3.6}$$

From (1.3), we get

$$\begin{aligned} x(t) &= y(t) - \int_{a_1}^{b_1} F_1(t,\tau_1,x(\theta_1(t,\tau_1)))\triangle\tau_1 \\ &\geq y(t) - \int_{a_1}^{b_1} P_1(t,\tau_1)\,x(\theta_1(t,\tau_1))\triangle\tau_1 \\ &= y(t) - \int_{a_1}^{b_1} P_1(t,\tau_1)\left[y(\theta_1(t,\tau_1)) - \int_{a_1}^{b_1} F_1(t,\tau_1,x(\theta_1(t,\tau_1)))\triangle\tau_1\right]\triangle\tau_1 \\ &\geq y(t) - \int_{a_1}^{b_1} P_1(t,\tau_1)\,y(\theta_1(t,\tau_1))\triangle\tau_1. \end{aligned}$$

Using the fact that the function $y(t)$ is increasing for $t \geq t_2$, we have

$$x(t) \geq \left(1 - \int_{a_1}^{b_1} P_1(t,\tau_1)\triangle\tau_1\right) y(t) \;\text{ for } t \geq t_2. \tag{3.7}$$

Using (3.7) in (2.1), we get

$$\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\triangle} + \int_{a_2}^{b_2} P_2(t,\tau_2)\left(1-\int_{a_1}^{b_1} P_1(\theta_2(t,\tau_2),\tau_1)\triangle\tau_1\right)^{\lambda}(y(\theta_2(t,\tau_2)))^{\lambda}\triangle\tau_2 \le 0$$

for $t \ge t_2$. Using the fact that θ_2 is decreasing in the second variable and (3.1) and (3.2) we obtain

$$\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\triangle} + Q(t)\,y^{\lambda}(g(t)) \le 0 \text{ for } t \ge t_2. \tag{3.8}$$

By Lemma 2.1, the equation

$$\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\triangle} + Q(t)\,y^{\lambda}(g(t)) = 0$$

has a nonoscillatory solution. But this is impossible by the hypothesis. This completes the proof. □

For all sufficiently large $t_0 \ge 0$, we set

$$\beta(t) = \frac{\int_{t_0}^{g(t)} a^{-1/\alpha}(s)\triangle s}{\int_{t_0}^{t} a^{-1/\alpha}(s)\triangle s} \text{ and } \eta(t) = \left(\int_{t_0}^{t} a^{-1/\alpha}(s)\triangle s\right)^{-1} \text{ for } t \ge t_0. \tag{3.9}$$

Theorem 3.2. *Let conditions* (i)–(iv) *hold and* (3.3) *hold. Assume that there exists a positive non-decreasing, delta-differentiable function* $\delta(t)$ *such that for all sufficiently large* $t_0 \ge 0$ *we have*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\beta^{\lambda}(s)\,\delta(s)\,Q(s) - \frac{\left(\frac{\alpha}{\lambda}\right)^{\alpha} a(s)\left(\delta^{\triangle}(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left(\gamma(s)\,\delta(s)\right)^{\alpha}}\right]\triangle s = \infty \tag{3.10}$$

for $t_1 \ge t_0$*, where* g *and* Q *are as* (2.1) *and* (2.1),

$$\gamma(t) = \begin{cases} c_1, & c_1 \text{ is some positive constant if } \lambda > \alpha \\ 1, & \text{if } \lambda = \alpha \\ c_2\left(\eta^{\sigma}(t)\right)^{\frac{\alpha-\lambda}{\alpha}}, & c_2 \text{ is some positive constant if } \lambda < \alpha. \end{cases} \tag{3.11}$$

Then the equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$. Then $x(\theta_i(t,\tau_i)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$, $b_i \ge \tau_i \ge a_i$, $i = 1,2$. Proceeding as in the proof of Theorem 3.1 we obtain the inequality (3.8). We set

$$W(t) = \frac{\delta(t)\,a(t)\left(y^{\triangle}(t)\right)^{\alpha}}{y^{\lambda}(t)} \text{ for } t \ge t_2. \tag{3.12}$$

