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LYAPUNOV TYPE INEQUALITIES FOR SECOND ORDER SUB AND SUPER-HALF-LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In the case of oscillatory potential, we present a Lyapunov type inequality for second order differential equations of the form

$$(r(t)\Phi_\beta(x'(t)))' + q(t)\Phi_\gamma(x(t)) = 0,$$

in the sub-half-linear ($0 < \gamma < \beta$) and the super-half-linear ($0 < \beta < \gamma < 2\beta$) cases where $\Phi_*(s) = |s|^{*-1}s$.

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1. INTRODUCTION

Consider the second-order differential equation

$$(r(t)\Phi_\beta(x'(t)))' + q(t)\Phi_\gamma(x(t)) = 0; \qquad a \le t \le b, \tag{1.1}$$

with the Dirichlet boundary conditions

$$x(a) = x(b) = 0, \tag{1.2}$$

where r, q are real-valued functions with $r(t) > 0$. Our interest is to establish a new Lyapunov type inequality for Eq. (1.1) which does not assume that the potential q be of definite sign.

When $\beta = \gamma$, Prb. (1.1)–(1.2) reduces to the half-linear equation

$$(r(t)\Phi_\gamma(x'(t)))' + q(t)\Phi_\gamma(x(t)) = 0; \qquad a \le t \le b, \tag{1.3}$$

satisfying (1.2).

In 2003, Yang [47] showed that there is a striking similarity between linear and half-linear equations and obtained the following result.

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Theorem 1.1 (Lyapunov Type Inequality). *If $x(t)$ is a nontrivial solution of Eq.* (1.3) *satisfying* (1.2), *where a, $b \in \mathbb{R}$ are consecutive zeros, then the inequality*

$$\int_a^b q^+(t)\mathrm{d}t > 2^{\gamma+1}\left(\int_a^b r^{-1/\gamma}(t)\mathrm{d}t\right)^{-\gamma} \tag{1.4}$$

holds, where $q^+(t) = \max\{q(t), 0\}$ and γ is positive constant.

Similar Lyapunov type inequalities with (1.4) were obtained by O. Došlý and P. Řehák [8, p. 190], Lee et al. [20] and Pinasco [33, 34].

In 2010, Sim and Lee [37] proved the following inequality

$$\int_a^b (b-t)^\gamma (t-a)^\gamma q(t)\mathrm{d}t \geq 2^{1-\gamma}(b-a)^\gamma \tag{1.5}$$

for Eq. (1.3) with $r(t) = 1$ and $q(t) > 0$. However, Ineq. (1.5) is not correct: Ineq.'s (2.4) and (2.6) obtained in [37, pp. 4] are on the intervals $[a, (a+b)/2]$ and $[(a+b)/2, b]$, respectively, and they cannot be just added to get Ineq. (1.5).

In 2011, Wang [45], and Tang and He [39], removing the positivity condition on $q(t)$, extended Ineq.'s (1.4) and (1.5) to

$$\int_a^b I_a^\gamma(t) I_b^\gamma(t) q^+(t)\mathrm{d}t > C_\gamma \left[I_a(t) + I_b(t)\right]^\gamma = C_\gamma \left(\int_a^b r^{-1/\gamma}(t)\mathrm{d}t\right)^\gamma \tag{1.6}$$

and

$$\int_a^b \left[I_a^{-\gamma}(t) + I_b^{-\gamma}(t)\right]^{-1} q^+(t)\mathrm{d}t > 1 \tag{1.7}$$

respectively, where

$$I_a(t) = \int_a^t r^{-1/\gamma}(s)\mathrm{d}s, \quad I_b(t) = \int_t^b r^{-1/\gamma}(s)\mathrm{d}s$$

and

$$C_\gamma = \begin{cases} 1, & 0 < \gamma \leq 1; \\ 2^{1-\gamma}, & \gamma > 1. \end{cases}$$

Here, we note that the Ineq. (1.7) is sharper than Ineq. (1.6) in both cases i.e. $\gamma \in (0,1)$ and $\gamma \in (1, \infty)$ that follows from the inequality:

$$\eta_1^\mu + \eta_2^\mu \geq (\eta_1 + \eta_2)^\mu \times \begin{cases} 1, & 0 < \mu \leq 1; \\ 2^{1-\mu}, & \mu > 1; \end{cases} \qquad \eta_1, \eta_2 \in [0, \infty).$$

We note that the above inequality is an immediate consequence of following lemma.

