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MÖNCH TYPE RESULTS FOR MAPS WITH WEAKLY SEQUENTIALLY CLOSED GRAPHS

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ABSTRACT. In this paper we present fixed point results of Mönch type and a homotopy result for maps with weakly sequentially closed graphs.

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1. INTRODUCTION

In this paper we discuss Mönch type maps [1, 6]. We begin in Theorem 2.2 and present a new fixed point result for Mönch type self maps with weakly sequentially closed graph. Next we define the notion of an essential map which is of Mönch type and has weakly sequentially closed graph. We use this notion to present a homotopy type result for this class of maps.

2. FIXED POINT THEORY

First we recall the following result [7].

Theorem 2.1. *Let Q be a nonempty, convex, weakly compact subset of a metrizable locally convex linear topological space E. Suppose $F : Q \to K(Q)$ has weakly sequentially closed graph; here $K(Q)$ denotes the family of nonempty, convex, weakly compact subsets of Q. Then F has a fixed point in Q.*

We now prove a result which will be needed in Section 3.

Theorem 2.2. *Let Q be a nonempty, closed, convex subset of a metrizable locally convex linear topological space E and let $x_0 \in Q$. Suppose $F : Q \to K(Q)$ has weakly sequentially closed graph and F takes relatively weakly compact sets into relatively*

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weakly compact sets. Also assume the following hold:

$$\text{(2.1)}\qquad \begin{cases} A \subseteq Q, A = co(\{x_0\} \cup F(A)) \text{ with } \overline{A}^w = \overline{C}^w \\ \text{and } C \subseteq A \text{ countable, implies } \overline{A}^w \text{ is weakly compact} \end{cases}$$

$$\text{(2.2)}\qquad \begin{cases} \text{for any relatively compact subset } A \text{ of } E \text{ there} \\ \text{exists a countable set } B \subseteq A \text{ with } \overline{B}^w = \overline{A}^w \end{cases}$$

and

$$\text{(2.3)}\qquad \begin{cases} \text{if } A \text{ is a weakly compact subset of } E \\ \text{then } \overline{co}(A) \text{ is weakly compact.} \end{cases}$$

Then F has a fixed point in Q.

Remark 2.3. If E is a Banach space then (2.3) holds from the Krein-Šmulian theorem [3, pg. 434, 5 pg. 82]. Note (2.3) holds if a Krein-Šmulian type theorem holds (for example E could be a quasicomplete locally convex linear topological space); for examples see [4 pp 553, 5 pp 82].

Remark 2.4. If K is a weakly compact subset of E and K with the relative weak topology is metrizable (for example E could be a Banach space whose dual E^* is separable) then (2.2) holds (recall compact metric spaces are separable).

Proof. Let

$$D_0 = \{x_0\}, \quad D_n = co\left(\{x_0\} \cup F(D_{n-1})\right) \quad \text{for} \quad n = 1, 2, \dots \text{ and } D = \bigcup_{n=0}^{\infty} D_n.$$

Now for $n = 0, 1, \dots$ notice D_n is convex and

$$D_0 \subseteq D_1 \subseteq \cdots \subseteq D_{n-1} \subseteq D_n \cdots \subseteq Q.$$

Note D is convex and since (D_n) is increasing we have

$$\text{(2.4)}\qquad D = \bigcup_{n=1}^{\infty} co\left(\{x_0\} \cup F(D_{n-1})\right) = co(\{x_0\} \cup F(D)).$$

We claim D_n is relatively weakly compact for $n \in \{0, 1, \dots\}$. Certainly it is true if $n = 0$. Now suppose D_k is relatively weakly compact for some $k \in \{0, 1, \dots\}$. Now since F takes relatively weakly compact sets into relatively weakly compact sets then $F(D_k)$ is relatively weakly compact. This together with (2.3) guarantees D_{k+1} is relatively weakly compact.

