

NONLINEAR BOUNDARY VALUE PROBLEMS FOR p-LAPLACIAN FRACTIONAL DIFFERENTIAL SYSTEMS

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ABSTRACT. In this paper, we study the existence results of positive solutions for p-Laplacian fractional differential systems by means of fixed point theorems on cones. As an application, an example is given to demonstrate our main results.

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1. INTRODUCTION

In this paper, we study the existence of positive solutions to the following fractional differential systems involving the p-Laplacian fractional operator

$$(1.1) \quad D^{\beta_1}(\phi_{p_1}(D^{\alpha_1}u(t))) + f_1(t, v(t)) = 0, \quad t \in (0, 1),$$

$$(1.2) \quad D^{\beta_2}(\phi_{p_2}(D^{\alpha_2}v(t))) + f_2(t, u(t)) = 0, \quad t \in (0, 1),$$

$$(1.3) \quad \begin{cases} a_1u(0) - b_1u'(0) = \int_0^1 u(s)dA(s), \\ c_1u(1) + d_1u'(1) = \int_0^1 u(s)dB(s), \\ D^{\alpha_1}u(0) = 0, \end{cases}$$

$$(1.4) \quad \begin{cases} a_2v(0) - b_2v'(0) = \int_0^1 v(s)dA(s), \\ c_2v(1) + d_2v'(1) = \int_0^1 v(s)dB(s), \\ D^{\alpha_2}v(0) = 0, \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < \alpha_i \leq 2$, $0 < \beta_i \leq 1$, for $i = 1, 2$, $f_i : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are continuous functions, a_i, b_i, c_i, d_i are nonnegative constants satisfying $b_i > \frac{2-\alpha_i}{\alpha_i-1}a_i$ with $D_i = a_i c_i + a_i d_i + b_i c_i > 0$, D^{α_i} and D^{β_i} for $i = 1, 2$ are the Caputo fractional derivatives, $A, B : [0, 1] \rightarrow \mathbb{R}^+$ are nondecreasing functions of bounded variation and the integrals are the Riemann-Stieltjes integrals.

Differential equations of fractional order occur in different research areas and engineering, such as mechanics, electricity, chemistry, biology, economics, control theory. For details, see [5,6,7,10] and the references therein.

To our knowledge, very few authors studied the existence result of positive solution for p-Laplacian fractional order differential systems with boundary conditions involving the Riemann-Stieltjes integrals. For the case of nonlinear fractional differential systems, we would like to mention the papers [1,2,4,6,8].

The rest of the paper is organized as follows. In section 2, we present some preliminaries and lemmas that will be used to prove our main results. We also develop some properties of the Green's function. In section 3, we discuss the existence of positive solution for the BVP (1.1)–(1.4). In section 4, we study the nonexistence of positive solution for the problem (1.1)–(1.4). Finally, in section 5, one example is also included to illustrate the main results.

The proof of our main results is based on the well-known Guo-Krasnosel'skii fixed point theorem, which we present now.

Theorem 1.1 ([10]). *Let B be a Banach space, and let $\wp \subset B$ be a cone in B . Assume Ω_1, Ω_2 are open subsets of B with $\theta \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$ and let*

$$A : \wp \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow \wp$$

be a completely continuous operator such that, either

- (i) $\|Ay\| \leq \|y\|$, $y \in \wp \cap \partial\Omega_1$, and $\|Ay\| \geq \|y\|$, $y \in \wp \cap \partial\Omega_2$; or
- (ii) $\|Ay\| \geq \|y\|$, $y \in \wp \cap \partial\Omega_1$, and $\|Ay\| \leq \|y\|$, $y \in \wp \cap \partial\Omega_2$.

Then A has at least one fixed point in $\wp \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. PRELIMINARIES AND LEMMAS

In this section, we give the necessary definitions and lemmas from fractional calculus theory. These definitions and lemmas can be found in [11] and [12].

Definition 2.1. The Riemann Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ for a continuous function $h : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

where $\Gamma(\cdot)$ is the Euler Gamma function, provided that the integral exists.

Definition 2.2. If $h \in C^n[0, 1]$, then the Caputo fractional derivative of order α is defined by

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds = I^{n-\alpha} h^{(n)}(t),$$

$n-1 < \alpha < n$, $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of the real number α .