Lemma 1.2 (Jensen's Inequality [23]). *Let $f(u)$ be a convex function defined on an interval $J \subset \mathbb{R}$. If $u_k \in J$ and $c_k \in \mathbb{R}^+$, $k = 1, \dots, n$ with $\sum_{k=1}^n c_k = 1$, then the inequality*

$$f\left(\sum_{k=1}^n c_k u_k\right) \leq \sum_{k=1}^n c_k f(u_k) \tag{1.8}$$

holds.

In 2012, Tiryaki et al. [42] improved Ineq. (1.7) and obtained similar Lyapunov type inequalities for equations of the form

$$(r(t)|x'(t)|^{\alpha-2}x'(t))' + q(t)|x(t)|^{\alpha_*-2}x(t) = 0,$$

where α^* is the conjugate exponent of $\alpha > 1$, i.e., $(\alpha^* - 1)(\alpha - 1) = 1$.

Basically, Ineq. (1.4) is a generalization of the classical Lyapunov inequality

$$\int_a^b |q(t)| \mathrm{d}t > 4(b-a)^{-1}, \tag{1.9}$$

established by Lyapunov [21] for Hill's equation

$$x''(t) + q(t)x(t) = 0; \qquad a \le t \le b \tag{1.10}$$

satisfying (1.2). Here the constant 4 in the right hand side of Ineq. (1.9) is the best possible largest number (see [21] and [15, Thm. 5.1]). In fact, the best Lyapunov inequality for Eq. (1.10) satisfying (1.2) is

$$\int_a^b (b-t)(t-a)q^+(t)\mathrm{d}t > b - a, \tag{1.11}$$

obtained by Hartman [15]. Note that Ineq.'s (1.6) and (1.7) reduce to Ineq. (1.11) when $r(t) = 1$ and $\gamma = 1$.

The Lyapunov inequality (1.9) and its several generalizations have proved to be useful tools in oscillation and Sturmian theory, disconjugacy, asymptotic theory, eigenvalue problems, boundary value problems and various properties of the solutions of (1.1) and associated equations, see for instance [1–3, 6, 7, 11, 15, 16, 19, 20, 22, 24–26, 29, 35, 36, 38, 46] and the references cited therein. For some extensions to Hamiltonian systems, higher order differential equations, nonlinear and half-linear differential equations, difference and dynamic equations, functional and impulsive differential equations, we refer in particular to [4–10, 12–14, 17, 18, 27, 28, 30–32, 40, 43, 44, 48, 49].

Motivated by the half-linear equations, we attempt to obtain in this work the analogous Lyapunov type inequalities for second-order equations of the form (1.1) satisfying the Dirichlet boundary condition (1.2). Moreover we shall show how they can be carried over to linear, half-linear, sub-linear and super-linear equations.

It is clear that two special cases of Eq. (1.1) are the sub-half-linear equation ($0 < \gamma < \beta$) and the super-half-linear equation ($0 < \beta < \gamma < 2\beta$). Although the nonlinear results require the coefficient functions to be non-negative, as an important feature of our work, we allow $q(t)$ to be negative in equation (1.1). Further, we note that letting $\beta = \gamma$ results in (1.3), and as a consequence our results extend some of the existing Lyapunov type results for half-linear equations. We also give two new Lyapunov type inequalities for the Emden-Fowler equations

$$(r(t)x'(t))' + q(t)\Phi_\gamma(x(t)) = 0, \tag{1.12}$$

in the sub-linear ($0 < \gamma < 1$) and the super-linear ($1 < \gamma < 2$) cases by taking $\beta = 1$.

We note that the Lyapunov type inequalities of this type have been studied by many authors, but to the best of our knowledge, there is no result in the literature for nonlinear equations of type (1.1) and (1.12), see for instance the survey papers of Cheng [6] and Tiryaki [41] and the references therein.