Then we get

$$W^{\triangle}(t)=\left(\frac{\delta(t)}{y^{\lambda}(t)}\right)\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\triangle}+\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\sigma}\left(\frac{\delta(t)}{y^{\lambda}(t)}\right)^{\triangle}$$
$$\leq-\delta(t)\,Q(t)\left(\frac{y(g(t))}{y(t)}\right)^{\lambda}+\delta^{\triangle}(t)\,\frac{\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\sigma}}{y^{\lambda}(\sigma(t))}$$
$$-\,\delta(t)\,\frac{\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\sigma}\left(y^{\lambda}(t)\right)^{\triangle}}{y^{\lambda}(t)\,y^{\lambda}(\sigma(t))}. \tag{3.13}$$

Using the fact that $y(t)$ is increasing and $a(t)\left(y^{\triangle}(t)\right)^{\alpha}$ is non-increasing for $t\geq t_2$ we have

$$y(g(t))>y(g(t))-y(t_2)=\int_{t_2}^{g(t)}a^{-1/\alpha}(s)\left(a(s)\left(y^{\triangle}(s)\right)^{\alpha}\right)^{1/\alpha}\triangle s$$
$$\geq a^{1/\alpha}(g(t))\,y^{\triangle}(g(t))\int_{t_2}^{g(t)}a^{-1/\alpha}(s)\,\triangle s,$$

or

$$\frac{a^{1/\alpha}(g(t))\,y^{\triangle}(g(t))}{y(g(t))}\leq\left(\int_{t_2}^{g(t)}a^{-1/\alpha}(s)\,\triangle s\right)^{-1}. \tag{3.14}$$

Also, we find

$$y(t)=y(g(t))+\int_{g(t)}^{t}a^{-1/\alpha}(s)\left(a(s)\left(y^{\triangle}(s)\right)^{\alpha}\right)^{1/\alpha}\triangle s$$
$$\leq y(g(t))+a^{1/\alpha}(g(t))\left(y^{\triangle}(g(t))\right)\int_{g(t)}^{t}a^{-1/\alpha}(s)\,\triangle s$$

and so

$$\frac{y(t)}{y(g(t))}\leq 1+\frac{a^{1/\alpha}(g(t))\,y^{\triangle}(g(t))}{y(g(t))}\int_{g(t)}^{t}a^{-1/\alpha}(s)\,\triangle s. \tag{3.15}$$

Using (3.14) in (3.15) we have

$$\frac{y(g(t))}{y(t)}\geq\frac{\int_{t_2}^{g(t)}a^{-1/\alpha}(s)\,\triangle s}{\int_{t_2}^{t}a^{-1/\alpha}(s)\,\triangle s}:=\beta(t),\ \ t\geq t_2. \tag{3.16}$$

By applying the chain rule on time scales Lemma 2.4, we have

$$\left(y^{\lambda}(t)\right)^{\triangle}=\lambda y^{\triangle}(t)\int_{0}^{1}\left[y(t)+h\mu(t)\,y^{\triangle}(t)\right]^{\lambda-1}dh$$
$$\geq\begin{cases}\lambda\left(y^{\sigma}(t)\right)^{\lambda-1}y^{\triangle}(t) & ,0<\lambda\leq 1\\ \lambda\left(y(t)\right)^{\lambda-1}y^{\triangle}(t) & ,\lambda>1.\end{cases} \tag{3.17}$$

Using (3.16) and (3.17) in (3.13) and for all $\lambda > 0$ we obtain

$$\begin{aligned} W^{\triangle}(t) &\leq -\delta(t) Q(t) \beta^{\lambda}(t) + \delta^{\triangle}(t) \left(\frac{W(t)}{\delta(t)}\right)^{\sigma} \\ &\quad - \lambda\delta(t) \frac{\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\sigma}}{y^{\sigma}(t)^{\lambda+1}} y^{\triangle}(t) \\ &\leq -\delta(t) \beta^{\lambda}(t) Q(t) + \delta^{\triangle}(t) \left(\frac{W(t)}{\delta(t)}\right)^{\sigma} - \lambda\delta(t) \frac{y^{\triangle}(t)}{y^{\sigma}(t)} \left(\frac{W(t)}{\delta(t)}\right)^{\sigma}. \end{aligned}$$

Using the fact that $a(t)\left(y^{\triangle}(t)\right)^{\alpha}$ is decreasing for $t \geq t_2$ we get

$$\begin{aligned} \frac{y^{\triangle}(t)}{y^{\sigma}(t)} &\geq \frac{\left(\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\sigma}\right)^{1/\alpha}}{a^{1/\alpha}(t)\left(y^{\sigma}(t)\right)^{\lambda/\alpha}} \left(y^{\sigma}(t)\right)^{\frac{\lambda-\alpha}{\alpha}} \\ &= \left(\frac{W^{\sigma}(t)}{\delta^{\sigma}(t)}\right)^{1/\alpha} a^{-1/\alpha}(t) \left(y^{\sigma}(t)\right)^{\frac{\lambda-\alpha}{\alpha}} \text{ for } t \geq t_{2.}. \end{aligned}$$