Remark 2.3. If $\alpha = n \in \mathbb{N}_0$, then the Caputo derivative coincides with a conventional n -th order derivative of the function $h(t)$.

Lemma 2.4. *Let $\alpha > 0$ and $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. If $h(t) \in C[0, 1]$, then the homogeneous fractional differential equation*

$${}^c D^\alpha h(t) = 0$$

has a solution

$$h(t) = c_1 + c_2 t + c_3 t^2 + \dots + c_n t^{n-1},$$

where $c_i \in \mathbb{R}$, $(i = 1, 2, \dots, n)$.

Lemma 2.5. *Let $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. If $y(t) \in C^n[0, 1]$, then*

$$(I^{\alpha c} D^\alpha y)(t) = y(t) - \sum_{i=0}^{n-1} \frac{y^{(i)}(0)}{i!} t^i.$$

Lemma 2.6 ([3]). *Let $1 < \alpha \leq 2$ and $h \in C[0, 1]$. Then the fractional order boundary-value problem*

$$(2.1) \quad \begin{cases} D^{\alpha_1} u(t) + h(t) = 0, & t \in (0, 1), \\ a_1 u(0) - b_1 u'(0) = 0, \\ c_1 u(1) + d_1 u'(1) = 0 \end{cases}$$

has a unique solution

$$(2.2) \quad u(t) = \int_0^1 G_1(t, s) h(s) ds,$$

where

$$(2.3) \quad G_1(t, s) = \begin{cases} \frac{-(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} + \frac{b_1 + a_1 t}{D_1} \left[\frac{c_1(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} + \frac{d_1(1-s)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \right], & s \leq t; \\ \frac{b_1 + a_1 t}{D_1} \left[\frac{c_1(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} + \frac{d_1(1-s)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \right], & t \leq s. \end{cases}$$

For convenience, we list the following condition.

$$(H1) \quad 0 \leq \int_0^1 dA(s) < a_i \text{ and } 0 \leq \int_0^1 dB(s) < c_i \text{ for } i = 1, 2.$$

where A, B are nondecreasing functions of bounded variation.

Lemma 2.7. *Let (H1) be satisfied, then the fractional order integral boundary-value problem*

$$(2.4) \quad \begin{cases} D^{\alpha_1} u(t) + h(t) = 0, & t \in (0, 1) \\ a_1 u(0) - b_1 u'(0) = \int_0^1 u(s) dA(s), \\ c_1 u(1) + d_1 u'(1) = \int_0^1 u(s) dB(s) \end{cases}$$

has a unique solution

$$(2.5) \quad u(t) = \int_0^1 H_1(t, s) h(s) ds,$$

where

$$(2.6) \quad H_1(t, s) = G_1(t, s) + G_2(t, s), \quad t, s \in [0, 1].$$

Here, $G_1(t, s)$ is given by (2.3) and

$$(2.7) \quad G_2(t, s) = \frac{1}{\delta} [c_1 + d_1 - \int_0^1 s dB(s)] \int_0^1 G_1(\tau, s) dA(\tau) + \frac{1}{\delta} [a_1 - \int_0^1 dA(s) + \int_0^1 dB(s)] t \int_0^1 G_1(\tau, s) dB(\tau), \quad t, s \in [0, 1],$$

and

$$(2.8) \quad \delta = \begin{vmatrix} a_1 - \int_0^1 dA(s) & -(b_1 + \int_0^1 s dA(s)) \\ c_1 - \int_0^1 dB(s) & c_1 + d_1 - \int_0^1 s dB(s) \end{vmatrix} > 0.$$

Proof. Let

$$(2.9) \quad w(t) = \int_0^1 G_1(t, s) h(s) ds.$$

Then by Lemma 2.6, $w(t)$ satisfies

$$(2.10) \quad \begin{cases} D^{\alpha_1} w(t) + h(t) = 0, & t \in (0, 1) \\ a_1 w(0) - b_1 w'(0) = 0, \\ c_1 w(1) + d_1 w'(1) = 0. \end{cases}$$

Assume that $u(t)$ is a solution of (2.4), and let

$$(2.11) \quad z(t) = u(t) - w(t), \quad t \in [0, 1].$$

Then,

$$(2.12) \quad \begin{cases} D^{\alpha_1} z(t) = 0, & t \in (0, 1) \\ a_1 z(0) - b_1 z'(0) = \int_0^1 z(s) dA(s) + \int_0^1 w(s) dA(s), \\ c_1 z(1) + d_1 z'(1) = \int_0^1 z(s) dB(s) + \int_0^1 w(s) dB(s) \end{cases}$$

