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NECESSARY CONDITIONS OF OPTIMALITY FOR STOCHASTIC SWITCHING CONTROL SYSTEMS

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ABSTRACT. This paper is devoted to optimal control problem of stochastic switching systems. Dynamics of this processes governed by stochastic differential equations with control terms in the drift and diffusion coefficients. Necessary conditions for optimality of described systems with the restrictions in each interval are obtained. The constraints on the transitions are described by the set of functional inclusions. Ekeland's variational principle are applied to prove maximum principle in general form.

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1. INTRODUCTION

Optimal control problems of stochastic systems have a multitude practical applications in fields such as pricing an option,forecasting the growth of population and determining optimal portfolio of investments,etc. [1, 2, 3, 4, 5]. Modern theory of stochastic optimal control in the main has been developed along the two lines: maximum principle and dynamic programming [6, 7]. The analogue of maximum principle for stochastic systems has been first obtained by Kushner [8]. Earlier results on the developments of Pontryagin's maximum principle for stochastic control systems are met in [9, 10, 11, 12]. Investigation of stochastic maximum principle by using random convex analysis was obtained by Bismut [13]. In [14, 15, 16] are obtained the modern presentation of maximum principle for stochastic systems with backward stochastic differential equations . Many real systems have unpredictable structural changes in their behavior from causes of random failures, sudden disturbances, abrupt variation of the connecting points on a mechanisms. These processes have been described by the collection of stochastic differential equations [17, 18] are known as hybrid systems. A switching systems are special class of hybrid systems and have the advantage of modeling nature phenomena with the continuous changing law of system's structure. Therefore optimization problems of switching systems provide both theoretical and practical interest [19, 20, 21, 22, 23].

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This paper is dedicated to the stochastic optimal control problems of switching systems with controlled drift and controlled diffusion coefficients. We obtain necessary condition of optimality in the form of a maximum principle for such systems, where the restrictions on transitions are described by functional constraints in the each of constituent interval.

In present paper, backward stochastic differential equations have been used to prove a maximum principle for stochastic optimal control problems of switching systems. Optimal control problems of stochastic switching systems with uncontrolled diffusion coefficients have been considered by the authors in [24, 25, 26, 27]. The problem with controlled diffusion coefficients without endpoint constraints is studied in [28]. Stochastic switching systems with controlled diffusion and with the special type of restrictions were investigated in [29, 30].

This paper contains five sections. Notations, definitions and the statement of main problem are given in Section 2. Section 3 is devoted to problem of optimality of stochastic switching systems with controlled coefficients. In section 4 stochastic optimal control problem of switching system with endpoint restrictions is treated. It is proved some important facts for our goal and is established necessary condition of optimality in form of maximum principle. Conclusion finalizes the present paper.

2. DESCRIPTION OF MAIN PROBLEM

Following notations are used throughout present paper. R^m represents the m dimensional real vector space; $|\cdot|$ denotes the Euclidean norm. We use N as notation for some positive constant; $\overline{1,r}$ denotes the set of integer numbers $1,\dots,r$. Assume that σ-algebras $F_t^l = \bar{\sigma}(w_t^l, t_{l-1} \leq t \leq t_l)$ are generated by independent Wiener processes $w_t^1, w_t^2, \dots, w_t^r$. Let (Ω, F, P) be a complete probability space with filtration $\{F_t, t \in [0,T]\}$, where $F_t = \bar{\sigma}(F_t^l, l = \overline{1,r})$. $L^2_{F^l}(0,T;R^m)$ denotes the space of all predictable processes $x_t(\omega)$ in R^m such that $E\int\limits_0^T |x_t(\omega)|^2 dt < +\infty$. $R^{k\times m}$ represents the space of all linear transformations from R^k to R^m. Let $O_l \subset R^{n_l}$, $Q_l \subset R^{m_l}$, be open sets. Unless specified otherwise, we will use following notations: $\mathbf{t} = (t_0,\dots,t_r)$, $\mathbf{u} = (u^1,\dots,u^r)$ and $\mathbf{x} = (x^1,\dots,x^r)$

Consider the following stochastic control system:

$$dx_t^l = g^l(x_t^l, u_t^l, t)dt + f^l(x_t^l, u_t^l, t)dw_t^l, \quad t \in (t_{l-1}, t_l], l = \overline{1,r}; \tag{2.1}$$

$$x_{t_l}^{l+1} = \Phi^l(x_{t_l}^l, t_l)\, l = \overline{1, r-1}\,; x_{t_0}^1 = \xi, \tag{2.2}$$

$$u_t^l \in U_\partial^l \equiv \{u^l(\cdot,\cdot) \in L^2_{F^l}(t_{l-1}, t_l; R^{m_l}) | u^l(t,\cdot) \in U \subset R^{m_l} \; a.c.\} \tag{2.3}$$

where $\{t_k\}$ denote the time that x_t is heavily changed, which are a series of unknown moments satisfying $t_1 < t_2 < t_3 < \cdots$.

Elements of $U_\partial = U_\partial^1 \times U_\partial^2 \times \cdots \times U_\partial^r$ are called admissible controls. Our main goal is to find optimal inputs $(x^1, x^2, \ldots, x^r, u^1, u^2, \ldots, u^r)$ and switching sequence $t_1, t_2, \ldots, t_r$, which are minimize following cost functional:

$$J(\mathbf{u}) = E\left[\varphi(x_{t_r}^r) + \sum_{l=1}^{r} \int_{t_{l-1}}^{t_l} p^l(x_t^l, u_t^l, t)dt\right] \tag{2.4}$$

on the decisions of the system (2.1)-(2.3), which are generated by all admissible controls at conditions:

$$Eq^l(x_{t_l}^l) \in G^l, \quad l = \overline{1, r}\,. \tag{2.5}$$

G^l are a closed convex sets in R^1.

To establish necessary condition of optimality for the stochastic control problem (2.1)–(2.5) we need to the following assumptions.

(H1) Functions $g^l, f^l, p^l, \Phi^l, q^l$ are twice continuously differentiable with respect to x for each $l = 1, 2, \ldots, r$.

(H2) For each $l = 1, 2, \ldots, r$ functions g^l, f^l, p^l, and all their derivatives are continuous in (x, u). $g_x^l, g_{xx}^l, f_x^l, f_{xx}^l, p_{xx}^l$ are bounded and hold the linear growth conditions.

(H3) Functions $\varphi(x) : R^{n_r} \to R$ are twice continuously differentiable and hold the condition:

$$|\varphi(x)| + |\varphi_x(x)| \le N(1 + |x|), \quad |\varphi_{xx}(x)| \le N.$$

(H4) For each $l = 1, 2, \ldots, r - 1$ functions $\Phi^l(x, t) : R^{n_l} \times \mathrm{T} \to R^1$ are twice continuously differentiable with respect to (t) and satisfy the condition:

$$\left|\Phi^l(x, t)\right| + \left|\Phi_x^l(x, t)\right| \le N(1 + |x|), \quad \left|\Phi_{xx}^l(x, t)\right| \le N.$$

(H5) Functions $q^l\,(x) : R^{n_l} \to R^1, l = \overline{1, r}$ are continuously differentiable with respect to (x) and meet the condition:

$$\left|q^l(x)\right| + \left|q_x^l(x)\right| \le N(1 + |x|), \quad \left|q_{xx}^l(x)\right| \le N.$$

Consider the sets:

$$A_i = \mathrm{T}^{i+1} \times \prod_{j=1}^{i} O_j \times \prod_{j=1}^{i} Q_j, \quad i = \overline{1, r},$$

with the elements

$$\pi^i = (t_0, t_1, \ldots, t_i, x_t^1, x_t^2, \ldots, x_t^i, u^1, u^2, \ldots, u^i).$$

Definition 2.1. Collection of stochastic processes $\left\{x_t^l = x^l(t, \pi^l)\right\}$, $t \in [t_{l-1}, t_l]\,, l = \overline{1, r}$ is called a solution of variable structure system (2.1)–(2.2) corresponding to an element $\pi^r \in \mathrm{A}_r$, if stochastic process $x_t^l \in O_l$ almost certainly continuous on the interval $[t_{l-1}, t_l]$, holds the condition (2.2) at point t_l and satisfies the equation (2.1) almost everywhere.