2. MAIN RESULTS

Throughout this paper we shall assume that the function $r(t) \in \mathrm{L}^{(-1/\beta)}[a,b]$, $\beta > 0$, and the potential $q(t) \in \mathrm{L}^1[a,b]$.

We will need the following lemma.

Lemma 2.1. *If A is positive, and B, z are nonnegative, then*

$$Az^{2\tau} - Bz^\sigma + \Gamma_{\sigma\tau} A^{-\sigma/(2-\sigma)} B^{2\tau/(2-\sigma)} \geq 0 \tag{2.1}$$

for any $\sigma \in (0, 2\tau)$, where

$$\Gamma_{\sigma\tau} = (2\tau - \sigma)\sigma^{\sigma/(2\tau-\sigma)}\tau^{-2\tau/(2\tau-\sigma)}2^{-2\tau/(2\tau-\sigma)}$$

with equality holding if and only if $B = z = 0$.

Proof. Let

$$\mathcal{G}(z) = Az^{2\tau} - Bz^\sigma; \qquad z \geq 0 \tag{2.2}$$

where $A > 0$ and $B \geq 0$. Clearly, when $z = 0$ or $B = 0$, (2.1) is obvious. On the other hand, if $B > 0$, then it is easy to see that $\mathcal{G}$ attains its minimum at $z_0 = (\sigma A^{-1}B/(2\tau))^{1/(2\tau-\sigma)}$ and

$$\mathcal{G}_{\min} = -\Gamma_{\sigma\tau} A^{-\sigma/(2\tau-\sigma)} B^{2\tau/(2\tau-\sigma)}.$$

Thus, (2.1) holds. Note that if $B > 0$, then Ineq. (2.1) is strict. □

Now we state and prove our first result.

Theorem 2.2 (Lyapunov Type Inequality)**.** *If $x(t)$ is a nontrivial solution of Eq.* (1.1) *satisfying* (1.2)*, where a, $b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then the inequality*

$$\int_a^b q^+(t)\mathrm{d}t > \frac{2^\beta}{\sqrt{\Gamma_{\gamma\beta}}} \left(\int_a^b r^{-1/\beta}(t)\mathrm{d}t \right)^{-\beta} \tag{2.3}$$

holds, where $\gamma \in (0, 2\beta)$ and

$$\Gamma_{\gamma\beta} = (2\beta - \gamma)\gamma^{\gamma/(2\beta-\gamma)}\beta^{-2\beta/(2\beta-\gamma)}2^{-2\beta/(2\beta-\gamma)} > 0. \tag{2.4}$$

Proof. Let $x(t)$ be a nontrivial solution of Eq. (1.1) satisfying (1.2), where $a,\ b \in \mathbb{R}$ with $a < b$ are consecutive zeros. Without loss of generality, we may assume that $x(t) > 0$ for $t \in (a, b)$. In fact, if $x(t) < 0$ for $t \in (a, b)$, then we can consider $-x(t)$, which is also a solution. Let $c \in (a, b)$ be the least point of the local maxima of $x(t)$ in (a, b), i.e., $x'(c) = 0$ and $x'(t) > 0$ on $[a, c)$. Then we have

$$
\begin{aligned}
x^{\beta+1}(c) &= \left(\int_a^c x'(t)\mathrm{d}t\right)^{\beta+1} \\
&= \left(\int_a^c r^{-1/(\beta+1)}(t)\, r^{1/(\beta+1)}(t)x'(t)\mathrm{d}t\right)^{\beta+1} \\
&\le \left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t\right)^{\beta} \int_a^c r(t)[x'(t)]^{\beta+1}\mathrm{d}t \qquad (2.5)
\end{aligned}
$$

by Hölder's inequality. On the other hand, multiplying (1.1) by $x(t)$ and integrating from a to c by parts, we get

$$
\int_a^c r(t)[x'(t)]^{\beta+1}\mathrm{d}t = \int_a^c q(t)x^{\gamma+1}(t)\mathrm{d}t \le x^{\gamma+1}(c)\int_a^c q^+(t)\mathrm{d}t. \qquad (2.6)
$$

Now using (2.5) and (2.6) we obtain $x^\beta(c) \le \mathcal{K}_0 x^\gamma(c)$, where

$$
\mathcal{K}_0 = \left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t\right)^{\beta}\int_a^c q^+(t)\mathrm{d}t.
$$