By Lemma 2.4, we have

$$(2.13) \quad z(t) = k_1 + k_2 t, \quad t \in [0, 1].$$

Substituting $z(t)$ into (2.12), we obtain that

$$k_1(a_1 - \int_0^1 dA(s)) - k_2(b_1 + \int_0^1 s dA(s)) = \int_0^1 w(s) dA(s)$$

$$k_1(c_1 - \int_0^1 dB(s)) + k_2(c_1 + d_1 - \int_0^1 s dB(s)) = \int_0^1 w(s) dB(s).$$

It follows from (H1) that

$$\delta = \left| \begin{matrix} a_1 - \int_0^1 dA(s) & -(b_1 + \int_0^1 s dA(s)) \\ c_1 - \int_0^1 dB(s) & c_1 + d_1 - \int_0^1 s dB(s) \end{matrix} \right| > 0,$$

$$(2.14) \quad k_1 = \frac{1}{\delta} [c_1 + d_1 - \int_0^1 s dB(s)] \int_0^1 w(s) dA(s) > 0$$

and

$$(2.15) \quad k_2 = \frac{1}{\delta} [a_1 - \int_0^1 dA(s) + \int_0^1 dB(s)] \int_0^1 w(s) dB(s) > 0.$$

Substituting (2.14) and (2.15) into (2.13), we have

$$(2.16) \quad z(t) = \frac{1}{\delta} \left[c_1 + d_1 - \int_0^1 s dB(s) \right] \int_0^1 \int_0^1 G_1(\tau, s) h(s) ds dA(\tau) + \frac{1}{\delta} \left[a_1 - \int_0^1 dA(s) + \int_0^1 dB(s) \right] t \int_0^1 \int_0^1 G_1(\tau, s) h(s) ds dB(\tau). \quad \square$$

Lemma 2.8. *Let (H1) be satisfied and $1 < \alpha_1 \leq 2$, $0 < \beta_1 \leq 1$. Then the fractional order boundary value problem*

$$(2.17) \quad \begin{cases} D^{\beta_1}(\phi_{p_1}(D^{\alpha_1}u(t))) + h(t) = 0, & t \in (0, 1) \\ a_1u(0) - b_1u'(0) = \int_0^1 u(s)dA(s) \\ c_1u(1) + d_1u'(1) = \int_0^1 u(s)dB(s) \\ D^{\alpha_1}u(0) = 0 \end{cases}$$

has a unique solution

$$(2.18) \quad u(t) = \int_0^1 H_1(t, s) \frac{1}{\Gamma(\beta_1)^{q_1-1}} \left(\int_0^s (s - \tau)^{\beta_1-1} h(\tau) d\tau \right)^{q_1-1} ds.$$

Proof. By Lemma 2.5, the equation $D^{\beta_1}(\phi_{p_1}(D^{\alpha_1}u(t))) + h(t) = 0$ subject to the boundary conditions given by (1.3) can be written as

$$\phi_{p_1}(D^{\alpha_1}u(t)) = -I^{\beta_1}h(t) - c_1t^{\beta-1}.$$

Using boundary condition $D^{\alpha_1}u(0) = 0$, we get $c_1 = 0$. Hence, we obtain

$$-D^{\alpha_1}u(t) = \phi_{q_1}(I^{\beta_1}h(t)).$$

Thus, the boundary value problem (2.17) is equivalent to the following problem:

$$\begin{cases} -D^{\alpha_1}u(t) = \phi_{q_1}(I^{\beta_1}h(t)), & t \in (0, 1), \\ a_1u(0) - b_1u'(0) = \int_0^1 u(s)dA(s), \\ c_1u(1) + d_1u'(1) = \int_0^1 u(s)dB(s). \end{cases}$$

Lemma 2.7 implies that boundary value problem (2.17) has a unique solution,

$$u(t) = \int_0^1 H_1(t, s) \frac{1}{\Gamma(\beta_1)^{q_1-1}} \left(\int_0^s (s - \tau)^{\beta_1-1} h(\tau) d\tau \right)^{q_1-1} ds. \quad \square$$