Definition 2.2. $\pi^r \in A_r$ is called the admissible element if pairs $(x_t^l, u_t^l), t \in [t_{l-1}, t_l]$, $l = \overline{1,r}$ satisfy (2.1)–(2.3) and conditions (2.5).

Definition 2.3. Let A_r^0 be the set of admissible elements. The element $\tilde{\pi}^r \in A_r^0$, is called an optimal solution of problem (2.1)–(2.5) if there exist admissible controls $\tilde{u}_t^l$, $t \in [t_{l-1}, t_l]$ and corresponding solutions of system (2.1)–(2.2) such that pairs $(\tilde{x}_t^l, \tilde{u}_t^l)$, $l = \overline{1,r}$ minimize the functional (2.4).

3. MAXIMUM PRINCIPLE OF STOCHASTIC SWITCHING SYSTEMS

Applying the similar technique as in [28] following necessary condition of optimality for stochastic control system (2.1)–(2.4) is obtained.

Theorem 3.1. *Suppose that,*

$$\pi^r = (t_0, t_1, \dots, t_r, x_t^1, \dots, x_t^r, u^1, \dots, u^r)$$

is an optimal solution of problem (2.1)–(2.4) *and conditions* (H1)–(H4) *hold . Then,*
a) there exist stochastic processes $(\psi_t^l, \beta_t^l) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l x n_l})$ *and* $(\Psi_t^l, \mathrm{K}_t^l) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l x n_l})$ *which are the solutions of the following conjugate equations:*

$$\begin{cases} d\psi_t^l = -H_x^l(\psi_t^l, x_t^l, u_t^l, t)dt + \beta_t^l dw_t^l, \;\; t_{l-1} \le t < t_l, l = \overline{1,r}, \\ \psi_{t_l}^l = \psi_{t_l}^{l+1} \Phi_x^l(x_{t_l}^l, t_l), \; l = \overline{1, r-1}, \\ \psi_{t_r}^r = -\varphi_x(x_{t_r}^r), \end{cases} \tag{3.1}$$

$$\begin{cases} d\Psi_t^l = -[\boldsymbol{H}_x^l(\Psi_t^l, x_t^l, u_t^l, t) + H_{xx}^l(\psi_t^l, x_t^l, u_t^l, t) \\ +f_x^{l*}(x_t^l, u_t^l, t)\Psi_t^l f_x^l(x_t^l, u_t^l, t)] \; dt + \mathrm{K}_t^l dw_t^l, t \in [t_{l-1}, t_l) \\ \Psi_{t_l}^l = \Psi_{t_l}^{l+1} \Phi_{xx}^l(x_{t_l}^l, t_l), \;\; l = \overline{1, r-1}, \\ \Psi_{t_r}^r = -\varphi_{xx}(x_{t_r}^r) \end{cases} \tag{3.2}$$

b) almost everywhere in $\theta \in [t_{l-1}, t_l]$, *and* $\forall \; \tilde{u}^l \in U^l$, $l = \overline{1,r}$, *almost certainly (a.c.) fulfills the maximum principle:*

$$H^l(\psi_\theta^l, x_\theta^l, u^l, \theta) - H^l(\psi_\theta^l, x_\theta^l, u_\theta^l, \theta) + 0.5\Delta_{u^l} f^{l*}(x_\theta^l, u_\theta^l, \theta)\Psi_\theta^l \Delta_{u^l} f^l(x_\theta^l, u_\theta^l, \theta) \le 0 \tag{3.3}$$

c) following transversality conditions hold:

$$\psi_{t_l}^{l+1} \Phi_t^l(x_{t_l}^l, t_l) = 0 \, , \; l = \overline{1, r-1} \, , \; a.c. \tag{3.4}$$

Here we used following notations:

$$H^l(\psi_t, x_t, u_t, t) = \psi_t g^l(x_t, u_t, t) + \beta_t f^l(x_t, u_t, t) - p^l(x_t, u_t, t),$$

$$\mathbf{H}^l(\Psi_t, x_t, u_t, t) = \Psi_t g^l(x_t, u_t, t) + g^{l*}(x_t, u_t, t)\,\Psi_t + \mathrm{K}_t f^l(x_t, u_t, t) + f^{l*}(x_t, u_t, t)\,\mathrm{K}_t.$$

Proof. Let $\bar{u}_t^l = u_t^l + \Delta\bar{u}_t^l$ $l = \overline{1,r}$ represent some admissible controls and $\bar{x}_t^l = x_t^l + \Delta\bar{x}_t^l$ $l = \overline{1,r}$ represent the corresponding trajectories of system (2.1)–(2.3). Let $0 = t_0 < t_1 < \cdots < t_r = T$ be switching sequence. Then following identities are obtained for some sequence $0 = \bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_r = T$:

(3.5)

$$\begin{cases} d\Delta\bar{x}_t^l = \left[\Delta_{\bar{u}^l} g^l(x_t^l, u_t^l, t) + g_x^l(x_t^l, u_t^l, t)\Delta\bar{x}_t^l + 0.5\Delta x_t^* g_{xx}^l(x_t^l, u_t^l, t)\Delta x_t\ \right] dt + \\ \left[\Delta_{\bar{u}^l} f^l(x_t^l, u_t^l, t) + f_x^l(x_t^l, u_t^l, t)\Delta\bar{x}_t^l + 0.5\Delta x_t^* f_{xx}^l(x_t^l, u_t^l, t)\Delta x_t\ \right] dw_t^l + \eta_t^1\ ,\ t \in (t_{l-1}, t_l] \\ \Delta\bar{x}_{t_0}^1 = 0, \\ \Delta\bar{x}_{t_{l-1}}^l = \Phi^{l-1}(\bar{x}_{t_{l-1}}^{l-1}, \bar{t}_{l-1}) - \Phi^{l-1}(x_{t_{l-1}}^{l-1}, t_{l-1}\,), l = \overline{2,r}. \end{cases}$$

where

$$\eta_t^1 = \left\{ \int_0^1 \left[g_x^{l*}(x_t^l + \mu\Delta\bar{x}_t^l, \bar{u}_t^l, t) - g_x^{l*}(x_t^l, u_t^l, t)\right]\Delta\bar{x}_t^l d\mu \right.$$

$$\left. + 0.5 \cdot \int_0^1 \Delta\bar{x}_t^{l*}\left[g_{xx}^{l*}(x_t^0 + \mu\Delta\bar{x}_t^l, u_t^l, t) - g_{xx}^*(x_t^l, u_t^l, t)\right]\Delta\bar{x}_t^l d\mu \right\} dt$$

$$+ \left\{ \int_0^1 \left[f_x^{l*}(x_t^l + \mu\Delta\bar{x}_t^l, \bar{u}_t^l, t) - f_x^{l*}(x_t^l, u_t^l, t)\right]\Delta\bar{x}_t^l d\mu \right.$$

$$\left. + 0.5 \cdot \int_0^1 \Delta\bar{x}_t^{l*}\left[f_{xx}^{l*}(x_t^0 + \mu\Delta\bar{x}_t^l, u_t^l, t) - f_{xx}^*(x_t^l, u_t^l, t)\right]\Delta\bar{x}_t^l d\mu \right\} dw_t^l.$$