Now (2.2) implies that for each $n \in \{0, 1, \dots\}$ there exists C_n with C_n countable, $C_n \subseteq D_n$, and $\overline{C_n}^w = \overline{D_n}^w$. Let $C = \bigcup_{n=0}^{\infty} C_n$. Now since

$$\bigcup_{n=0}^{\infty} D_n \subseteq \bigcup_{n=0}^{\infty} \overline{D_n}^w \subseteq \overline{\bigcup_{n=0}^{\infty} D_n}^w$$

we have

$$\overline{\bigcup_{n=0}^{\infty} \overline{D_n}^w}^w = \overline{\bigcup_{n=0}^{\infty} D_n}^w = \overline{D^w} \text{ and } \overline{\bigcup_{n=0}^{\infty} \overline{D_n}^w}^w = \overline{\bigcup_{n=0}^{\infty} \overline{C_n}^w}^w = \overline{\bigcup_{n=0}^{\infty} C_n}^w = \overline{C^w}.$$

Thus $\overline{C^w} = \overline{D^w}$ so from (2.1) (see (2.4)) we have that $\overline{D^w}$ is weakly compact.

Consider the map $F^\star : \overline{D^w} \to K(\overline{D^w})$ given by

$$F^\star(x) = F(x) \cap \overline{D^w}.$$

We first show $F^\star(x) \neq \emptyset$ for each $x \in \overline{D^w}$. Note from (2.4) that $F(D) \subseteq D \subseteq \overline{D^w}$ so $D \subseteq F^{-1}(\overline{D^w})$. Now let $x \in \overline{D^w}$. Now since $\overline{D^w}$ is weakly compact the Eberlein-Šmulian theorem [4 pg. 549] guarantees that there is a sequence (x_n) in D with $x_n \rightharpoonup x$ (here $\rightharpoonup$ denotes weak convergence). Take any $y_n \in F(x_n)$. Now since $F(D) \subseteq D$ we have $y_n \in D$. Also since $\overline{D^w}$ is weakly compact the Eberlein-Šmulian theorem [4 pg. 549] guarantees that we may assume without loss of generality that $y_n \rightharpoonup y$ for some $y \in \overline{D^w}$. Note $y_n \in F(x_n)$, $x_n \rightharpoonup x$, $y_n \rightharpoonup y$ implies $y \in F(x)$ since F has weakly sequentially closed graph. Thus $y \in F(x) \cap \overline{D^w}$ so $x \in F^{-1}(\overline{D^w})$. As a result $\overline{D^w} \subseteq F^{-1}(\overline{D^w})$ i.e. $F^\star(x) \neq \emptyset$ for each $x \in \overline{D^w}$.

Note $F^\star : \overline{D^w} \to K(\overline{D^w})$ has weakly sequentially closed graph. Now Theorem 2.1 guarantees a $x \in \overline{D^w}$ with $x \in F^\star(x) \subseteq F(x)$. □

Remark 2.5. Theorem 2.2 improves [1, Theorem 3.3].

3. HOMOTOPY RESULTS

In this section let E be a metrizable locally convex linear topological space, C a closed convex subset of E, and U a weakly open subset of C with $0 \in U$.

Definition 3.1. $F \in A(\overline{U^w}, C)$ if $F : \overline{U^w} \to K(C)$ has weakly sequentially closed graph, F takes relatively weakly compact sets into relatively weakly compact sets, and F satisfies the following condition: if $D \subseteq \overline{U^w}$ and $D \subseteq \overline{co}(\{0\} \cup F(D))$ with $\overline{D^w} = \overline{C^w}$ and $C \subseteq D$ countable, then $\overline{D^w}$ is weakly compact.

Definition 3.2. We say $F \in A_{\partial U}(\overline{U^w}, C)$ if $F \in A(\overline{U^w}, C)$ with $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the weak boundary of U in C.