Using this and Ineq. (2.1) in Lemma 2.1 with $A = B = 1$, we have the quadratic inequality

$$
\mathcal{K}_0 x^{2\beta}(c) - x^\beta(c) + \mathcal{K}_0\Gamma_{\gamma\beta} > 0.
$$

This is possible only if $\mathcal{K}_0^2\Gamma_{\gamma\beta} > 1/4$, i.e.,

$$
\int_a^c q^+(t)\mathrm{d}t > \frac{1}{2\sqrt{\Gamma_{\gamma\beta}}}\left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta}. \qquad (2.7)
$$

Similarly, if d is the greatest point of local maxima of $x(t)$ in (a, b), i.e., $x'(d) = 0$ and $x'(t) < 0$ on (d, b), we have

$$
\int_d^b q^+(t)\mathrm{d}t > \frac{1}{2\sqrt{\Gamma_{\gamma\beta}}}\left(\int_d^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta}. \qquad (2.8)
$$

Consequently, adding up (2.7) and (2.8) we obtain

$$
\begin{aligned}
\int_a^b q^+(t)\mathrm{d}t &\ge \int_a^c q^+(t)\mathrm{d}t + \int_d^b q^+(t)\mathrm{d}t \\
&> \frac{1}{2\sqrt{\Gamma_{\gamma\beta}}}\left\{\left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} + \left(\int_d^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta}\right\}.
\end{aligned}
$$

On the other hand, since the function $f(u) = u^{-\beta}$ is convex for $u > 0$, Ineq. (1.8) in Lemma 1.2 with $u_1 = \int_a^c r^{-1/\beta}(t)\mathrm{d}t$, $u_2 = \int_d^b r^{-1/\beta}(t)\mathrm{d}t$ and $c_1 = c_2 = 1/2$ implies

$$\left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} + \left(\int_d^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta}$$
$$\geq 2^{\beta+1}\left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t + \int_d^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta}$$
$$\geq 2^{\beta+1}\left(\int_a^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta},$$

which completes the proof of Theorem 2.2. □

Remark 2.3. Thm. 2.2 is interesting because it covers all sub-half-linear, half-linear and super-half-linear equations i.e., the cases $\gamma \in (0, \beta)$, $\gamma = \beta$ and $\gamma \in (\beta, 2\beta)$, respectively. Moreover when $\gamma = \beta$, it is easy to see that Thm. 2.2 improves the main result of Yang [47], i.e., Thm. 1.1. In fact, since $\Gamma_{\gamma\gamma} = 1/4$, condition (2.3) of Thm. 2.2 reduces to condition (1.4) of Thm. 1.1.

Remark 2.4. Thm. 2.2 is of particular interest, in fact, it improves the classical Lyapunov's inequality not only for the Hill's equation, but also the Sturm-Liouville equation satisfying (1.2) for $\beta = \gamma = 1$.

Next we present a new result where Eq. (1.1) has two special type of nonlinearities, namely Emden-Fowler sub-linear ($0 < \gamma < 1$) and super-linear ($1 < \gamma < 2$) equations.

Theorem 2.5 (Lyapunov Type Inequality). *If $x(t)$ is a nontrivial solution of Eq.* (1.12) *satisfying* (1.2)*, where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then the inequality*

$$\int_a^b q^+(t)\mathrm{d}t \geq \frac{2^{(3-\gamma)/(2-\gamma)}}{\gamma^{\gamma/(4-2\gamma)}\sqrt{2-\gamma}}\left(\int_a^b r^{-1}(t)\mathrm{d}t\right)^{-1}$$

holds, where $\gamma \in (0, 2)$.

We note that when $\gamma = 1$, classical results can be obtained from Thm. 2.5.

We conclude this paper with the following remark. The results obtained in this paper for Eq. (1.1) can be easily extended to the second order equations with negative coefficients

$$(r(t)\Phi_\beta(x'(t)))' - q(t)\Phi_\gamma(x(t)) = 0.$$

The formulation of the results are left to the reader.