Thus, for $\lambda > 0$ we obtain

$$\begin{aligned} W^{\triangle}(t) &\leq -\delta(t) \beta^{\lambda}(t) Q(t) + \delta^{\triangle}(t) \left(\frac{W(t)}{\delta(t)}\right)^{\sigma} \\ &\quad - \lambda\delta(t) a^{-1/\alpha}(t) \left(\left(\frac{W(t)}{\delta(t)}\right)^{\sigma}\right)^{\frac{\alpha+1}{\alpha}} \left(y^{\sigma}(t)\right)^{\frac{\lambda-\alpha}{\alpha}} \text{ for } t \geq t_{2.}. \end{aligned} \tag{3.18}$$

Now, we consider the following three cases:

Case (I) : $\lambda > \alpha$.

In this case, since $y^{\triangle}(t) > 0$ for $t \geq t_2$, then there exists a $t_3 \geq t_2$ such that $y^{\sigma}(t) \geq y(t) \geq b > 0$ for $t \geq t_3$. This implies that

$$\left(y^{\sigma}(t)\right)^{\frac{\lambda-\alpha}{\alpha}} \geq b^{\frac{\lambda-\alpha}{\alpha}} := c_1.$$

Case (II) : $\lambda = \alpha$.

In this case, we see that $\left(y^{\sigma}(t)\right)^{\frac{\lambda-\alpha}{\alpha}} = 1$.

Case (III) : $\lambda < \alpha$.

Since $a(t)\left(y^{\triangle}(t)\right)^{\alpha}$ is decreasing, then there exists a constant $b_1 > 0$ such that

$$y^{\triangle}(t) \leq b_1^{1/\alpha} a^{-1/\alpha}(t) \text{ for } t \geq t_2.$$

Integrating this inequality from t_2 to t, we have

$$y(t) \leq y(t_2) + b_1^{1/\alpha} \int_{t_2}^{t} a^{-1/\alpha}(s) \triangle s.$$

Thus, there exist a constant $b_2 > 0$ and $t_3 \geq t_2$ such that

$$y(t) \leq b_2 \eta^{-1}(t) \text{ for } t \geq t_3,$$

and hence

$$\left(y^{\sigma}(t)\right)^{\frac{\lambda-\alpha}{\alpha}} \geq c_2 \left(\eta^{\sigma}\right)^{\frac{\alpha-\lambda}{\alpha}}(t) \text{ for } t \geq t_3,$$

where $c_2 = (b_2)^{\frac{\lambda-\alpha}{\alpha}}$.

Using these three cases and definitions of $\gamma(t)$ in (3.11), we get

$$W^{\triangle}(t) \le -\delta(t)\beta^{\lambda}(t)Q(t) + \delta^{\triangle}(t)\left(\frac{W(t)}{\delta(t)}\right)^{\sigma} - \lambda\delta(t)a^{-1/\alpha}(t)\gamma(t)\left(\left(\frac{W(t)}{\delta(t)}\right)^{\sigma}\right)^{\frac{\alpha+1}{\alpha}} \quad \text{for } t \ge t_3. \tag{3.19}$$

Define

$$X = (\lambda\gamma(t)\delta(t))^{\frac{\alpha}{\alpha+1}} a^{\frac{-1}{\alpha+1}}(t)\left(\frac{W(t)}{\delta(t)}\right)^{\sigma}, \ \beta = \frac{\alpha+1}{\alpha} > 1,$$
$$Y = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\left(\frac{\delta^{\triangle}(t)}{\delta^{\sigma}(t)}\right)^{\alpha}\left[(\lambda\gamma(t)\delta(t))^{-\frac{\alpha}{\alpha+1}}\delta^{\sigma}(t)a^{\frac{1}{\alpha+1}}(t)\right]^{\alpha}$$

and using Lemma 2.3 we obtain

$$\lambda\gamma(t)\delta(t)a^{-1/\alpha}(t)\left(\left(\frac{W(t)}{\delta(t)}\right)^{\sigma}\right)^{\frac{\alpha+1}{\alpha}} - \delta^{\triangle}(t)\left(\frac{W(t)}{\delta(t)}\right)^{\sigma} \ge -\frac{\left(\frac{\alpha}{\lambda}\right)^{\alpha}a(t)\left(\delta^{\triangle}(t)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\gamma(t)\delta(t))^{\alpha}} \quad \text{for } t \ge t_3. \tag{3.20}$$