Definition 3.3. A map $F \in A_{\partial U}(\overline{U^w}, C)$ is essential in $A_{\partial U}(\overline{U^w}, C)$ if for every $G \in A_{\partial U}(\overline{U^w}, C)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

Theorem 3.4 (Homotopy Property)**.** *Let E be a metrizable locally convex linear topological space, C a closed convex subset of E, U a weakly open subset of C with $0 \in U$. Suppose $F \in A(\overline{U^w}, C)$ and assume the following conditions hold:*

(3.1) *the zero map is essential in $A_{\partial U}(\overline{U^w}, C)$*

and

$$x \notin \lambda\, F\, x \text{ for every } x \in \partial U \text{ and } \lambda \in (0,1]. \tag{3.2}$$

Then F is essential in $A_{\partial U}(\overline{U^w}, C)$.

Proof. Let $H \in A_{\partial U}(\overline{U^w}, C)$ with $H|_{\partial U} = F|_{\partial U}$. We must show H has a fixed point in U. Consider

$$B = \{x \in \overline{U^w} : x \in tH(x) \text{ for some } t \in [0,1]\}.$$

Now $B \neq \emptyset$ since $0 \in U$. Also B is weakly sequentially closed. To see this let (x_n) be sequence of B which converges weakly to some $x \in \overline{B^w}$ (in particular $x \in \overline{U^w}$) and let (λ_n) be a sequence of $[0,1]$ satisfying $x_n \in \lambda_n H x_n$. Then for each n there is a $z_n \in Hx_n$ with $x_n = \lambda_n z_n$. By passing to a subsequence if necessary, we may assume that (λ_n) converges to some $\lambda \in [0,1]$ and without loss of generality assume $\lambda_n \neq 0$ for all n. This implies that the sequence (z_n) converges weakly to some $z \in \overline{U^w}$ with $x = \lambda z$. Since F has weakly sequentially closed graph then $z \in H(x)$. Hence $x \in \lambda H x$ and therefore $x \in B$. Thus B is weakly sequentially closed.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in B. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in $[0,1]$ with $x_n \in t_n H x_n$ and we may assume without loss of generality that $t_n \to t \in [0,1]$. Let $C = \{x_n\}_{n=1}^{\infty}$. Note C is countable and $C \subseteq co(H(C) \cup \{0\})$. Since $H \in A(\overline{U^w}, C)$ then $\overline{C^w}$ is weakly compact. The Eberlein-Šmulian theorem [4 pg. 549] guarantees that there is a subsequence N of $\{1, 2, \dots\}$ and a $x \in \overline{C^w}$ with $x_n \rightharpoonup x$ as $n \to \infty$ in N. Now since B is weakly sequentially closed we have $x \in B$. Consequently B is weakly sequentially compact, so weakly compact by the Eberlein-Šmulian theorem [3, pg. 430] .

Now $B \cap \partial U = \emptyset$ since (3.2) holds; note $H|_{\partial U} = F|_{\partial U}$ and $0 \in U$. Now $E = (E, w)$, the space E endowed with the weak topology, is completely regular. This there exists a weakly continuous map $\mu : \overline{U^w} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map $R_\mu : \overline{U^w} \to K(C)$ by $R_\mu(x) = \mu(x)H(x)$. Note R_μ has weakly sequentially closed graph (since H has weakly sequentially closed graph) and R_μ takes relatively weakly compact sets into relatively weakly compact sets. [If $A \subseteq \overline{U^w}$ is weakly compact and $y_n \in R_\mu(A)$, then $y_n = \mu(x_n)z_n$ where $z_n \in H(x_n)$ and $x_n \in A$. Without loss of generality we may assume there exists $x \in A$ and $z \in \overline{H(A)}^w$ with $x_n \rightharpoonup x$ and $z_n \rightharpoonup z$ (recall A and $\overline{H(A)}^w$ are weakly compact; in fact from a standard result [7] we note that $H : A \to K(C)$ has weakly closed graph and from another standard result [1] we have that $H(A)$ is weakly compact). Then $z \in H(x)$ since H has weakly sequentially closed graph. Let $y = \mu(x)z$. Then $y_n \rightharpoonup y$ and $y \in R_\mu(A)$. As a result $R_\mu(A)$ is weakly compact.] Next suppose $D \subseteq \overline{U^w}$ with $D \subseteq \overline{co}\,(\{0\} \cup R_\mu(D))$ and with $\overline{D^w} = \overline{C^w}$ and $C \subseteq D$ countable. Then since $R_\mu(D) \subseteq co(\{0\} \cup H(D))$ and