Lemma 2.9 ([3]). *The Green's function $G_1(t, s)$ defined by (2.3) is continuous on $[0, 1] \times [0, 1)$. Assume $b_1 > \frac{2-\alpha_1}{\alpha_1-1}a_1$, then $G_1(t, s)$ also have the following properties:*

- (1): $G_1(t, s) > 0$ for $(t, s) \in (0, 1) \times (0, 1)$;
- (2): $G_1(t, s) \leq G_1(s, s)$ for $(t, s) \in (0, 1) \times (0, 1)$;
- (3): *there exists a positive number λ_1 such that $G_1(t, s) \geq \lambda_1 G_1(s, s)$, for $(t, s) \in [0, 1] \times [0, 1)$, where*

$$\lambda_1 = \min\left\{\frac{4a_1c_1d_1((\alpha_1-2)a_1 + (\alpha_1-1)b_1)}{((\alpha_1-1)a_1d_1 + a_1c_1 - b_1c_1) + 4a_1c_1((\alpha_1-1)b_1d_1 + b_1c_1)}, \frac{4a_1b_1c_1d_1((\alpha_1-2)a_1 + (\alpha_1-1)b_1)}{((\alpha_1-1)a_1d_1 + a_1c_1 - b_1c_1) + 4a_1c_1((\alpha_1-1)b_1d_1 + b_1c_1)}\right\} < 1.$$

Lemma 2.10. *For $(t, s) \in (0, 1) \times (0, 1)$, $H_1(t, s)$ satisfies the following inequalities:*

- (i): $H_1(t, s) > 0$;
- (ii): $H_1(t, s) \leq \Psi_1 G_1(s, s)$;
- (iii): $H_1(t, s) \geq \Theta_1 G_1(s, s)$,

where

$$(2.19) \quad \Psi_1 = 1 + \frac{1}{\delta}[c_1 + d_1 - \int_0^1 s dB(s)] \int_0^1 dA(\tau) + [a_1 - \int_0^1 dA(s) + \int_0^1 dB(s)] \int_0^1 dB(\tau)$$

$$(2.20) \quad \Theta_1 = \lambda_1[1 + \frac{1}{\delta}(c_1 + d_1 - \int_0^1 s dB(s)) \int_0^1 dA(\tau).$$

Proof. i) It is easy to see that assumption (i) holds. Now, we will verify property (ii).

ii) By (2, 6) and Lemma 2.9, we have

$$G_2(t, s) \leq \frac{G_1(s, s)}{\delta} \left[[c_1 + d_1 - \int_0^1 s dB(s)] \int_0^1 dA(\tau) + [a_1 - \int_0^1 dA(s) + \int_0^1 dB(s)] \int_0^1 dB(\tau) \right]$$

So, we have

$$H_1(t, s) \leq G_1(s, s) \left[1 + \frac{1}{\delta}(c_1 + d_1 - \int_0^1 s dB(s)) \int_0^1 dA(\tau) + (a_1 - \int_0^1 dA(s) + \int_0^1 dB(s)) \int_0^1 dB(\tau) \right].$$

$$\begin{aligned} \text{iii) } G_2(t, s) &\geq \frac{1}{\delta}[c_1 + d_1 - \int_0^1 s dB(s)] \int_0^1 G_1(\tau, s) dA(\tau) \\ &\geq \frac{\lambda_1 G_1(s, s)}{\delta} [c_1 + d_1 - \int_0^1 s dB(s)] \int_0^1 dA(\tau). \end{aligned}$$

Therefore, we have

$$\begin{aligned} H_1(t, s) &= G_1(t, s) + G_2(t, s) \\ &\geq \lambda_1 G_1(s, s) + \frac{\lambda_1 G_1(s, s)}{\delta} \left[c_1 + d_1 - \int_0^1 s dB(s) \right] \int_0^1 dA(\tau) \\ &= \lambda_1 G_1(s, s) \left[1 + \frac{1}{\delta} \left(c_1 + d_1 - \int_0^1 s dB(s) \right) \right] \int_0^1 dA(\tau) \\ &= G_1(s, s) \Theta_1. \end{aligned}$$

In a similar manner, the results of the Green functions $H_2(t, s) = \overline{G}_1(t, s) + \overline{G}_2(t, s)$ and $\overline{G}_1(t, s)$ for the homogeneous BVP's corresponding to the fractional order BVP (1.2), (1.4) are obtained. □