It will be of interest to find a Hartman type inequality for Eq. (1.1), and similar results for the mixed nonlinear equations with positive and negative coefficients

$$(r(t)\Phi_\beta(x'(t)))' \pm p(t)\Phi_\alpha(x(t)) \mp q(t)\Phi_\gamma(x(t)) = 0$$

for $0 < \gamma < \beta < \alpha$ or Emden-Fowler super-linear Eq. (1.12) for $\gamma \in [2, \infty)$.

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REFERENCES

[1] A. Beurling, Un théoréme sur les fonctions bornées et uniformément continues sur l'axe réel, *Acta Math.*, 77:127–136, 1945.

[2] G. Borg, On a Liapunoff criterion of stability, *Amer. J. Math.*, 71:67–70, 1949.

[3] R. C. Brown and D. B. Hinton, Opials inequality and oscillation of 2nd order equations, *Proc. Amer. Math. Soc.*, 125:1123–1129, 1997.

[4] D. Cakmak, Lyapunov-type integral inequalities for certain higher order differential equations, *Appl. Math. Comput.*, 216:368–373, 2010.

[5] S. S. Cheng, A discrete analogue of the inequality of Lyapunov, *Hokkaido Math.*, 12:105–112, 1983.

[6] S. S. Cheng, Lyapunov inequalities for differential and difference equations, *Fasc. Math.*, 23:25–41, 1991.

[7] R. S. Dahiya and B. Singh, A Liapunov inequality and nonoscillation theorem for a second order nonlinear differential-difference equations, *J. Math. Phys. Sci.*, 7:163–170, 1973.

[8] O. Došlý and P. Řehák, *Half-Linear Differential Equations*, Elsevier Ltd., Heidelberg, 2005.

[9] A. Elbert, A half-linear second order differential equation, *Colloq. Math. Soc. Jnos Bolyai*, 30:158–180, 1979.

[10] S. B. Eliason, A Lyapunov inequality for a certain non-linear differential equation, *J. London Math. Soc.*, 2: 461–466, 1970.

[11] S. B. Eliason, Lyapunov inequalities and bounds on solutions of certain second order equations, *Canad. Math. Bull.*, 17:499–504, no. 4, 1974.

[12] S. B. Eliason, Lyapunov type inequalities for certain second order functional differential equations, *SIAM J. Appl. Math.*, 27:180–199, 1974.

[13] G. Sh. Guseinov and B. Kaymakcalan, Lyapunov inequalities for discrete linear Hamiltonian systems, *Comput. Math. Appl.*, 45:1399–1416, 2003.

[14] G. Sh. Guseinov and A. Zafer, Stability criteria for linear periodic impulsive Hamiltonian systems, *J. Math. Anal. Appl.*, 35:1195–1206, 2007.

[15] P. Hartman, *Ordinary Differential Equations*, Wiley, New York, 1964 and Birkhuser, Boston, 1982.

[16] H. Hochstadt, A new proof of stability estimate of Lyapunov, *Proc. Amer. Math. Soc.*, 14:525–526, 1963.

[17] L. Jiang and Z. Zhou, Lyapunov inequality for linear Hamiltonian systems on time scales, *J. Math. Anal. Appl.*, 310:579–593, 2005.

[18] Z. Kayar and A. Zafer, Stability criteria for linear Hamiltonian systems under impulsive perturbations, *Appl. Math. Comput.*, 230:680–686, 2014.

[19] M. K. Kwong, On Lyapunov's inequality for disfocality, *J. Math. Anal. Appl.*, 83:486–494, 1981.

[20] C. Lee, C. Yeh, C. Hong and R. P. Agarwal, Lyapunov and Wirtinger inequalities, *Appl. Math. Lett.*, 17:847–853, 2004.

[21] A. M. Liapunov, Probleme général de la stabilité du mouvement, (French Translation of a Russian paper dated 1893), *Ann. Fac. Sci. Univ. Toulouse 2 (1907), 27–247, Reprinted as Ann. Math. Studies, No. 17*, Princeton, 1947.

[22] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives, Mathematics and its Applications (East European Series)*, 53. Kluwer Academic Publishers Group, Dordrecht, 1991.

[23] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis. Mathematics and its Applications (East European Series)*, 61. Kluwer Academic Publishers Group, Dordrecht, 1993.