Using (3.20) in (3.19) we have

$$W^{\triangle}(t) \le -\delta(t)\beta^{\lambda}(t)Q(t) + \frac{\left(\frac{\alpha}{\lambda}\right)^{\alpha}a(t)\left(\delta^{\triangle}(t)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\gamma(t)\delta(t))^{\alpha}}. \tag{3.21}$$

Integrating (3.21) from t_3 to t, we get

$$\int_{t_3}^{t}\left[\delta(s)\beta^{\lambda}(s)Q(s) - \frac{\left(\frac{\alpha}{\lambda}\right)^{\alpha}a(s)\left(\delta^{\triangle}(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\gamma(s)\delta(s))^{\alpha}}\right]\triangle s \le W(t_3) - W(t) \le W(t_3).$$

Taking lim sup of both sides of this inequality as $t \to \infty$, we obtain a contradiction to condition (3.10). This completes the proof. □

Theorem 3.3. *Let conditions* (i)–(iv) *and* (2.1) *hold. If in addition suppose that there exists a positive non-decreasing delta-differentiable function* $\delta(t)$ *such that, for all sufficiently large* $t_0 \ge 0$

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\delta(s)\beta^{\lambda}(s)Q(s) - \gamma_1(s)\eta^{\alpha}(s)\delta^{\triangle}(s)\right]\triangle s = \infty \tag{3.22}$$

for $t \ge t_1$, *where*

$$\gamma_1(t) = \begin{cases} c_1, c_1 \text{ is some positive constant}, & \text{when } \lambda > \alpha \\ 1, & \text{when } \lambda = \alpha \\ c_2\eta^{\lambda-\alpha}(t), c_2 \text{ is some positive constant}, & \text{when } \lambda < \alpha, \end{cases} \tag{3.23}$$

then the equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1, say $x(t) > 0$ for $t \geq t_0 \geq 0$. Proceeding as in the proof of Theorem 3.1, we obtain the inequality (3.18), which becomes

$$W^{\triangle}(t) \leq -\delta(t)\beta^{\lambda}(t)Q(t) + \delta^{\triangle}(t)\left(\frac{W(t)}{\delta(t)}\right)^{\sigma} \text{ for } t \geq t_2,$$

or

$$\begin{aligned} W^{\triangle}(t) &\leq -\delta(t)\beta^{\lambda}(t)Q(t) + \delta^{\triangle}(t)\frac{\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\sigma}}{y^{\lambda}(\sigma(t))} \\ &\leq -\delta(t)\beta^{\lambda}(t)Q(t) + \delta^{\triangle}(t)\frac{a(t)\left(y^{\triangle}(t)\right)^{\alpha}}{y^{\lambda}(t)} \\ &= -\delta(t)\beta^{\lambda}(t)Q(t) + \delta^{\triangle}(t)a(t)\left(\frac{y^{\triangle}(t)}{y(t)}\right)^{\alpha} y^{\alpha-\lambda}(t) \text{ for } t \geq t_2. \end{aligned} \tag{3.24}$$

Now

$$\begin{aligned} y(t) &= y(t_1) + \int_{t_1}^{t} a^{-1/\alpha}(s)\left(a(s)\left(y^{\triangle}(s)\right)^{\alpha}\right)^{1/\alpha}\triangle s \\ &\geq a^{1/\alpha}(t)\,y^{\triangle}(t)\int_{t_2}^{t} a^{-1/\alpha}(s)\triangle s. \end{aligned}$$

Thus,

$$\left(\frac{y^{\triangle}(t)}{y(t)}\right)^{\alpha} \leq \frac{1}{a(t)}\left(\int_{t_2}^{t} a^{-1/\alpha}(s)\triangle s\right)^{-\alpha} = \frac{\eta^{\alpha}(t)}{a(t)} \quad \text{for } t \geq t_2. \tag{3.25}$$

Using (3.25) in (3.25), we have

$$W^{\triangle}(t) \leq -\delta(t)\beta^{\lambda}(t)Q(t) + \delta^{\triangle}(t)(\eta(t))^{\alpha} y^{\alpha-\lambda}(t) \text{ for } t \geq t_2. \tag{3.26}$$

As in the proof of Theorem 3.2, we consider the three cases. The cases (I) and (II) are similar, to that of Theorem 3.2 and hence is omitted.