Remark 2.11. It is easy to see the following inequalities:

$$H_1(t, s) \leq \Psi G_1(s, s) \text{ and } H_2(t, s) \leq \Psi \overline{G}_1(s, s),$$

for all $(t, s) \in (0, 1) \times (0, 1)$, where $\Psi = \max\{\Psi_1, \Psi_2\}$.

$$H_1(t, s) \geq \Theta G_1(s, s) \text{ and } H_2(t, s) \geq \Theta \overline{G}_1(s, s)$$

for all $(t, s) \in (0, 1) \times (0, 1)$, where $\Theta = \min\{\Theta_1, \Theta_2\}$.

Let $C = C([0, 1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, 1] \rightarrow \mathbb{R}$ endowed with the norm defined by $\|u\| = \sup_{t \in [0, 1]} |u(t)|$, and define a cone \wp in $C([0, 1], \mathbb{R})$ by

$$\wp = \{u \in C([0, 1], \mathbb{R}^+) : \min_{t \in [0, 1]} u(t) \geq \frac{\Theta}{\Psi} \|u\|\}.$$

It is well known that (u, v) is a solution of the system (1.1)–(1.4) if and only if (u, v) is a solution of the following nonlinear integral system

$$(2.21) \quad \begin{cases} u(t) = \frac{1}{\Gamma(\beta_1)^{q_1-1}} \int_0^1 H_1(t, s) \left(\int_0^s (s-\tau)^{\beta_1-1} f_1(\tau, v(\tau)) d\tau \right)^{q_1-1} ds, \\ v(t) = \frac{1}{\Gamma(\beta_2)^{q_2-1}} \int_0^1 H_2(t, s) \left(\int_0^s (s-\tau)^{\beta_2-1} f_2(\tau, u(\tau)) d\tau \right)^{q_2-1} ds. \end{cases}$$

Now define the operator

$$(Au)(t) = \frac{1}{\Gamma(\beta_2)^{q_2-1}} \int_0^1 H_2(t, s) \left(\int_0^s (s-\tau)^{\beta_2-1} f_2(\tau, u(\tau)) d\tau \right)^{q_2-1} ds$$

then the integral system (2.21) is equivalent to the following nonlinear integral equation

$$u(t) = \frac{1}{\Gamma(\beta_1)^{q_1-1}} \int_0^1 H_1(t, s) \left(\int_0^s (s-\tau)^{\beta_1-1} f_1(\tau, Au(\tau)) d\tau \right)^{q_1-1} ds.$$

Next, for any $u \in \varphi$, define an operator T as follows.

$$Tu(t) = \frac{1}{(\Gamma(\beta_1))^{q_1-1}} \int_0^1 H_1(t, s) \left(\int_0^s (s - \tau)^{\beta_1-1} f_1(\tau, Au(\tau)) d\tau \right)^{q_1-1} ds.$$

Lemma 2.12. *Suppose that (H1) hold. Then $T(\varphi) \subset \varphi$ and $T : \varphi \rightarrow \varphi$ is completely continuous.*

Proof. From the expression of T , it is easy to see that $T(\varphi) \subset \varphi \forall u \in \varphi$. Besides, using the Arzela Ascoli theorem and the standard arguments, one can easily show that $T : \varphi \rightarrow \varphi$ is completely continuous operator. □

3. EXISTENCE OF ONE POSITIVE SOLUTIONS

In this section, we impose growth conditions on f_1 and f_2 which allow us to apply Theorem 1.1 to establish the existence of one positive solution for the BVP (1.1)–(1.4). Now, we begin by introducing some notation.

$$f_{10} = \lim_{u \rightarrow 0^+} \inf \min_{t \in [0,1]} \frac{f_1(t, u)}{\phi_{p_1}(u)}, \quad f_{20} = \lim_{u \rightarrow 0^+} \inf \min_{t \in [0,1]} \frac{f_2(t, u)}{\phi_{p_2}(u)},$$

$$f_1^0 = \lim_{u \rightarrow 0^+} \sup \max_{t \in [0,1]} \frac{f_1(t, u)}{\phi_{p_1}(u)}, \quad f_2^0 = \lim_{u \rightarrow 0^+} \sup \max_{t \in [0,1]} \frac{f_2(t, u)}{\phi_{p_2}(u)},$$