According to Ito's formula implies that following identities are true:

(3.6)

$$\begin{aligned} &d(\psi_t^{l*}\Delta\bar{x}_t^l\Delta\bar{t}_l) = d\psi_t^{l*}\Delta\bar{x}_t^l\Delta\bar{t}_l + \psi_t^{l*}d\Delta\bar{x}_t^l\Delta\bar{t}_l + \{\ \beta_t^{l*}[\Delta_{\bar{u}^l} f^l(x_t^l, u_t^l, t) + f_x^l(x_t^l, u_t^l, t)\Delta\bar{x}_t^l \\ &+0.5\Delta\bar{x}_t^{l*} f_{xx}^l(x_t^l, u_t^l, t)\Delta\bar{x}_t^l]\Delta\bar{t}_l + \beta_t^{l*}\int_0^1 \left[\ f_x^l(x_t^0 + \mu\Delta\bar{x}_t^l, \bar{u}_t^l, t) -\ f_x^l(x_t^l, \bar{u}_t^l, t)\right]\Delta\bar{x}_t^l\Delta\bar{t}_l d\mu \\ &+0.5\beta_t^{l*}\int_0^1 \Delta\bar{x}_t^{l*}\left[\ f_{xx}^l(x_t^0 + \mu\Delta\bar{x}_t^l, u_t^l, t) -\ f_{xx}^l(x_t^0, \bar{u}_t^l, t)\right]\Delta\bar{x}_t^l\Delta\bar{t}_l\ \}\, dt \end{aligned}$$

and

(3.7)

$$\begin{aligned} &d(\Delta\bar{x}_t^{l*}\Psi_t^l\Delta\bar{x}_t^l\Delta\bar{t}_l) = \Delta\bar{x}_t^{l*}\Psi_t^l\Delta\bar{x}_t^l + \Delta\bar{x}_t^{l*}d\Psi_t^l\Delta x_t^l\Delta\bar{t}_l + \Delta x_t^{l*}\Psi_t^l d\Delta\bar{x}_t^l\Delta\bar{t}_l + \\ &d\Delta\bar{x}_t^{l*}\Psi_t^l\Delta\bar{x}_t^l\Delta\bar{t}_l + \{\mathrm{K}_t^{l*}[\Delta_{\bar{u}} f^l(x_t^l, u_t^l, t) + f_x^l(x_t^l, u_t^l, t)\Delta\bar{x}_t^l + 0.5\Delta\bar{x}_t^{l*} f_{xx}^l(x_t^l, u_t^l, t)\Delta\bar{x}_t^l]\Delta\bar{t}_l \\ &+[\Delta_{\bar{u}} f_x^l(x_t^l, u_t^l, t) + f_x^l(x_t^l, u_t^l, t)\Delta\bar{x}_t^l + 0.5\Delta\bar{x}_t^{l*} f_{xx}^l(x_t^l, u_t^l, t)\Delta\bar{x}_t^l]\Psi_t^l \cdot \\ &[\Delta_{\bar{u}} f^l(x_t^l, u_t^l, t) + f_x^l(x_t^l, u_t^l, t)\Delta\bar{x}_t^l + 0.5\Delta\bar{x}_t^{l*} f_{xx}^l(x_t^l, u_t^l, t)\Delta\bar{x}_t^l]\Delta\bar{t}_l\}dt \end{aligned}$$

Note that linear terms in (3.5) can be handled in the following way. Consider the following matrix-valued equations:

$$d\boldsymbol{Z}_t = A_t\boldsymbol{Z}_t dt + B_t\boldsymbol{Z}_t dw_t,$$

$$\boldsymbol{Z}_0 = I,$$

which have a unique solution $\boldsymbol{Z}_t$ with $E\sup\left\|\boldsymbol{Z}_t\right\|^{2s}<\infty, s\geq 1$, if A_t and B_t are the predictable and bounded matrices (see [11]). It is easy to show that the matrix $\boldsymbol{Z}_t$ has an inverse and $\mathbf{G}_t=\boldsymbol{Z}_t^{-1}$ is a solution of the equation:

$$\begin{aligned} d\mathbf{G}_t &= -\left(\mathbf{G}_tA_t-\mathbf{G}_tB_tB_t\right)-\mathbf{G}_tB_tdw_t,\\ \mathbf{G}_0 &= I. \end{aligned}$$

In order to establish the existence and uniqueness of solution of adjoint stochastic differential equations, it is enough to follow the method described in the article [10] and to make use the independence of Wiener processes $w_t^1,\ldots,w_t^r$ in the each interval $[t_{l-1},t_l]$, $l=1,\ldots,r$. The stochastic processes ψ_t^l,Ψ_t^l, $l=\overline{1,r}$, at the points $t_1,t_2,\ldots,t_r$ are defined as:

$$\psi_{t_l}^l=\psi_{t_l}^{l+1}\Phi_x^l(x_{t_l}^l,t_l),\ l=\overline{1,r-1};\quad \psi_{t_r}^r=-\varphi_x(x_{t_r}^r) \tag{3.8}$$

and

$$\Psi_{t_l}^l=\Psi_{t_l}^{l+1}\Phi_{xx}^l(x_{t_l}^l,t_l),\ l=\overline{1,r-1};\quad \Psi_{t_r}^r=-\varphi_{xx}(x_{t_r}^r) \tag{3.9}$$

Using the expressions (3.5)-(3.9) for the increment of a functional (2.4) we obtain the form as indicated below:

(3.10)

$$\begin{aligned} &\Delta J(\ \mathbf{u}\ )=E\left\{\varphi(\bar{x}_{t_r}^r)-\varphi(x_{t_r}^r)+\sum_{l=1}^{r}\int_{t_{l-1}}^{t_l}\left[p^l(\bar{x}_t^l,\bar{u}_t^l,t)-p^l(x_t^l,u_t^l,t)\right]dt\right\}=\\ &-\sum_{l=1}^{r}E\int_{t_{-1}}^{t_l}\left\{\Delta_{\bar{u}^l}H^l(\psi_t^l,x_t^l,u_t^l,t)+H_x^l(\psi_t^l,x_t^l,u_t^l,t)\Delta\bar{x}_{t_l}^l+0.5\cdot\Delta_{\bar{u}^l}f^{l*}(x_t^l,u_t^l,t)\Psi_t^l\times\right.\\ &\Delta_{\bar{u}^l}f^l(x_t^l,u_t^l,t)-0.5\Delta\bar{x}_{t_l}^{l*}f^{l*}(x_t^l,u_t^l,t)\Psi_t^lf_x^l(x_t^l,u_t^l,t)\Delta\bar{x}_{t_l}^l+\Delta\bar{x}_{t_l}^{l*}\Delta_{\bar{u}^l}g^l(x_t^l,u_t^l,t)\,\Psi_t^l\Delta\bar{x}_{t_l}^l\\ &-\Delta\bar{x}_{t_l}^{l*}g_x^l(x_t^l,u_t^l,t)\,\Psi_t^l\Delta\bar{x}_{t_l}^l+\Delta\bar{x}_{t_l}^{l*}\Delta_{\bar{u}^l}f^l(x_t^l,u_t^l,t)\,K_t^l\Delta\bar{x}_{t_l}^l-\Delta\bar{x}_{t_l}^{l*}f_x^l(x_t^l,u_t^l,t)\,K_t^l\Delta\bar{x}_{t_l}^l+\\ &\left.+\psi_t^{l*}\Delta_{\bar{u}^l}g_x^l(x_t^l,u_t^l,t)\Delta\bar{x}_{t_l}^l+\beta_t^{l*}\Delta_{\bar{u}^l}f_x^l(x_t^l,u_t^l,t)\Delta\bar{x}_{t_l}^l-\Delta_{\bar{u}^l}p_x^l(x_t^l,u_t^l,t)\,\Delta\bar{x}_{t_l}^l\right\}\Delta\bar{t}_ldt\\ &+\sum_{l=1}^{r-1}\psi_{t_l}^{l+1}\Phi_t(x_{t_l}^l,t_l)\Delta\bar{t}_ldt\ +\eta_0+\sum_{l=1}^{r}\eta_{t_{l-1}}^{t_l}, \end{aligned}$$

