Dynamic Systems and Applications 24 (2015) 283-294

EXISTENCE OF SOLUTIONS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH m-POINT INTEGRAL BOUNDARY CONDITIONS

TUGBA SENLIK CERDIK, NUKET AYKUT HAMAL, AND FULYA YORUK DEREN

Department of Mathematics, Ege University, 35100, Bornava, Izmir

ABSTRACT. In this paper, we consider m-point integral boundary value problems for fractional differential equations involving the Riemann Liouville fractional derivative. The existence results of solutions are established via the application of fixed point theorem.

AMS (MOS) Subject Classification. 34B10; 34B15.

1. INTRODUCTION

Fractional calculus has recently devolved as an interesting and important field of research. As a generalization of differentiation and integration to arbitrary non-integer order, fractional calculus is a significant tool for solving complex problems from various fields such as engineering, science, pure and applied mathematics. Much attention has been focused on the study of the existence and multiplicity of solutions or positive solutions for boundary value problems of fractional differential equations by using techniques of nonlinear analysis. For more details, we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9] and the references therein.

Although the boundary value problems of the fractional differential equation with m-point fractional boundary conditions have been studied in many literature, only a few papers can be found in the literature on the existence of solutions for the fractional differential equation with m-point fractional integral boundary conditions, see [10, 11, 12]. In particular, we would like to mention some excellent results.

In [2], Ahmad, Ntouyas and Alsaedi considered the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations of order $q \in (1, 2]$ with three-point integral boundary conditions given by

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ x(0) = 0, \quad x(1) = a \displaystyle\int_0^\eta x(s)ds, \quad 0 < \eta < 1, \end{cases}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order $q, f : [0, 1] \times X \to X$ is continuous, and $a \in \mathbb{R}$ is such that $a \neq 2/\eta^2$. Here, $(X, \|\cdot\|)$ is a Banach space and

Received April 2, 2015

$\mathcal{C} = C([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

In [3], Boucherif and Ntouyas considered the following first order initial value problem for fractional differential equations with nonlocal conditions

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad 0 < t < T, \quad 0 < q \leq 1, \\ x(0) + \displaystyle\sum_{j=1}^{m} \gamma_i x(t_j) = 0, \end{cases}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order $q, f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $t_j, (j = 0, 1, \ldots, m)$ are given point with $0 \leq t_1 \leq \cdots \leq t_m \leq T$ and γ_j are real numbers with $1 + \displaystyle\sum_{j=1}^{m} \gamma_j \neq 0$.

In this study, we discuss the existence and uniqueness of solutions of Riemann-Liouville fractional differential equation involving the p-Laplacian operator with m-point integral boundary conditions given by

$$\begin{cases} -D^{\beta}(\phi_p(D^{\alpha} x(t))) = f(t, x(t)), \quad 1 < \alpha \leq 2, \quad t \in [0,1], \\ D^{\delta} x(0) = 0, \quad D^{\delta} x(1) = \displaystyle\sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} D^{\delta} x(s) ds, \quad D^{\alpha} x(0) = 0, \end{cases} \tag{1.1}$$

where $0 < \delta \leq 1$, $1 < \alpha - \delta < 2$, $0 < \beta \leq 1$, $0 = \eta_0 < \eta_1 < \cdots < \eta_{m-2} < \eta_{m-1} = 1$, and $\alpha_i \geq 0$ for $i \in 0, \ldots, m-1$, $\sum_{i=1}^{m-1} \alpha_i \frac{\eta_i^{\alpha-\delta} - \eta_{i-1}^{\alpha-\delta}}{\alpha - \delta} \neq 1$. Here, $D^{(.)}$ denotes the Riemann-Liouville fractional derivative of order (.), f is given continuous function and $\phi_p(s)$ is a p-Laplacian operator, *i.e.*, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

We note that the m-point boundary condition is related to $m-1$ intervals of the area under the curve of solution $u(t)$ from $t = \eta_{i-1}$ to $t = \eta_i$ for $i = 1, \ldots, m-1$.