$$f_{1\infty} = \lim_{u \rightarrow \infty} \inf \min_{t \in [0,1]} \frac{f_1(t, u)}{\phi_{p_1}(u)}, \quad f_{2\infty} = \lim_{u \rightarrow \infty} \inf \min_{t \in [0,1]} \frac{f_2(t, u)}{\phi_{p_2}(u)},$$

$$f_1^\infty = \lim_{u \rightarrow \infty} \sup \max_{t \in [0,1]} \frac{f_1(t, u)}{\phi_{p_1}(u)}, \quad f_2^\infty = \lim_{u \rightarrow \infty} \sup \max_{t \in [0,1]} \frac{f_2(t, u)}{\phi_{p_2}(u)},$$

$$A_1 = \frac{1}{\Gamma(\beta_1 + 1)^{(q_1-1)}} \Psi \int_0^1 G_1(s, s) s^{\beta_1(q_1-1)} ds,$$

$$A_2 = \frac{\Theta^2}{\Psi \Gamma(\beta_1 + 1)^{(q_1-1)}} \int_0^1 G_1(s, s) s^{\beta_1(q_1-1)} ds,$$

$$B_1 = \frac{1}{\Gamma(\beta_2 + 1)^{(q_2-1)}} \Psi \int_0^1 \overline{G}_1(s, s) s^{\beta_2(q_2-1)} ds,$$

$$B_2 = \frac{\Theta^2}{\Psi \Gamma(\beta_2 + 1)^{(q_2-1)}} \int_0^1 \overline{G}_1(s, s) s^{\beta_2(q_2-1)} ds.$$

Theorem 3.1. *Suppose that (H1) is satisfied. In addition, we assume that $f_1^0 < \phi_{p_1}(\frac{1}{A_1})$, $f_2^0 < \phi_{p_2}(\frac{1}{B_1})$ and $f_{1\infty} > \phi_{p_1}(\frac{1}{A_2})$, $f_{2\infty} > \phi_{p_2}(\frac{1}{B_2})$. Then the BVP (1.1)–(1.4) has at least one positive solution.*

Proof. In view of $f_1^0 < \phi_{p_1}(\frac{1}{A_1})$, $f_2^0 < \phi_{p_2}(\frac{1}{B_1})$, there exists $\epsilon_1 > 0$ such that

$$(3.1) \quad (f_1^0 + \epsilon_1) \leq \phi_{p_1}(\frac{1}{A_1}), \quad (f_2^0 + \epsilon_1) \leq \phi_{p_2}(\frac{1}{B_1}).$$

By the definition of f_1^0 and f_2^0 , we may choose $\sigma_1 > 0$ so that

$$(3.2) \quad f_1(t, u) \leq (f_1^0 + \epsilon_1)\phi_{p_1}(u),$$

$$f_2(t, u) \leq (f_2^0 + \epsilon_1)\phi_{p_2}(u), \quad t \in [0, 1], u \in [0, \sigma_1].$$

Set $\Omega_1 = \{u \in E \mid \|u\| < \sigma_1\}$. It follows from (3.1) and (3.2) that for any $u \in \wp \cap \partial\Omega_1$, we get

$$(3.3) \quad \begin{aligned} Au(t) &= \frac{1}{(\Gamma(\beta_2))^{q_2-1}} \int_0^1 H_2(t, s) \left(\int_0^s (s - \tau)^{\beta_2-1} f_2(\tau, u(\tau)) d\tau \right)^{q_2-1} ds \\ &\leq \frac{\|u\|}{(\Gamma(\beta_2))^{q_2-1}} (f_2^0 + \epsilon_1)^{q_2-1} \Psi \int_0^1 \overline{G}_1(s, s) \left(\frac{s^{\beta_2}}{\beta_2} \right)^{q_2-1} ds \\ &= \frac{\|u\|}{(\Gamma(\beta_2 + 1))^{q_2-1}} (f_2^0 + \epsilon_1)^{q_2-1} \Psi \int_0^1 \overline{G}_1(s, s) s^{\beta_2(q_2-1)} ds \\ &\leq \|u\|. \end{aligned}$$