where

$$\begin{aligned} \eta_0&=-E\int_0^1(1-\mu)\left[\varphi_x^*(x_{t_r}^r+\mu\Delta\bar{x}_{t_r}^r)-\varphi_x^*(x_{t_r}^r)\right]\Delta\bar{x}_{t_r}^rd\mu-\\ &E\int_0^1(1-\mu)\Delta\bar{x}_{t_r}^{r*}\left[\varphi_{xx}^*(x_{t_r}^r+\mu\Delta\bar{x}_{t_r}^r)-\varphi_{xx}^*(x_{t_r}^r)\right]\Delta\bar{x}_{t_r}^rd\mu \end{aligned} \tag{3.11}$$

and

(3.12)

$$\eta_{t_{l-1}}^{t_l} = E\int\limits_{t_{l-1}}^{t_l}\int\limits_0^1 (1-\mu)\left[H_x^l(\psi_t^l, x_t^l+\mu\Delta\bar{x}_t^l, u_t^l, t) - H_x^l(\psi_t^l, x_t^l u_t^l, t)\right]\Delta\bar{x}_t^l\ \Delta\bar{t}_l d\mu dt-$$
$$-E\int\limits_0^1 (1-\mu)\psi_{t_l}^{l+1}\left[\Phi_x^l(x_t^l+\mu\Delta\bar{x}_t^l, t_l) - \Phi_x^l(x_t^l, t_l)\right]\Delta x_t^l\Delta\bar{t}_l d\mu$$
$$E\int\limits_{t_{l-1}}^{t_l}\int\limits_0^1 (1-\mu)\Delta\bar{x}_t^{l*}\left[H_{xx}^l(\psi_t^l, x_t^l+\mu\Delta\bar{x}_t^l, u_t^l, t) - H_{xx}^l(\psi_t^l, x_t^l u_t^l, t)\right]\Delta\bar{x}_t^l\ \Delta\bar{t}_l d\mu dt-$$
$$-E\int\limits_0^1 (1-\mu)\Delta\bar{x}_{t_l}^{l*}\Psi_{t_l}^{l+1}\left[\Phi_{xx}^l(x_t^l+\mu\Delta\bar{x}_t^l, t_l) - \Phi_{xx}^l(x_t^l, t_l)\right]\Delta x_t^l\Delta\bar{t}_l d\mu$$

According to (3.1), (3.2), (3.8) and (3.9) , through the simple transformations expression (3.10) may be written as:

(3.13)

$$\Delta J(\ \mathbf{u}\ ) = -\sum_{l=1}^{r} E\int\limits_{t_{l-1}}^{t_l}\left[\Delta_{\overline{u}^l} H^l\ (\psi_t^l, x_t^l, u_t^l, t) + \Delta_{\overline{u}^l} H_{x^l}^l(\psi_t^l, x_t^l, u_t^l, t)\Delta\bar{x}_t^l +\right.$$
$$+0.5\cdot\Delta_{\bar{u}^l} f^{l*}(x_t^l, u_t^l, t)\cdot\Psi_t^l\Delta_{\bar{u}^l} f^l(x_t^l, u_t^l, t) - 0.5\Delta\bar{x}_{t_l}^{l*} f^{l*}(x_t^l, u_t^l, t)\Psi_t^l f_x^l(x_t^l, u_t^l, t)\Delta\bar{x}_{t_l}^l +$$
$$+\Delta\bar{x}_{t_l}^{l*}\Delta_{\bar{u}^l} g^l(x_t^l, u_t^l, t)\,\Psi_t^l\Delta\bar{x}_{t_l}^l - \Delta\bar{x}_{t_l}^{l*} g_x^l(x_t^l, u_t^l, t)\,\Psi_t^l\Delta\bar{x}_{t_l}^l + \Delta\bar{x}_{t_l}^{l*}\Delta_{\bar{u}^l} f^l(x_t^l, u_t^l, t)\,K_t^l\Delta\bar{x}_{t_l}^l -$$
$$\left.-\Delta\bar{x}_{t_l}^{l*} f_x^l(x_t^l, u_t^l, t)\,K_t^l\Delta\ \bar{x}_{t_l}^l\right]\Delta\bar{t}_l dt + \sum_{l=1}^{r-1}\psi_{t_l}^{l+1}\Phi_t(x_{t_l}^l, t_l)\Delta\bar{t}_l dt\ + \eta_0 + \sum_{l=1}^{r}\eta_{t_{l-1}}^{t_l}$$

Based on fact that $\pi^r = (\mathbf{t}, \mathbf{x}, \mathbf{u})$ is optimal solution, using the independence of increments respective to different arguments and assumption (H4) from expression (3.13), we obtain that (3.4) is true.

Consider the following spike variations:

$$\Delta u_t^l = \Delta u_{t,\varepsilon^l}^{\theta^l} = \begin{cases} 0,\ t\notin[\theta_l, \theta_l+\varepsilon_l), \varepsilon_l > 0,\ \ \theta_l\in[t_{l-1}, t_l) \\ \tilde{u}^l - u_t^l,\ \ t\in[\theta_l, \theta_l+\varepsilon_l), \tilde{u}^l\in L^2(\Omega, F^{\theta_l}, P; R^m) \end{cases}$$

where ε_l are enough small numbers. Then the expression (3.13) takes the form of:

$$\begin{aligned} \Delta_\theta J(\ \mathbf{u}) = \sum_{l=1}^{r} E\int\limits_{\theta_l}^{\theta_l+\varepsilon_l} [\ \Delta_{\tilde{u}^l} H^l(\psi_t^l, x_t^l, u_t^l, t)\ + \Delta_{\bar{u}^l} H_{x^l}^l(\psi_t^l, x_t^l, u_t^l, t)\Delta\bar{x}_t^l + \\ +\,0.5\Delta_{\bar{u}^l} f^{l*}(x_t^l, u_t^l, t)\cdot\Psi_t^l\Delta_{\bar{u}^l} f^l(x_t^l, u_t^l, t)\ ]\,dt\ + \sum_{l=1}^{r}\eta_{\theta_l}^{\theta_l+\varepsilon_l} \end{aligned} \tag{3.14}$$

In order to obtain estimation for increment (3.14), we introduce following lemma.

Lemma 3.2 (Gronwall's inequality [6]). *Let $m(t)$ is a continuous function satisfying: $0\le m(t)\le h(t)+\int\limits_s^t g(\tau)m(\tau)d\tau, s\le t\le t_1$, here $g(t)$ is continuous,$h(t)$ is bounded functions and $\int\limits_s^{t_1} g(t)dt < +\infty$. Then following holds:*

$$m(t)\le h(t)+\int\limits_s^t g(\tau)h(\tau)\exp[\int\limits_s^t g(u)du]d\tau, s\le t\le t_1$$

Proof of following lemma can be found in [29]. Here, the brief proof will be given in due to make comprehensible content for this paper.