$\{0\} \cup co(\{0\} \cup H(D)) = co(\{0\} \cup H(D))$ we have

$$D \subseteq \overline{co}\left(\{0\} \cup R_\mu(D)\right) \subseteq \overline{co}\left(co\left(\{0\} \cup H(D)\right)\right) = \overline{co}\left(\{0\} \cup H(D)\right).$$

Then $\overline{D^w}$ is weakly compact since $H \in A(\overline{U^w}, C)$. Thus $R_\mu \in A(\overline{U^w}, C)$ with $R_\mu|_{\partial U} = \{0\}$. Now (3.1) guarantees that there exists $x \in U$ with $x \in R_\mu(x)$. As a result $x \in B$, so $\mu(x) = 1$ i.e. $x \in H(x)$. □

Next we discuss (3.1).

Theorem 3.5. *Let E be a metrizable locally convex linear topological space, C a closed convex subset of E, U a weakly open subset of C with $0 \in U$. Suppose* (2.2) *and* (2.3) *hold. Then* (3.1) *holds.*

Proof. Let $\theta \in A_{\partial U}(\overline{U^w}, C)$ with $\theta|_{\partial U} = \{0\}$. We must show θ has a fixed point in U. Let

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U^w} \\ \{0\}, & \text{otherwise.} \end{cases}$$

Note $J : C \to K(C)$ has weakly sequentially closed graph. Now suppose $A \subseteq C$, $A = co(\{0\} \cup J(A))$ with $\overline{A^w} = \overline{D^w}$ and $D \subseteq A$ countable. Then

$$A \subseteq \overline{co}(\{0\} \cup \theta(U \cap A)) \tag{3.3}$$

and so

$$U \cap A \subseteq \overline{co}(\{0\} \cup \theta(U \cap A)). \tag{3.4}$$

Notice $D \cap U$ is countable, $D \cap U \subseteq A \cap U$ and

$$\overline{D \cap U^w} = \overline{A \cap U^w} \tag{3.5}$$

since

$$D \cap U \subseteq A \cap U \subseteq \overline{A^w} \cap U = \overline{D^w} \cap U \subseteq \overline{D \cap U^w}$$

(note for sets D_0 and D_1 of C with D_0 weakly open in C, then $D_0 \cap \overline{D_1}^w \subseteq \overline{D_0 \cap D_1}^w$). Now (3.4) and (3.5) and $\theta \in A(\overline{U^w}, C))$ implies that $\overline{A \cap U^w}$ is weakly compact. Also since $\theta \in A(\overline{U^w}, C)$ (θ takes relatively weakly compact sets into relatively weakly compact sets) we have that $\theta(A \cap U)$ is relatively weakly compact. Now (2.3) guarantees that $\overline{co}(\{0\} \cup \overline{\theta(A \cap U)}^w)$ is weakly compact. This together with (3.3) implies that $\overline{A^w}$ is weakly compact.

Now Theorem 2.2 guarantees that there exists $x \in C$ with $x \in J(x)$. If $x \notin U$ we have $x \in J(x) = \{0\}$, which is a contradiction since $0 \in U$. Thus $x \in U$ so $x \in J(x) = \theta(x)$. □

Combining Theorem 3.4 and Theorem 3.5 yields the following nonlinear alternative of Leray-Schauder type.

Theorem 3.6. *Let E be a metrizable locally convex linear topological space, C a closed convex subset of E, and U a weakly open subset of C with $0 \in U$. Suppose* (2.2) *and* (2.3) *hold. Also suppose $F \in A(\overline{U^w}, C)$ satisfies* (3.2)*. Then F is essential in $A_{\partial U}(\overline{U^w}, C)$ (in particular F has a fixed point in U).*

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