The rest of the paper is organized as follows. In Section 2, we present some definitions and lemmas that will be used to prove our main results. In Section 3, we prove an existence and uniqueness results by using the Banach's fixed point theorem. Finally, as an application, the results are demonstrated with an example.

2. PRELIMINARIES

In this section, we give some basic definitions and lemmas which are useful for the presentation of our main results.

Definition 2.1 ([13, 14])**.** The Riemann Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a function $h : (0, \infty) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

provided that the right hand side is defined pointwise.

Definition 2.2 ([13, 14])**.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$D_{0+}^{\alpha} f(t) = \Big(\frac{d}{dt}\Big)^{n} I_{0+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^{n} \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where n is the smallest integer greater than or equal to α, provided that the right-hand side is defined pointwise. In particular, for $\alpha = n$, $D_{0+}^{n} f(t) = f^{(n)}(t)$.

Lemma 2.3 ([14])**.** *The equality $D_{0+}^{\gamma} I_{0+}^{\gamma} f(t) = f(t)$ with $\gamma > 0$ holds for $f \in L^1(0,1)$.*

Lemma 2.4 ([14])**.** *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0+}^{\alpha} u = 0$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, \dots, n$, where n is the smallest integer greater than or equal to α.

Lemma 2.5 ([14])**.** *Let $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order α $(\alpha > 0)$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, \dots, n$, where n is the smallest integer greater than or equal to α.

The basic properties of the p-Laplacian operator which will be used in the following studies are listed below.

(i) If $1 < p < 2, xy > 0$, and $|x|, |y| \geq m > 0$, then

$$|\phi_p(x) - \phi_p(y)| \leq (p-1) m^{p-2} |x-y|.$$

(ii) If $p \geq 2, |x|, |y| \leq M$, then

$$|\phi_p(x) - \phi_p(y)| \leq (p-1) M^{p-2} |x-y|.$$

Observe that the substitution $x(t) = I^{\delta} y(t) = D^{-\delta} y(t)$ transforms the boundary value problem (1.1) to the following form:

$$\begin{cases} -D^{\beta}(\phi_p(D^{\alpha-\delta} y(t))) = f(t, y(t)), \quad t \in [0,1], \\ y(0) = 0, \quad y(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} y(s) ds \quad D^{\alpha-\delta} y(0) = 0. \end{cases} \tag{2.1}$$

To define the solutions of the problem (2.1), we need the following lemma.

Lemma 2.6. *For any $h \in C[0,1]$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} -D^{\alpha-\delta}y(t) = h(t), \quad t \in (0,1), \\ y(0) = 0, \quad y(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} y(s)ds \end{cases} \tag{2.2}$$

is

$$y(t) = -I^{\alpha-\delta}h(t) + \frac{t^{\alpha-\delta-1}}{1-\sum_{i=1}^{m-1}\alpha_i \frac{\eta_i^{\alpha-\delta}-\eta_{i-1}^{\alpha-\delta}}{\alpha-\delta}}\Big(I^{\alpha-\delta}h(1) - \sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} I^{\alpha-\delta}h(s)ds\Big).$$

Proof. It is well known by means of Lemma 2.5 that the solution of fractional differential equation in (2.2) can be written as

$$y(t) = -I^{\alpha-\delta}h(t) - c_1 t^{\alpha-\delta-1} - c_2 t^{\alpha-\delta-2}, \tag{2.3}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Using the boundary condition $y(0) = 0$, we conclude that $c_2 = 0$. Then we have

$$y(1) = -I^{\alpha-\delta}h(1) - c_1.$$

Hence, by the boundary condition $y(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} y(s)ds$, we obtain that

$$c_1 = \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \frac{\eta_i^{\alpha-\delta}-\eta_{i-1}^{\alpha-\delta}}{\alpha-\delta}}\Big[-I^{\alpha-\delta}h(1) + \sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} I^{\alpha-\delta}h(s)ds\Big].$$