Lemma 3.3. *Suppose that conditions* (H1)–(H2) *are satisfied. Then, the following is obtained:*

$$\lim_{\varepsilon_l \to 0} E\left|x^{\theta_l}_{t,\varepsilon_l} - x^l_t\right|^2 \le N\varepsilon_l, \ \text{a.e. in } t \in [t_{l-1}, t_l), \quad l = \overline{1,r}.$$

Here $x^{\theta_l}_{t,\varepsilon_l}$ *are the solutions of system* (2.1)–(2.2)*, corresponding to the controls* $u^{\theta_l}_{t,\varepsilon_l} = u^l_t + \Delta u^{\theta_l}_{t,\varepsilon_l}$.

Proof. Let's denote the following: $\tilde{x}^l_{t,\varepsilon_l} = x^{\theta_l}_{t,\varepsilon_l} - x^l_t$. It is clear that $\forall t \in [t_{l-1}, \theta_l)$ $\tilde{x}^l_{t,\varepsilon_l} = 0$, $l = \overline{1,r}$. Then for $\forall t \in [\theta_l, \theta_l + \varepsilon_l)$

$$d\tilde{x}^l_{t,\varepsilon_l} = \left[g^l(x^l_t + \varepsilon^l \tilde{x}^l_{t,\varepsilon_l}, \tilde{u}^l, t) - g^l(x^l_t, u^l_t, t)\right] dt + \left[f^l(x^l_t + \varepsilon_l \tilde{x}^l_{t,\varepsilon_l}, \tilde{u}^l, t) - f^l(x^l_t, u^l_t, t)\right] dw^l_t$$

$$\tilde{x}^l_{\theta_l,\varepsilon_l} = -(g^l(x^l_{\theta_l}, \tilde{u}^l, \theta_l) - g(x^l_{\theta_l}, u^l_{\theta_l}, \theta_l))$$

or

$$\begin{aligned}
\tilde{x}^l_{\theta_l+\varepsilon_l,\varepsilon_l} = & \int_{\theta_l}^{\theta_l+\varepsilon_l} \left[g^l(x^l_s + \varepsilon_l \tilde{x}^l_{s,\varepsilon_l}, u^l, s) - g^l(x^l_s, u^l_s, s)\right] ds \\
& + \int_{\theta_l}^{\theta_l+\varepsilon_l} \left[g^l(x^l_\theta, u^l_{\theta_l}, \theta_l) - g^l(x^l_s, u^l_s, s)\right] ds \\
& + \int_{\theta_l}^{\theta_l+\varepsilon_l} \left[f^l(x^l_s + \varepsilon_l \tilde{x}^l_{s,\varepsilon_l}, u^l_s, s) - f^l(x^l_s, u^l_s, s)\right] dw^l_s \\
& + \int_{\theta_l}^{\theta_l+\varepsilon_l} \left[g^l(x^l_s, \tilde{u}^l, s) - g^l(x^l_{\theta_l}, \tilde{u}^l, \theta_l)\right] ds
\end{aligned}$$

Therefore from the conditions (H1)–(H2) and using the Lemma 3.2 we have

$$\begin{aligned}
E\left|\tilde{x}^l_{\theta_l+\varepsilon_l,\varepsilon_l}\right|^2 \le N \Bigg[& \varepsilon_l^2 \sup_{\theta_l \le t \le \theta_l+\varepsilon_l} E\left|x^{\theta_l}_{t,\varepsilon_l} - x^l_t\right|^2 \\
& + \varepsilon_l^2 \sup_{\theta_l \le t \le \theta_l+\varepsilon_l} E\left|x^l_t - x^l_{\theta_l}\right|^2 \\
& + \sup_{\theta_l \le t \le \theta_l+\varepsilon_l} \varepsilon_l^2 E\left|g^l(x^l_t, \tilde{u}^l, t) - g^l(x^l_{\theta_l}, \tilde{u}^l, \theta_l)\right|^2 \\
& + \varepsilon_l E \int_{\theta_l}^{\theta_l+\varepsilon_l} \left|f^l(x^l_t, u^l_t, t) - f^l(x^l_{\theta_l}, u^l_{\theta_l}, \theta_l)\right|^2 dt \\
& + \varepsilon_l^2 E \int_{\theta_l}^{\theta_l+\varepsilon_l} \left|g^l(x^l_t, u^l_t, t) - g^l(x^l_{\theta_l}, u^l_{\theta_l}, \theta_l)\right|^2 dt \Bigg]
\end{aligned}$$

Hence: $E\left|\tilde{x}^l_{t+\varepsilon_l,\varepsilon_l}\right|^2 \leq \varepsilon_l N$, $\varepsilon_l \to 0$, $\forall t \in [\theta_l, \theta_l+\varepsilon_l)$. According to identity (3.5) for special spike variation of control $\forall t \in [\theta_l+\varepsilon_l, t_l]$:

$$d\tilde{x}^l_{t,\varepsilon} = \left[g^l(x^l_t+\varepsilon_l\tilde{x}^l_{t,\varepsilon_l}u^l_t,t) - g^l(x^l_t,u^l_t,t)\right] dt + \left[f^l(x^l_t+\varepsilon_l\tilde{x}^l_{t,\varepsilon_l},u^l_t,t) - f(x^l_t,u^l_t,t)\right] dw^l_t$$

which can be rewritten as follow:

$$d\tilde{x}^l_{t,\varepsilon_l} = \int\limits_0^1 g^l_x(x^l_t+\mu\varepsilon_l\tilde{x}^l_{t,\varepsilon_l},u^l_t,t)\tilde{x}^l_{t,\varepsilon_l}d\mu dt + \int\limits_0^1 f^l_x(x^l_t+\mu\varepsilon_l\tilde{x}^l_{t,\varepsilon_l},u^l_t,t)\tilde{x}^l_{t,\varepsilon_l}d\mu dw_t$$

$$\tilde{x}^l_{\theta_l+\varepsilon_l,\varepsilon_l}\varepsilon_l = -(g^l(x^l_{\theta_l+\varepsilon_l},u^l_{\theta_l+\varepsilon_l},\theta_l) - g(x^l_{\theta_l+\varepsilon_l},\tilde{u}^l,\theta_l)).$$

Hence: $E\left|\tilde{x}^l_{t,\varepsilon_l}\right|^2 \leq \varepsilon_l N$, for $\forall t \in [\theta_l+\varepsilon_l, t_l]$, if $\varepsilon_l \to 0$.

Thus: $\sup\limits_{t_{l-1}\leq t\leq t_l} E\left|\tilde{x}^l_{t,\varepsilon_l}\right|^2 \leq N\varepsilon_l,\ l=\overline{1,r}.$ □

From the expression (3.12), due to Lemma 3.3 for each l implies following estimation:

$$\eta^{\theta_l+\varepsilon_l}_{\theta_l} = o(\varepsilon_l).$$

Then according to fact that $\bar{\mathbf{u}}_t = (\bar{u}^1_t, \bar{u}^2_t, \ldots, \bar{u}^r_t)$ is optimal of control from (3.14) for each l it follows that:

$$\begin{aligned}\Delta_{\theta^l} J(\mathbf{u}) = -E\big[&\psi^{l*}_{\theta_l}\Delta_{\tilde{u}^l}\ g^l(x^l_{\theta_l},u^l_{\theta_l},\theta_l) - \Delta_{\tilde{u}^l}p^l(x^l_{\theta_l},u^l_{\theta_l},\theta_l)\\ &+0.5\Delta_{\tilde{u}^l}f^{l*}(x^l_{\theta_l},u^l_{\theta_l},\theta_l)\Psi^l_{\theta_l}\Delta_{\bar{u}^l}f^l(x^l_{\theta_l},u^l_{\theta_l},\theta_l)\ \big]\Delta\bar{t}_l + o(\varepsilon_l) \geq 0\end{aligned}$$

Finally, due to the smallness and arbitrariness of ε_l (3.3) is achieved. □

4. NECESSARY CONDITION OF OPTIMALITY FOR SWITCHING SYSTEMS WITH CONSTRAINTS

First, we recall notion Ekeland's variational principle to use in our main result.