[24] P. L. Napoli and J. P. Pinasco, Estimates for eigenvalues of quasilinear elliptic systems, *J. Differential Equations*, 227:102–115, 2006.

[25] Z. Nehari, Some eigenvalue estimates, *J. Anal. Math.*, 7:79–88, 1959.

[26] Z. Nehari, *On an inequality of Lyapunov, in: Studies in Mathematical Analysis and Related Topics*, Stanford University Press, Stanford, CA, 1962.

[27] B. G. Pachpatte, On Lyapunov-type inequalities for certain higher order differential equations *J. Math. Anal. Appl.*, 195:527–536, 1995.

[28] B. G. Pachpatte, Lyapunov type integral inequalities for certain differential equations *Georgian Math. J.*, 4:139–148, 1997.

[29] B. G. Pachpatte, Inequalities related to the zeros of solutions of certain second order differential equations, *Facta Univ. Ser. Math. Inform.*, 16:35–44, 2001.

[30] S. Panigrahi, Lyapunov-type integral inequalities for certain higher order differential equations, *Electron. J. Differential Equations*, 2009:1–14, 2009.

[31] N. Parhi and S. Panigrahi, On Liapunov-type inequality for third-order differential equations, *J. Math. Anal. Appl.*, 233:445–460, 1999.

[32] N. Parhi and S. Panigrahi, Liapunov-type inequality for higher order differential equations, *Math. Slovaca*, 52:31–46, 2002.

[33] J. P. Pinasco, Lower bounds for eigenvalues of the one-dimensional p-Laplacian, *Abstr. Appl. Anal.*, 2004:147–153, 2004.

[34] J. P. Pinasco, Comparison of eigenvalues for the p-Laplacian with integral inequalities, *Appl. Math. Comput.*, 182:1399–1404, 2006.

[35] T. W. Reid, A matrix equation related to an non-oscillation criterion and Lyapunov stability *Quart. Appl. Math. Soc.*, 23:83–87, 1965.

[36] T. W. Reid, A matrix Lyapunov inequality, *J. Math. Anal. Appl.*, 32:424–434, 1970.

[37] I. Sim and Y. H. Lee, Lyapunov inequalities for one-dimensional p-Laplacian problems with a singular weight function *J. Inequal. Appl.*, Art. ID 865096:9, 2010.

[38] B. Singh, Forced oscillation in general ordinary differential equations, *Tamkang J. Math.*, 6:5–11, 1975.

[39] X. H. Tang and X. He, Lower bouns for generalized eigenvalues of the quasilinear systems, *J. Math. Anal. Appl.*, 385:72–85, 2012.

[40] A. Tiryaki, M. Unal and D. Cakmak, Lyapunov-type inequalities for nonlinear systems, *J. Math. Anal. Appl.*, 332: 497–511, 2007.

[41] A. Tiryaki, Recent developments of Lyapunov-type inequalities, *Advances in Dynam. Sys. Appl.*, 5:231–248, 2010.

[42] A. Tiryaki, D. Cakmak and M.F. Aktas, Lyapunov-type inequalities for a certain class of nonlinear systems, *Comput. Math. Appl.*, 64:1804–1811, 2012.

[43] M. Unal, D. Cakmak and A. Tiryaki, A discrete analogue of Lyapunov-type inequalities for nonlinear systems, *Comput. Math. Appl.*, 55:2631–2642, 2008.

[44] M. Unal and D. Cakmak, Lyapunov-type inequalities for certain nonlinear systems on time scales, *Turkish J. Math.*, 32:255–275, 2008.

[45] X. Wang, Lyapunov type inequalities for second-order half-linear differential equations, *J. Math. Anal. Appl.*, 382:792–801, 2011.

[46] A. Wintner, On the nonexistence of conjugate points, *Amer. J. Math.*, 73:368–380, 1951.

[47] X. Yang, On inequalities of Lyapunov type, *Appl. Math. Comput.*, 134:293–300, 2003.

[48] X. Yang, On Liapunov-type inequality for certain higher-order differential equations, *Appl. Math. Comput.*, 134:307–317, 2003.

[49] X. Yang, Lyapunov-type inequality for a class of even-order differential equations, *Appl. Math. Comput.*, 215:3884–3890, 2010.