Substituting these values in (2.3) yields

$$y(t) = -I^{\alpha-\delta}h(t) + \frac{t^{\alpha-\delta-1}}{1-\sum_{i=1}^{m-1}\alpha_i \frac{\eta_i^{\alpha-\delta}-\eta_{i-1}^{\alpha-\delta}}{\alpha-\delta}}\Big(I^{\alpha-\delta}h(1) - \sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} I^{\alpha-\delta}h(s)ds\Big).$$

This completes the proof. □

Note that, by Lemma 2.5, the equation $-D^{\beta}(\phi_p(D^{\alpha-\delta}y(t))) = h(t)$ subject to the boundary conditions given by (2.1) can be written as

$$\phi_p(D^{\alpha-\delta}y(t)) = -I^{\beta}h(t) - c_1 t^{\beta-1}.$$

Using boundary condition $D^{\alpha-\delta}y(0) = 0$, we get $c_1 = 0$. Hence, we obtain

$$-D^{\alpha-\delta}y(t) = \phi_q(I^{\beta}h(t)). \tag{2.4}$$

Thus, the boundary value problem (2.1) is equivalent to the following problem:

$$\begin{cases} -D^{\alpha-\delta}y(t) = \phi_q(I^{\beta}h(t)), \quad t \in (0,1), \\ y(0) = 0, \quad y(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} y(s)ds, \end{cases}$$

Lemma 2.6 implies that boundary value problem (2.1) has a unique solution,

$$y(t) = -I^{\alpha-\delta}\phi_q(I^{\beta}h(t)$$

$$+\frac{t^{\alpha-\delta-1}}{1-\sum_{i=1}^{m-1}\alpha_i\frac{\eta_i^{\alpha-\delta}-\eta_{i-1}^{\alpha-\delta}}{\alpha-\delta}}\Big(I^{\alpha-\delta}\phi_q(I^\beta h(1))-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}I^{\alpha-\delta}\phi_q(I^\beta h(s))ds\Big).$$

Then, the solution of $-D^\beta(\phi_p(D^\alpha x(t)))=h(t)$ given by boundary conditions (1.1),

$$\begin{aligned}
x(t)&=I^\delta y(t)\\
&=I^\delta\Big[-I^{\alpha-\delta}\phi_q(I^\beta h(t))+\frac{t^{\alpha-\delta-1}}{1-\sum_{i=1}^{m-1}\alpha_i\frac{\eta_i^{\alpha-\delta}-\eta_{i-1}^{\alpha-\delta}}{\alpha-\delta}}\Big(I^{\alpha-\delta}\phi_q(I^\beta h(1))\\
&\quad-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}I^{\alpha-\delta}\phi_q(I^\beta h(s))ds\Big)\Big]\\
&=-I^\alpha\phi_q(I^\beta h(t))+\frac{1}{1-\sum_{i=1}^{m-1}\alpha_i\frac{\eta_i^{\alpha-\delta}-\eta_{i-1}^{\alpha-\delta}}{\alpha-\delta}}\Big(I^{\alpha-\delta}\phi_q(I^\beta h(1))\\
&\quad-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}I^{\alpha-\delta}\phi_q(I^\beta h(s))ds\Big)\frac{1}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}s^{\alpha-\delta-1}ds\\
&=-I^\alpha\phi_q(I^\beta h(t))+\frac{1}{1-\sum_{i=1}^{m-1}\alpha_i\frac{\eta_i^{\alpha-\delta}-\eta_{i-1}^{\alpha-\delta}}{\alpha-\delta}}\Big(I^{\alpha-\delta}\phi_q(I^\beta h(1))\\
&\quad-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}I^{\alpha-\delta}\phi_q(I^\beta h(s))ds\Big)\Big\{\frac{t^{\alpha-1}}{\Gamma(\delta)}\int_0^1(1-\nu)^{\delta-1}\nu^{\alpha-\delta-1}d\nu\Big\}.
\end{aligned}$$