Theorem 4.1 (Ekeland's variational principle [31])**.** *Let $(\boldsymbol{X}, d)$ is complete metric space and $f : \boldsymbol{X} \longrightarrow R\bigcup(+\infty)$ be a semi-continuous function from below. ε, λ are positive numbers and for some point $x_0 \in \boldsymbol{X}$ is satisfied: $f(x_0) \leq \min\limits_{x\in\boldsymbol{X}} f(x) + \varepsilon\lambda$, there exist $\bar{x} \in \boldsymbol{X}$ such that:*
1) $f(\bar{x}) \leq f(x_0)$,
2) $d(\bar{x}, x_0) \leq \lambda$,
3) $\forall x \in \boldsymbol{X}, f(\bar{x}) \leq f(x) + \varepsilon\lambda d(\bar{x}, x)$.

By applying the Theorem 3.1 and Theorem 4.1 the necessary condition of optimality for stochastic control problem of switching systems (2.1)–(2.5) is obtained.

Theorem 4.2. *Assume that, assumptions* (H1)–(H5) *satisfy and* $\pi^r = (\boldsymbol{t},\boldsymbol{x},\boldsymbol{u})$ *is an optimal solution of problem* (2.1)–(2.5). *Then,*
a) there exist non-zero vector $(\lambda_0, \lambda_1, \dots, \lambda_r) \in R^{r+1}$ *and stochastic functions* $(\psi_t^l, \beta_t^l) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l x n_l})$ *and* $(\Psi_t^l, \mathrm{K}_t^l) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l x n_l})$ *which are the solutions of the following conjugate equations:*

$$\left\{\begin{array}{l} d\psi_t^l = -H_x^l(\psi_t^l, x_t^l, u_t^l, t)dt + \beta_t^l dw_t^l, \ \ t \in [t_{l-1}, t_l), l = \overline{1, r} \\ \psi_{t_l}^l = -\lambda_l q_x^l(x_{t_l}^l) + \psi_{t_{l+1}}^l \Phi_x^l(x_{t_l}^l, t_l), \ \ l = \overline{1, r-1}, \\ \psi_{t_r}^r = -\lambda_0 \varphi_x(x_{t_r}^r) - \lambda_r q_x^r(x_{t_r}^r); \end{array}\right. \tag{4.1}$$

$$\left\{\begin{array}{l} d\Psi_t^l = -[\boldsymbol{H}_x^l(\Psi_t^l, x_t^l, u_t^l, t) + H_{xx}^l(\psi_t^l, x_t^l, u_t^l, t) \\ + f_x^{l*}(x_t^l, u_t^l, t)\Psi_t^l f_x^l(x_t^l, u_t^l, t)] \ dt + \mathrm{K}_t^l dw_t^l, t \in [t_{l-1}, t_l) \\ \Psi_{t_l}^l = -\lambda_l q_{xx}^l(x_{t_l}^l) + \Psi_{t_{l+1}}^l \Phi_{xx}^l(x_{t_l}^l, t_l), \ \ l = \overline{1, r-1}, \\ \Psi_{t_r}^r = -\lambda_0 \varphi_{xx}(x_{t_r}^r) - \lambda_r q_{xx}^r(x_{t_r}^r). \end{array}\right. \tag{4.2}$$

b) a.e. $\theta \in [t_{l-1}, t_l]$ *and* $\forall\ \tilde{u}^l \in U^l$, $l = \overline{1, r}$, *a.c. holds the maximum principle:*
(4.3)

$$H^l(\psi_\theta^l, x_\theta^l, u^l, \theta) - H^l(\psi_\theta^l, x_\theta^l, u_\theta^l, \theta) + 0.5\Delta_{u^l} f^{l*}(x_\theta^l, u_\theta^l, \theta)\Psi_\theta^l \Delta_{u^l} f^l(x_\theta^l, u_\theta^l, \theta) \leq 0$$

c) following transversality conditions holds:

$$\psi_{t_l}^{l+1}\Phi_t^l(x_{t_l}^l, t_l) = 0, a.c. \tag{4.4}$$

Proof. First, we discuss the existence of uniquely solutions of adjoint equations (4.1) and (4.2). In fact from [10, 15, 16], the first-order adjoint processes (ψ_t^l, β_t^l) and second order adjoint processes $(\Psi_t^l, \mathrm{K}_t^l)$ described in a unique way by (4.1) and (4.2) respectively. Using Theorem 4.1 , the problem is convert into the sequence of unconstrained problems. Finally, we obtain maximum principle in the case when and endpoint constraints are imposed. Consider following approximating functional for any natural j :

$$I_j(\mathbf{u}) = S_j(E\varphi(x_{t_r}^r) + E\sum_{l=1}^{r}\int_{t_{l-1}}^{t_l} p^l(x_t^l, u_t^l, t)dt, Eq^1(x_{t_1}^1), \dots, Eq^r(x_{t_r}^r)) =$$

$$\min_{(c,\mathbf{y})\in\chi} \sqrt{\left|c - 1/j - EM(\mathbf{x}, \mathbf{u}, \mathbf{t})\right|^2 + \sum_{l=1}^{r} \left|y_l - Eq^l(x_{t_l}^l)\right|^2}$$

where $M(\mathbf{x}, \mathbf{u}, \mathbf{t}) = \varphi(x_{t_r}^r) + \sum\limits_{l=1}^{r}\int\limits_{t_{l-1}}^{t_l} p(x_t^l, u_t^l, t)dt$, $\mathbf{y} = (y_1, \dots, y_r)$ and $\chi = \{c : c \leq J^0,$ $y_1 \in G^1, \dots, y_r \in G^r\}$, J^0 minimal value of the functional in the problem (2.1)–(2.5). Introduce space of controls $V^l \equiv (U_\partial^l, d)$ obtained by means of the metric:

$$d(u^l, v^l) = (l \otimes P)\ \left\{(t, \omega) \in [t_{l-1}, t_l] \times \Omega : \nu_t^l \neq u_t^l\right\}.$$

$V^1, V^2, \dots, V^r$ are complete metric spaces [31]. It is easy to prove the following fact:

Lemma 4.3. *Assume that conditions* (H1)–(H4) *hold, for each* l $u_t^{l,n}$ *be the sequence of admissible controls from* V^l, *and* $x_t^{l,n}$ *be the sequence of corresponding trajectories of the system* (2.1)–(2.3). *If the following condition is met:* $d(u_t^{l,n}, u_t^l) \to 0$, *then,* $\lim_{n\to\infty}\left\{\sup_{t_{l-1}\le t\le t_l} E\left|x_t^{l,n} - x_t^l\right|^2\right\} = 0$, *where* x_t^l *is a trajectory corresponding to an admissible controls* u_t^l, $l = \overline{1, r}$.

Due to continuity of the functionals $I_j : V^1 \times \cdots V^r \to R^{n_l}$, according to Ekeland's variational principle, there are controls such as: $u_t^{l,j} : d(u_t^{l,j}, u_t^l) \le \sqrt{\varepsilon_j}$ and for $\forall u_t^l \in V^l$ follows: $I_j(\mathbf{u}^j) \le I_j(\mathbf{u}) + \sqrt{\varepsilon_j}\sum_{l=1}^{r} d(u^{l,j}, u^l), \varepsilon_j = \frac{1}{j}$.