Then, by (3.1), (3.2) and (3.3), we have

$$\begin{aligned} Tu(t) &= \frac{1}{(\Gamma(\beta_1))^{q_1-1}} \int_0^1 H_1(t, s) \left(\int_0^s (s - \tau)^{\beta_1-1} f_1(\tau, Au(\tau)) d\tau \right)^{q_1-1} ds \\ &\leq \frac{\|u\|}{(\Gamma(\beta_1 + 1))^{q_1-1}} (f_1^0 + \epsilon_1)^{q_1-1} \Psi \int_0^1 G_1(s, s) s^{\beta_1(q_1-1)} ds \\ &= \|u\|. \end{aligned}$$

Therefore,

$$(3.4) \quad \|Tu\| \leq \|u\|, \quad u \in \wp \cap \partial\Omega_1.$$

On the other hand, since $f_{1\infty} > \phi_{p_1}(\frac{1}{A_2})$, $f_{2\infty} > \phi_{p_2}(\frac{1}{B_2})$ there exists $\epsilon_2 > 0$ such that

$$(3.5) \quad (f_{1\infty} - \epsilon_2) \geq \phi_{p_1}(\frac{1}{A_2})$$

$$(3.6) \quad (f_{2\infty} - \epsilon_2) \geq \phi_{p_2}(\frac{1}{B_2}).$$

By the definition of $f_{1\infty}$ and $f_{2\infty}$, we may choose $\sigma'_2 > \sigma_1$ so that

$$(3.7) \quad f_1(t, u) \geq (f_{1\infty} - \epsilon_2)\phi_{p_1}(u),$$

$$f_2(t, u) \geq (f_{2\infty} - \epsilon_2)\phi_{p_2}(u), \quad t \in [0, 1], u \in [\sigma'_2, \infty).$$

Let $\sigma_2 = \max\{2\sigma_1, \frac{\sigma'_2 \Psi}{\Theta}\}$ and set $\Omega_2 = \{u \in E \mid \|u\| < \sigma_2\}$. Then $u \in \wp \cap \partial\Omega_2$ implies that $\sigma'_2 \leq \frac{\Theta}{\Psi} \|u\| \leq u(t)$ for any $t \in [0, 1]$. So, for $t \in [0, 1]$ in view of Remark 2.11, we

have

$$\begin{aligned} Au(t) &\geq \frac{\Theta^2}{\psi} \frac{\|u\|}{(\Gamma(\beta_2 + 1))^{q_2-1}} (f_{2\infty} - \epsilon_2)^{q_2-1} \int_0^1 \overline{G}_1(s, s) s^{\beta_2(q_2-1)} ds \\ &= \|u\|. \end{aligned}$$

Then, for $t \in (0, 1)$, by (3.5), (3.6), (3.7), we have

$$\begin{aligned} (3.8) \quad Tu(t) &\geq \frac{\Theta^2}{\psi} \frac{\|u\|}{(\Gamma(\beta_1 + 1))^{q_1-1}} (f_{1\infty} - \epsilon_2)^{q_1-1} \int_0^1 G_1(s, s) s^{\beta_1(q_1-1)} ds \\ &\geq \|u\|. \end{aligned}$$

Therefore,

$$(3.9) \quad \|Tu\| \geq \|u\|, \quad u \in \wp \cap \partial\Omega_2.$$

Hence, it follows from the first part of Theorem 1.1 that T has a fixed point $u \in \wp \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Consequently, the BVP (1.1) – (1.4) has a positive solution $(u_1, v_1) \in \wp \times \wp$, where

$$\begin{aligned} u_1(t) &> 0, v_1(t) > 0 \text{ for all } t \in (0, 1) \text{ and} \\ v_1(t) &= \frac{1}{\Gamma(\beta_2)^{q_2-1}} \int_0^1 H_2(t, s) \left(\int_0^s (s - \tau)^{\beta_2-1} f_2(\tau, u_1(\tau)) d\tau \right)^{q_2-1} ds. \end{aligned}$$

□

Theorem 3.2. *Suppose that (H1) is satisfied. In addition, we assume that $f_1^\infty < \phi_{p_1}(\frac{1}{A_1})$, $f_2^\infty < \phi_{p_2}(\frac{1}{B_1})$ and $f_{10} > \phi_{p_1}(\frac{1}{A_2})$, $f_{20} > \phi_{p_2}(\frac{1}{B_2})$. Then the BVP (1.1)–(1.4) has at least one positive solution.*

Proof. The proof is similar to Theorem 3.1 and therefore omitted. □

4. THE NONEXISTENCE OF A POSITIVE SOLUTION

Our last result corresponds to the case when the BVP (1.1)–(1.4) has no positive solution.