Here, we have used the substitution $s=t\nu$ in the integral of the last term. Using the relation for the Beta function $B(\cdot,\cdot)$,

$$B(\alpha,\beta)=\int_0^1(1-u)^{\alpha-1}u^{\beta-1}du=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

we get

$$\begin{aligned}
x(t)=-I^\alpha\phi_q(I^\beta h(t))&+\frac{t^{\alpha-1}\Gamma(\alpha-\delta)}{\Gamma(\alpha)(1-\sum_{i=1}^{m-1}\alpha_i\frac{\eta_i^{\alpha-\delta}-\eta_{i-1}^{\alpha-\delta}}{\alpha-\delta})}\\
&\times\Big(I^{\alpha-\delta}\phi_q(I^\beta h(1))-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}I^{\alpha-\delta}\phi_q(I^\beta h(s))ds\Big).
\end{aligned}$$

Then,

$$\begin{aligned}
x(t)=&-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s(s-\eta)^{\beta-1}h(\eta)d\eta\Big)ds\\
&+t^{\alpha-1}\theta\Big[\frac{1}{\Gamma(\alpha-\delta)}\int_0^1(1-s)^{\alpha-\delta-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s(s-\eta)^{\beta-1}h(\eta)d\eta\Big)ds\\
&-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\frac{1}{\Gamma(\alpha-\delta)}\int_0^s(s-\tau)^{\alpha-\delta-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\eta)^{\beta-1}h(\eta)d\eta\Big)d\tau ds\Big],
\end{aligned}$$

where

$$\theta = \frac{\Gamma(\alpha-\delta)}{\Gamma(\alpha)[1-\sum_{i=1}^{m-1}\alpha_i\frac{\eta_i^{\alpha-\delta}-\eta_{i-1}^{\alpha-\delta}}{\alpha-\delta}]}.$$

3. MAIN RESULTS

In this section, we will use the Banach contraction mapping principle to prove existence and uniqueness of the solution for the fractional boundary value problem (1.1).

Let $\mathcal{C} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \max_{t\in[0,1]}|x(t)|$. Now consider $T_i : C[0,1] \to C[0,1], i = 0, 1$, with

$$T_0x(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\eta)^{\beta-1}f(\eta, x(\eta))d\eta$$

and

$$\begin{aligned} T_1x(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q(x(s))ds \\ &+ t^{\alpha-1}\theta\Big[\frac{1}{\Gamma(\alpha-\delta)}\int_0^1 (1-s)^{\alpha-\delta-1}\phi_q(x(s))ds \\ &- \sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\frac{1}{\Gamma(\alpha-\delta)}\int_0^s (s-\tau)^{\alpha-\delta-1}\phi_q(x(\tau))d\tau ds\Big]. \end{aligned}$$

Then the operator $T : C[0,1] \to C[0,1]$, defined by $T = T_1 \circ T_0$ is continuous and compact. Clearly, a fixed point of the operator T is a solution of the problem (1.1).

Theorem 3.1. *Assume that* $1 < p \leq 2$ *holds and the following condition is satisfied.*

(A_1) $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ *is continuous function, there exist a nonnegative function* $w \in C[0,1]$ *and a constant* $L > 0$ *such that*

$$|f(t,x)| \leq w(t), \quad \textit{for any } (t,x) \in [0,1]\times\mathbb{R},$$

$$|f(t,x) - f(t,y)| \leq L|x-y|, \quad \textit{for } t \in [0,1] \textit{ and } x, y \in \mathbb{R}.$$

If

$$\begin{aligned} \Lambda_1 = \frac{L(q-1)M^{q-2}}{\Gamma(\beta+1)}\Big(&\frac{1}{\Gamma(\alpha+1)} + \theta\Big[\frac{1}{\Gamma(\alpha-\delta+1)} \\ &+ \frac{1}{\Gamma(\alpha-\delta+1)}\sum_{i=1}^{m-1}\alpha_i\frac{\eta_i^{\alpha-\delta+1}-\eta_{i-1}^{\alpha-\delta+1}}{\alpha-\delta+1}\Big]\Big) < 1, \end{aligned}$$