This fact can be treated following way: $(t_1, \dots, t_r, x_t^{1,j}, \dots, x_t^{r,j}, u_t^{1,j}, \dots, u_t^{r,j})$ is a solution of the following problem:

$$
\left\{
\begin{array}{l}
J_j(\mathbf{u}) = I_j(\mathbf{u}) + \sqrt{\varepsilon_j} E \sum\limits_{l=1}^{r} \int\limits_{t_{l-1}}^{t_l} \delta(u_t^l, u_t^{l,j}) dt \to \min \\
dx_t^l = g^l(x_t^l, u_t^l, t)dt + f^l(x_t^l, u_t^l, t)dw_t,\ l = \overline{1, r} \\
x_{t_l}^{l+1} = \Phi^l(x_{t_l}^l, t_l),\ l = \overline{1, r-1}\,; \\
x_{t_0}^1 = x_0, \\
u_t^l \in U_\partial^l
\end{array}
\right. \tag{4.5}
$$

Here

$$
\delta(u, v) = \left\{
\begin{array}{l}
0, u = v \\
1, u \ne v
\end{array}
\right.
$$

Then according to the Theorem 3.1, it is obtained as follows:
1) there exist the stochastic processes $(\psi_t^{l,j}, \beta_t^{l,j}) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l \times n_l})$, which are solutions of following system:

$$
\left\{
\begin{array}{l}
d\psi_t^{l,j} = -H_x^l(\psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t)dt + \beta_t^{l,j} dw_t, t \in [t_{l-1}, t_l)\ l = \overline{1, r}; \\
\psi_{t_l}^{l,j} = -\lambda_l^j q_x^l(x_{t_l}^{l,j}) + \psi_{t_{l+1}}^l \Phi_x^l(x_{t_l}^{l,j}, t_l), l = \overline{1, r-1} \\
\psi_{t_r}^{r_j} = -\lambda_0^j \varphi_x(x_{t_r}^{r,j}) - \lambda_r^j q_x^r(x_{t_r}^{r,j})\,.
\end{array}
\right. \tag{4.6}
$$

and the random processes $\Psi_t^{l,j} \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l})$, $\mathrm{K}_t^{l,j} \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l \times n_l})$, which are solutions of the system:

$$
\left\{
\begin{array}{l}
d\Psi_t^{l,j} = -[\mathbf{H}_x^l(\Psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t) + H_{xx}^l(\psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t) \\
\quad + f_x^{l*}(x_t^{l,j}, u_t^{l,j}, t)\Psi_t^{l,j} f_x^l(x_t^{l,j}, u_t^{l,j}, t)]\ dt + \mathrm{K}_t^{l,j} dw_t^l \\
\Psi_{t_l}^{l,j} = -\lambda_l^j q_{xx}^l(x_{t_l}^{l,j}) + \Psi_{t_{l+1}}^{l,j} \Phi_{xx}^l(x_{t_l}^{l,j}, t_l),\ l = \overline{1, r-1}, \\
\Psi_{t_r}^{r,j} = -\lambda_0^j \varphi_{xx}(x_{t_r}^{r,j}) - \lambda_r^j q_{xx}^r(x_{t_r}^{r,j}).
\end{array}
\right. \tag{4.7}
$$

where non-zero $(\lambda_0^j, \lambda_1^j, \dots, \lambda_r^j) \in R^{r+1}$ meet the following requirement:

$$
\begin{aligned}
(\lambda_0^j, \lambda_1^j, \dots, \lambda_r^j) = \Big(&\left[-c + 1/j + EM(\mathbf{x}^j, \mathbf{u}^j, \mathbf{t})\right], \\
&- y_1 + Eq^1(x_{t_1}^{1,j}), \dots, -y_r + Eq^r(x_{t_r}^{r,j})\Big)/J_j^0
\end{aligned} \tag{4.8}
$$

here

$$J_j^0 = \sqrt{\sum_{l=1}^{r} \left|y_l - Eq^l(x_{t_l}^l)\right|^2 + |c - 1/j - EM(\mathbf{x}, \mathbf{u}, \mathbf{t})|^2}$$

2) a.e. $t \in [t_{l-1}, t_l]$ and $\forall\ \tilde{u}^l \in V^l$, $l = \overline{1, r}$, a.c. is satisfied:

$$H^l\left(\psi_t^{l,j}, x_t^{l,j}, \tilde{u}_t^l, t\right) - H^l\left(\psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t\right) + 0.5\Delta_{\tilde{u}^l} f^{l*}(x_t^{l,j}, u_t^{l,j}, t)\Psi_t^{l,j}\Delta_{\tilde{u}^l} f^l(x_t^{l,j}, u_t^{l,j}, t) \leq 0 \tag{4.9}$$

3) following conditions of transversality satisfy:

$$\psi_{t_l}^{l+1,j}\Phi_{t_i}^l(x_{t_l}^{l,j}, t_l) = 0, a.c. \tag{4.10}$$

Since the following has existed $|(\lambda_0^j, \lambda_1^j, \ldots, \lambda_r^j)| = 1$, then according to (4.8) and conditions (H1)–(H5) it is implied that $(\lambda_0^j, \lambda_1^j, \ldots, \lambda_r^j) \to (\lambda_0, \lambda_1, \ldots, \lambda_r)$ if $j \to \infty$.

We now state the following results which will be needed in the future.

Lemma 4.4. *Let $\psi_{t_l}^l$ be a solution of system (4.1), $\psi_{t_l}^{l,j}$ be a solution of system (4.6). If $d(u_t^{l,j}, u_t^l) \to 0$, then*

$$\lim_{j\to\infty} E\int_{t_{l-1}}^{t_l} [|\psi_t^{l,j} - \psi_t^l|^2 + |\beta_t^{l,j} - \beta_t^l|^2]dt) = 0\ , l = \overline{1, r}.$$

Proof. It is clear that $\forall t \in [t_{l-1}, t_l]$:

$$\begin{aligned} d(\psi_t^{l,j} - \psi_t^l) &= -\left[H_x^l(\psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t) - H_x^l(\psi_t^l, x_t^l, u_t^l, t)\right] dt + (\beta_t^{l,j} - \beta_t^l)dw_t \\ &= \Big[\psi_t^{l,j} g_x^l(x_t^{l,j}, u_t^{l,j}, t) + \beta_t^{l,j} f_x^l(x_t^{l,j}, u_t^{l,j}, t) - p_x^l(x_t^{l,j}, u_t^{l,j}, t) - \psi_t^l g_x^l(x_t^l, u_t^l, t) - \\ &\quad -\beta_t^l f_x^l(x_t^l, u_t^l, t) + p_x^l(x_t^l, u_t^l, t)\Big] dt + (\beta_t^{l,j} - \beta_t^l)dw_t \end{aligned} \tag{4.11}$$

Squaring both sides of the equation, according to Ito formula $\forall s \in [t_{l-1}, t_l]$ we obtain:

$$\begin{aligned} E(\psi_{t_l}^{l,j} - \psi_{t_l}^l)^2 - E(\psi_s^{l,j} - \psi_s^l)^2 &= 2E\int_s^{t_l} [\psi_t^{l,j} - \psi_t^l][(g_x^{l*}(x_t^{l,j}, u_t^{l,j}, t) - g_x^{l*}(x_t^l, u_t^l, t))\psi_t^{l,j} + \\ &+ g_x^{l*}(x_t^l, u_t^l, t)(\psi_t^{l,j} - \psi_t^l) + (f_x^{l*}(x_t^{l,j}, u_t^{l,j}, t) - f_x^{l*}(x_t^l, u_t^l, t))\beta_t^{l,j} + f_x^{l*}(x_t^l, u_t^l, t) \times \\ &\times (\beta_t^{l,j} - \beta_t^l) - p^l(x_t^{l,j}, u_t^{l,j}, t) + p_x^l(x_t^l, u_t^l, t)]dt + E\int_s^{t_r} (\beta_t^{l,j} - \beta_t^l)^2 dt \end{aligned}$$