Case (III) : $\lambda < \alpha$.

Proceeding as in the proof of Theorem 3.2, there exists a constant $b_1 > 0$ and $t_3 \geq t_2$ such that

$$y(t) \leq b_1\eta^{-1}(t) \text{ for } t \geq t_3$$

and hence

$$y^{\alpha-\lambda}(t) \leq c_2\eta^{\lambda-\alpha}(t) \text{ for } t \geq t_3,$$

where $c_2 = b_1^{\alpha-\lambda}$. Using these three cases in (3.26), we have

$$W^{\triangle}(t) \leq -\delta(t)\beta^{\lambda}(t)Q(t) + \delta^{\triangle}(t)\gamma_1(t)(\eta(t))^{\alpha} \text{ for } t \geq t_3.$$

Integrating this inequality from t_3 to t, we have

$$0 < W(t) \leq W(t_3) - \int_{t_3}^{t}\left[\delta(s)\beta^{\lambda}(s)Q(s) - \delta^{\triangle}(s)\gamma_1(s)(\eta(s))^{\alpha}\right]\triangle s.$$

Taking lim sup of both sides of this inequality as $t \to \infty$, we obtain a contradiction to condition (3.22). This completes the proof. □

Theorem 3.4. *Let condition* (i)–(iv) *and* (2.1) *hold. If*

$$\limsup_{t\to\infty}\int_{g(t)}^{t}\left(\frac{g(s)}{2}\right)^{\lambda}a^{-1}(g(s))\,Q(s)\,\triangle s>0 \qquad \text{if } \lambda=\alpha \tag{3.27}$$

and

$$\int^{\infty}(g(s))^{\lambda}\,a^{-\lambda/\alpha}(g(s))\,Q(s)\,\triangle s=\infty \qquad \text{if } \lambda\neq\alpha, \tag{3.28}$$

then the equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t)>0$ for $t\geq t_0\geq 0$. Proceeding as in the proof of Theorem 3.1, we obtain the inequality (3.8)) holds for $t\geq t_2$. Now

$$\begin{aligned} y(t)&=y(t_2)+\int_{t_2}^{t}y^{\triangle}(s)\,\triangle s\geq (t-t_2)\,y^{\triangle}(t)\\ &\geq \frac{t}{2}y^{\triangle}(t)\ \text{ for } t\geq t_3\geq 2t_2. \end{aligned}$$

Thus inequality becomes

$$\left(a(t)\left(y^{\triangle}(t)\right)^{\alpha}\right)^{\triangle}+\left(\frac{g(t)}{2}\right)^{\lambda}Q(t)\left(y^{\triangle}(g(t))\right)^{\lambda}\leq 0 \text{ for } t\geq t_3.$$

Set $z(t)=a(t)\left(y^{\triangle}(t)\right)^{\alpha}$. Then

$$z^{\triangle}(t)+\left(\frac{g(t)}{2}\right)^{\lambda}a^{-\lambda/\alpha}(g(t))\,Q(t)\,z^{\lambda/\alpha}(g(t))\leq 0 \text{ for } t\geq t_3. \tag{3.29}$$

We consider the following two cases:

Case (I) : $\lambda=\alpha$.

Integrating (3.29) from $g(t)\geq t_3$ to t

$$z(t)-z(g(t))+\int_{g(t)}^{t}\left(\frac{g(s)}{2}\right)^{\lambda}a^{-1}(g(s))\,Q(s)\,z(g(s))\,\triangle s\leq 0. \tag{3.30}$$

Using the fact that $z>0$, decreasing on $[t_1,\infty)$ and $g(t)$ is nondecreasing in (3.30), we find

$$z(g(t))\int_{g(t)}^{t}\left(\frac{g(s)}{2}\right)^{\lambda}a^{-1}(g(s))\,Q(s)\,\triangle s\leq 0.$$

Using condition (3.27), we obtain a contradiction.

Case (II): $\lambda\neq\alpha$

First choose $\lambda<\alpha$.