where

$$M = \max_{t\in[0,1]}\frac{1}{\Gamma(\beta)}\int_0^t (t-\eta)^{\beta-1}w(\eta)d\eta$$

then the problem (1.1) *has a unique solution.*

Proof. Let $1 < p \leq 2$, then we have $q \geq 2$ from $\frac{1}{p} + \frac{1}{q} = 1$. By using (A_1), for any $t \in [0,1]$, $x \in \mathbb{R}$, we obtain

$$
\begin{aligned}
|T_0 x(t)| &= \Big|\frac{1}{\Gamma(\beta)} \int_0^t (t-\eta)^{\beta-1} f(\eta, x(\eta)) d\eta\Big| \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-\eta)^{\beta-1} |f(\eta, x(\eta))| d\eta \\
&\leq \max_{t\in[0,1]} \frac{1}{\Gamma(\beta)} \int_0^t (t-\eta)^{\beta-1} w(\eta) d\eta \\
&= M.
\end{aligned} \tag{3.1}
$$

By using (ii), (3.1) and (A_1), for any $x, y \in \mathbb{R}$, we have

$$
\begin{aligned}
\Big|\phi_q(T_0x(t)) - \phi_q(T_0y(t))\Big| &= \Big|\phi_q\Big(\frac{1}{\Gamma(\beta)} \int_0^t (t-\eta)^{\beta-1} f(\eta, x(\eta)) d\eta\Big) \\
&\quad - \phi_q\Big(\frac{1}{\Gamma(\beta)} \int_0^t (t-\eta)^{\beta-1} f(\eta, y(\eta)) d\eta\Big)\Big| \\
&\leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \Big| \int_0^t (t-\eta)^{\beta-1} f(\eta, x(\eta)) - f(\eta, y(\eta)) d\eta\Big| \\
&\leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \int_0^t (t-\eta)^{\beta-1} L|x(\eta) - y(\eta)| d\eta \\
&\leq \frac{(q-1)M^{q-2}}{\Gamma(\beta+1)} L\|x-y\|.
\end{aligned}
$$

Therefore, for any $x, y \in \mathbb{R}$

$$
\begin{aligned}
&|Tx(t) - Ty(t)| \\
&= |T_1(T_0x)(t) - T_1(T_0y)(t)| \\
&= \Big|\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\phi_q((T_0x)(s)) - \phi_q((T_0y)(s))) ds \\
&\quad + t^{\alpha-1}\theta\Big[\frac{1}{\Gamma(\alpha-\delta)} \int_0^1 (1-s)^{\alpha-\delta-1} (\phi_q((T_0x)(s)) - \phi_q((T_0y)(s))) ds \\
&\quad + \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} \frac{1}{\Gamma(\alpha-\delta)} \int_0^s (s-\tau)^{\alpha-\delta-1} (\phi_q((T_0x)(\tau)) - \phi_q((T_0y)(\tau))) d\tau ds\Big]\Big| \\
&\leq \frac{L(q-1)M^{q-2}}{\Gamma(\beta+1)} \Big(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \theta\Big[\frac{1}{\Gamma(\alpha-\delta)} \int_0^1 (1-s)^{\alpha-\delta-1} ds \\
&\quad + \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} \frac{1}{\Gamma(\alpha-\delta)} \int_0^s (s-\tau)^{\alpha-\delta-1} d\tau ds\Big]\Big) \|x-y\| \\
&\leq \frac{L(q-1)M^{q-2}}{\Gamma(\beta+1)} \Big(\frac{1}{\Gamma(\alpha+1)} + \theta\Big[\frac{1}{\Gamma(\alpha-\delta+1)} \\
&\quad + \frac{1}{\Gamma(\alpha-\delta+1)} \sum_{i=1}^{m-1} \alpha_i \frac{\eta_i^{\alpha-\delta+1} - \eta_{i-1}^{\alpha-\delta+1}}{\alpha-\delta+1}\Big]\Big) \|x-y\|
\end{aligned}
$$

$$= \Lambda_1 \|x - y\|.$$

Hence, for any $x, y \in \mathbb{R}$

$$\|Tx - Ty\| \leq \Lambda_1 \|x - y\|,$$

where $0 < \Lambda_1 < 1$. This implies that $T : C[0,1] \to C[0,1]$ is a contraction mapping. By Banach contraction mapping principle, T has a unique fixed point in $C[0,1]$ which is a solution of problem (1.1). □