Now, using the assumptions (H1)–(H5) we have:

$$\begin{aligned} E\int_s^{t_r} |\beta_t^{l,j} - \beta_t^l|^2 dt + E|\psi_s^{l,j} - \psi_s^l|^2 &\leq EN\int_s^{t_r} |\psi_t^{l,j} - \psi_t^l|^2 dt \\ &+ EN\varepsilon\int_s^{t_r} |\beta_t^{l,j} - \beta_t^l|^2 dt + E\left|\psi_{t_l}^{l,j} - \psi_{t_l}^l\right|^2 \end{aligned}$$

Hence, by the Lemma 3.2 we establish that:

$$E|\psi_s^{r,j} - \psi_s^r|^2 \le De^{N(t_r - s)} \text{ a.e. in } [t_{r-1}, t_r], \tag{4.12}$$

where $D = E|\psi_{t_r}^{r,j} - \psi_{t_r}^r|^2$. Hence from (4.1),(4.6) and conditions (H3),(H5) it follows that $\psi_{t_r}^{r,j} \to \psi_{t_r}^r$ and $D \to 0$. Consequently, from (4.12) we obtain that $\psi_s^{r,j} \to \psi_s^r$ in $L_F^2(t_{r-1}, t_r; R^{n_r})$ and thus $\beta_s^{r,j} \to \beta_s^r$ in $L_F^2(t_{r-1}, t_r; R^{n_r \times n_r})$.

Then from expression (4.11) in view of assumptions (H1)–(H5) and according to Lemma 3.2 we get:

$$E|\psi_s^{l,j} - \psi_s^l|^2 \le De^{N(t_l - s)} \text{ a.e. in } [t_{l-1}, t_l], \quad l = \overline{1, r-1},$$

here $D = E|\psi_{t_l}^{l,j} - \psi_{t_l}^l|^2$, which $D \to 0$ according to (4.1), (4.6) and conditions (H3)–(H4). Hence, from (4.12) implies that $\psi_s^{l,j} \to \psi_s^l$ in $L_{F^l}^2(t_{l-1}, t_l; R^n)$ and $\beta_s^{l,j} \to \beta_s^l$ in $L_{F^l}^2(t_{l-1}, t_l; R^{n\times n})$. □

Lemma 4.5. *Let $\Psi_{t_l}^{l,j}$ be a solution of system (4.2), and $\Psi_{t_l}^l$ be a solution of system (4.7). Then*

$$E\int_{t_{l-1}}^{t_l} |\Psi_t^{l,j} - \Psi_t^l|^2 dt + E\int_{t_{l-1}}^{t_l} |\mathrm{K}_t^{l,j} - \mathrm{K}_t^l|^2 dt \to 0\,, l = \overline{1, r},\ \textit{if } j \to \infty.$$

Proof. Due to Ito's formula from expressions (4.2) and (4.7) for $\forall s \in [t_{l-1}, t_l)$:

$$\begin{aligned} d(\Psi_t^{l,j} - \Psi_t^l) = &-\{(g_x^{l*}(x_t^{l,j}, u_t^{l,j}, t)\Psi_t^{l,j} - g_x^{l*}(x_t^j, u_t^j, t)\Psi_t^l) + (\Psi_t^{l,j} g_x^l(x_t^{l,j}, u_t^{l,j}, t) - \\ &-\Psi_t^l g_x^l(x_t^j, u_t^j, t)) + (f_x^{l*}(x_t^{l,j}, u_t^{l,j}, t)\Psi_t^{l,j} f_x^l(x_t^{l,j}, u_t^{l,j}, t) - f_x^{l*}(x_t^l, u_t^l, t)\Psi_t^l f_x^l(x_t^l, u_t^l, t)) \\ &+(f_x^{l*}(x_t^{l,j}, u_t^{l,j}, t)\mathrm{K}_t^{l,j} - f_x^{l*}(x_t^l, u_t^l, t)\mathrm{K}_t^l) + (\mathrm{K}_t^{l,j} f_x^l(x_t^{l,j}, u_t^{l,j}, t) - \mathrm{K}_t^l f_x^l(x_t^l, u_t^l, t)) + \\ &+H_{xx}^l(\psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t) - H_{xx}^l(\psi_t^l, x_t^l, u_t^l, t)\}\ dt + (\mathrm{K}_t^{l,j} - \mathrm{K}_t^l) dw_t^l \end{aligned}$$

Then with help simple transformations we obtain:

$$\begin{aligned} E\int_s^{t_l} |\mathrm{K}_t^{l,j} - \mathrm{K}_t^l|^2 dt + E|\Psi_t^{l,j} - \Psi_t^l|^2 \le EN\int_s^{t_l} |\Psi_t^{l,j} - \Psi_t^l|^2 dt \\ + EN\varepsilon\int_s^{t_1} |\mathrm{K}_t^{l,j} - \mathrm{K}_t^l|^2 dt + E|\Psi_{t_l}^{l,j} - \Psi_{t_l}^l|^2 \end{aligned}$$

According to Gronwall inequality a.e. in $[t_{l-1}, t_l)$ we have:

$$E|\Psi_s^{l,j} - \Psi_s^l|^2 \le De^{-N(t_l - s)}$$

where the constant D defined as:

$$D = E|\Psi_{t_l}^{l,j} - \Psi_{t_l}^l|^2 + EN\varepsilon\int_s^{t_l} |\mathrm{K}_t^{l,j} - \mathrm{K}_t^l|^2 dt$$

Follow the same steps as in Lemma 4.4 in view of (4.2), (4.7) and assumptions (H3), (H5) we establish that: $\Psi_{t_r}^{r,j} \to \Psi_{t_r}^{r}$. Further, according to assumptions (H1)–(H4) and expressions (4.2), (4.6) we obtain : $\Psi_t^{r,j} \to \Psi_t^r$ in $L^2_{F^r}(t_{r-1}, t_r; R^n)$ if $j \to \infty$.

According to sufficient smallness of ε follows that $D \to 0$. Consequently: $\Psi_t^{l,j} \to \Psi_t^l$ in $L^2_{F^l}(t_{l-1}, t_l; R^n)$ and $\mathrm{K}_t^{l,j} \to \mathrm{K}_t^l$ in $L^2_{F^l}(t_{l-1}, t_l; R^{n\times n})$,$l = \overline{1, r-1}$. □

Due to Lemma 4.4 and Lemma 4.5 it can be proceed to the limit in systems (4.6) , (4.7) and the fulfilment of (4.1),(4.2) are obtained. Follow a similar scheme by taking limit in (4.9) and (4.10) it is proved that (4.3),(4.4) are true. Theorem 4.2 is proved. □

5. CONCLUSION

A lot of theoretical and numerical advances have recently been realized in the field of modelling and control related with randomness [3, 4, 5, 32, 33]. Necessary conditions satisfied by an optimal solution, play an important role for investigation of optimization and optimal control problems. The present paper is devoted to optimal control problem of stochastic switching systems with the endpoint state restrictions in the form of functional constraints. The necessary conditions developed in this study can be viewed as a stochastic analogues of the problems that are formulated in [19, 20, 23]. However, Theorem 4.2 is a natural evolution of the results given in [28, 26, 29].

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