In this case, we have

$$z^{\triangle}(t)+\left(\frac{g(t)}{2}\right)^{\lambda}a^{-\lambda/\alpha}(g(t))\,Q(t)\,z^{\lambda/\alpha}(t)\leq 0$$

or

$$\left(\frac{g(t)}{2}\right)^{\lambda} a^{-\lambda/\alpha}(g(t))\,Q(t) \leq -z^{-\lambda/\alpha}(t)\,z^{\triangle}(t) \qquad \text{for } t \geq t_3.$$

Integrating this inequality from t_3 to t and using Lemma 2.2, we arrive at the desired contradiction.

Now, choose $\lambda > \alpha$.

Since $z(t) > 0$ and $\lambda/\alpha > 1$, from Lemma 2.2 we obtain

$$-\frac{z^{\triangle}(t)}{z(t)^{\lambda/\alpha}} \leq -\frac{z^{\triangle}(t)}{z(\sigma(t))^{\lambda/\alpha}} \qquad \text{for } t \geq t_3.$$

Then we can write (3.29) as

$$\left(\frac{g(t)}{2}\right)^{\lambda} a^{-\lambda/\alpha}(g(t))\,Q(t) \leq -\frac{z^{\triangle}(t)}{z^{\lambda/\alpha}(t)} \leq -\frac{z^{\triangle}(t)}{z(\sigma(t))^{\lambda/\alpha}} \qquad \text{for } t \geq t_{3.}$$

Integrating form t_3 to t and letting t to infinity, we get

$$\int_{t_3}^{\infty} \left(\frac{g(s)}{2}\right)^{\lambda} a^{-\lambda/\alpha}(g(s))\,Q(s)\,\triangle s \leq \int_{t_3}^{\infty} \frac{-z^{\triangle}(s)}{z(\sigma(s))^{\lambda/\alpha}}\triangle s.$$

Since the integral on the right is finite for $\lambda/\alpha > 1$ by Lemma 2.5 and the integral on the left is infinite by hypothesis, we arrive at the desired contradiction. This completes the proof. □

For illustration, we consider the following example.

Example 3.5. Consider the second order neutral nonlinear dynamic equations

$$(3.31)\qquad \left(\frac{1}{t}\left(\left(x(t) + \int_1^2 px(t-\tau_1)\,\triangle\tau_1\right)^{\triangle}\right)^{\frac{5}{3}}\right)^{\triangle} + \int_1^3 \frac{1}{\sqrt{t}} x^{\lambda}(t-\tau_2)\,\triangle\tau_2 = 0$$

where $0 < p < 1, t - \tau_i \in \mathbb{T}$ $i = 1, 2$ and

$$(3.32)\qquad \left(\frac{1}{t}\left(\left(x(t) + \frac{t+c-1}{t+c} x(\tau(t))\right)^{\triangle}\right)^{\alpha}\right)^{\triangle} + \int_1^3 \frac{1}{\sqrt{t}} x^{\lambda}(t-\tau_2)\,\triangle\tau_2 = 0$$

where $c > 0, \tau : \mathbb{T} \to \mathbb{T}$ is rd-continuous, $\lim_{t\to\infty} \tau(t) = \infty$ and λ is the ratio of positive odd integers.

One can easily see that the conditions of Theorem 3.2, Theorem 3.3 and Theorem 3.4 are fulfilled for all $\lambda > 0$, and hence we conclude that both equations (3.31) and (3.32) are oscillatory.

4. **Conclusions**

1. The results of this paper are presented in a form which is essentially new and of high degree of generality. It includes as a special case the neutral equation

$$\left(a\left(t\right)\left(x\left(t\right)+p\left(t\right)x\left(\tau\left(t\right)\right)^{\triangle}\right)^{\alpha}\right)^{\triangle}+q\left(t\right)f\left(x\left[g\left(t\right)\right]\right)=0$$

   where α and a are as in equation (1.1), $p,q:\mathbb{T}\to(0,\infty)$ are rd-continuous $\tau,g:\mathbb{T}\to\mathbb{T}$ are rd-continuous, $\lim_{t\to\infty}\tau\left(t\right)=\lim_{t\to\infty}g\left(t\right)=\infty$.
2. From the proofs of our results, one may establish new criteria for the oscillation of equation (2.1). The details are left to the reader.
3. The results of this paper are new for the continuous case ($\mathbb{T}=\mathbb{R}$) and the discrete case ($\mathbb{T}=\mathbb{Z}$). The formulation of our results for both cases are left to the reader.

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