Theorem 3.2. *Assume that $p > 2$ holds and the following condition is satisfied.*

(A_2) *$f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function, there exist $\lambda > 0$, $\delta \geq 0$, $L > 0$ such that $(\delta + \beta)(q-2) + 1 \geq 0$ and*

$$f(t,x) \geq \lambda t^{\delta}, \quad \textit{for any } (t,x) \in (0,1] \times \mathbb{R},$$

$$|f(t,x) - f(t,y)| \leq L|x - y|, \quad \textit{for } t \in [0,1] \textit{ and } x, y \in \mathbb{R}.$$

If

$$\begin{aligned}\Lambda_2 = \Delta\Big(&\frac{\Gamma((\delta+\beta)(q-2)+1)}{\Gamma(\alpha+(\delta+\beta)(q-2)+1)} + \theta\Big[\frac{\Gamma((\delta+\beta)(q-2)+1)}{\Gamma(\alpha-\delta+(\delta+\beta)(q-2)+1)} \\ &+ \sum_{i=1}^{m-1} \alpha_i(\eta_i - \eta_{i-1}) \frac{\Gamma((\delta+\beta)(q-2)+1)}{\Gamma(\alpha-\delta+(\delta+\beta)(q-2)+1)}\Big]\Big) \\ &< 1,\end{aligned}$$

where

$$\Delta = \frac{(q-1)L}{\Gamma(\beta+1)} \frac{\lambda^{q-2}\Gamma(\delta+1)^{q-2}}{\Gamma(\delta+\beta+1)^{q-2}}$$

then the problem (1.1) has a unique solution.

Proof. From the definition of operator T_0, for any $x, y \in C[0,1]$, we have $|\phi_q(T_0x(0)) - \phi_q(T_0y(0))| = 0$ and for $t > 0$, we obtain the following inequalities

$$\begin{aligned}(t-\eta)^{\beta-1}\lambda\eta^{\delta} &\leq (t-\eta)^{\beta-1} f(\eta, x(\eta)) \\ \frac{1}{\Gamma(\beta)}\int_0^t (t-\eta)^{\beta-1}\lambda\eta^{\delta} d\eta &\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-\eta)^{\beta-1} f(\eta, x(\eta)) d\eta \\ \frac{\lambda\Gamma(\delta+1)}{\Gamma(\delta+\beta+1)} t^{\delta+\beta} &\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-\eta)^{\beta-1} f(\eta, x(\eta)) d\eta.\end{aligned}$$

By using (i), (A_2) and the definition of the operator T_0, for any $x, y \in \mathbb{R}$, we have

$$\begin{aligned}&\Big|\phi_q(T_0x(t)) - \phi_q(T_0y(t))\Big| \\ &= \Big|\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^t (t-\eta)^{\beta-1} f(\eta, x(\eta)) d\eta\Big) - \phi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^t (t-\eta)^{\beta-1} f(\eta, y(\eta)) d\eta\Big)\Big| \\ &\leq (q-1)\Big[\frac{\lambda\Gamma(\delta+1)}{\Gamma(\delta+\beta+1)} t^{\delta+\beta}\Big]^{q-2} \Big|\frac{1}{\Gamma(\beta)}\int_0^t (t-\eta)^{\beta-1} f(\eta, x(\eta)) - f(\eta, y(\eta)) d\eta\Big|\end{aligned}$$

$$\le (q-1)\Big[\frac{\lambda\Gamma(\delta+1)}{\Gamma(\delta+\beta+1)}t^{\delta+\beta}\Big]^{q-2}\frac{1}{\Gamma(\beta)}\int_0^t (t-\eta)^{\beta-1}L|x(\eta)-y(\eta)|d\eta$$
$$\le \frac{(q-1)}{\Gamma(\beta+1)}\Big[\frac{\lambda\Gamma(\delta+1)}{\Gamma(\delta+\beta+1)}t^{\delta+\beta}\Big]^{q-2}L\|x-y\|.$$