Theorem 4.1. *Assume that (H1) holds and $f_1(t, u) < \phi_{p_1}(\frac{u}{A_1})$, $f_2(t, u) < \phi_{p_2}(\frac{u}{B_1})$ for all $t \in (0, 1)$, $u > 0$. Then the BVP (1.1)–(1.4) has no positive solution.*

Proof. Assume to the contrary that $u(t)$ is a positive solution of the BVP (1.1)–(1.4). Then $u(t) > 0$ for $t \in (0, 1)$ and

$$\begin{aligned} \|Au(t)\| &= \int_0^1 H_2(t, s) \phi_{q_2} \left(\frac{1}{\Gamma(\beta_2)} \int_0^s (s - \tau)^{\beta_2-1} f_2(\tau, u(\tau)) d\tau \right) ds \\ &< \Psi \frac{\|u\|}{B_1} \frac{1}{(\Gamma(\beta_2 + 1))^{q_2-1}} \int_0^1 \overline{G}_1(s, s) s^{\beta_2(q_2-1)} ds = \|u\|, \end{aligned}$$

and thus we have

$$\begin{aligned}
 u(t) &= \frac{1}{\Gamma(\beta_1)^{q_1-1}} \int_0^1 H_1(t, s) \left(\int_0^s (s - \tau)^{\beta_1-1} f_1(\tau, Au(\tau)) d\tau \right)^{q_1-1} ds \\
 &< \frac{\Psi \|Au\|}{(\Gamma(\beta_1 + 1))^{q_1-1} A_1} \int_0^1 G_1(s, s) s^{\beta_1(q_1-1)} ds < \| u \|,
 \end{aligned}$$

which is a contradiction, and this completes the proof. □

Similarly, we have the following results.

Theorem 4.2. *Assume that (H1) holds and $f_1(t, u) > \phi_{p_1}(\frac{u}{A_2})$, $f_2(t, u) > \phi_{p_2}(\frac{u}{B_2})$ for all $t \in (0, 1)$, $u > 0$. Then the BVP (1.1)–(1.4) has no positive solution.*

Proof. The proof is similar to Theorem 4.1 and therefore omitted. □

5. EXAMPLE

To illustrate how our main results can be used in practice, we present an example.

$$(5.1) \quad D^{\frac{1}{2}}(\phi_2(D^2u(t)) + f_1(t, u) = 0, t \in (0, 1),$$

$$(5.2) \quad D^{\frac{1}{2}}(\phi_2(D^2v(t)) + f_2(t, v) = 0, t \in (0, 1),$$

$$(5.3) \quad \begin{cases} u(0) - u'(0) = \int_0^1 u(s) dA(s), \\ u(1) - u'(1) = \int_0^1 u(s) dB(s), \\ D^2u(0) = 0, \end{cases}$$

$$(5.4) \quad \begin{cases} v(0) - v'(0) = \int_0^1 v(s) dA(s), \\ v(1) - v'(1) = \int_0^1 v(s) dB(s), \\ D^2v(0) = 0, \end{cases}$$

where $\beta_1 = \beta_2 = \frac{1}{2}$, $p = q = 2$, $\alpha_1 = \alpha_2 = 2$, $a_1 = b_1 = c_1 = d_1 = 1$ and $a_2 = b_2 = c_2 = d_2 = 1$, $A(s) = \frac{s^2}{2}$, $B(s) = \frac{s^3}{3}$,

$$f_1(t, u) = \frac{1}{1 + t^2} \left[\frac{u^2}{e^{u^2}} + \frac{3000u^3}{1 + u} \right],$$

$$f_2(t, u) = \frac{1}{2(1 + t^2)} \left[\frac{u^2}{e^{u^2}} + \frac{5000u^3}{1 + u} \right].$$

Then, by easy calculation, we have $\delta = \frac{127}{72}$, $\Psi = 1, 014$, $\Theta = 0, 476$, $A_1 = B_1 = 0, 564$, $A_2 = B_2 = 0, 022$. It is easy to compute that $f_1^0 = 1$, $f_{1\infty} = 3000$, $f_2^0 = \frac{1}{2}$, $f_{2\infty} = 2500$, which yields that the BVP (5.1)–(5.4) has at least one positive solution.

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