Hence, for any $t>0$, we can get

$$\begin{aligned}
&|Tx(t)-Ty(t)|\\
&=|T_1(T_0x)(t)-T_1(T_0y)(t)|\\
&=\Big|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\phi_q((T_0x)(s))-\phi_q((T_0y)(s)))ds\\
&\quad+t^{\alpha-1}\theta\Big[\frac{1}{\Gamma(\alpha-\delta)}\int_0^1 (1-s)^{\alpha-\delta-1}(\phi_q((T_0x)(s))-\phi_q((T_0y)(s)))ds\\
&\quad+\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\frac{1}{\Gamma(\alpha-\delta)}\int_0^s (s-\tau)^{\alpha-\delta-1}(\phi_q((T_0x)(\tau))-\phi_q((T_0y)(\tau)))d\tau ds\Big]\Big|\\
&\le \Delta\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{(\delta+\beta)(q-2)}ds+\theta\Big[\frac{1}{\Gamma(\alpha-\delta)}\int_0^1 (1-s)^{\alpha-\delta-1}s^{(\delta+\beta)(q-2)}ds\\
&\quad+\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\frac{1}{\Gamma(\alpha-\delta)}\int_0^s (s-\tau)^{\alpha-\delta-1}\tau^{(\delta+\beta)(q-2)}d\tau ds\Big]\Big)\|x-y\|\\
&\le \Delta\Big(\frac{1}{\Gamma(\alpha)}B(\alpha,(\delta+\beta)(q-2)+1)+\theta\Big[\frac{1}{\Gamma(\alpha-\delta)}B(\alpha-\delta,(\delta+\beta)(q-2)+1)\\
&\quad+\sum_{i=1}^{m-1}\alpha_i(\eta_i-\eta_{i-1})\frac{1}{\Gamma(\alpha-\delta)}B(\alpha-\delta,(\delta+\beta)(q-2)+1)\Big]\Big)\|x-y\|\\
&\le \Delta\Big(\frac{\Gamma((\delta+\beta)(q-2)+1)}{\Gamma(\alpha+(\delta+\beta)(q-2)+1)}+\theta\Big[\frac{\Gamma((\delta+\beta)(q-2)+1)}{\Gamma(\alpha-\delta+(\delta+\beta)(q-2)+1)}\\
&\quad+\sum_{i=1}^{m-1}\alpha_i(\eta_i-\eta_{i-1})\frac{\Gamma((\delta+\beta)(q-2)+1)}{\Gamma(\alpha-\delta+(\delta+\beta)(q-2)+1)}\Big]\Big)\|x-y\|\\
&\le \Lambda_2\|x-y\|.
\end{aligned}$$

Thus, for any $x,y\in\mathbb{R}$

$$\|Tx-Ty\|\le\Lambda_2\|x-y\|,$$

where $0<\Lambda_2<1$. This implies that $T:C[0,1]\to C[0,1]$ is a contraction mapping. By Banach contraction mapping principle, T has a unique fixed point in $C[0,1]$ which is a solution of problem (1.1). □

Theorem 3.3. *Assume that $p>2$ holds and the following condition is satisfied.*

(A_3) $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *is continuous function, there exist* $\lambda>0$, $\delta\ge 0$, $L>0$ *such that* $(\delta+\beta)(q-2)+1\ge 0$ *and*

$$f(t,x)\le-\lambda t^{\delta},\quad \text{for any } (t,x)\in(0,1]\times\mathbb{R},$$

$$|f(t,x)-f(t,y)|\le L|x-y|,\quad \textit{for } t\in[0,1] \textit{ and } x,y\in\mathbb{R}.$$

If

$$\Lambda_2=\Delta\Big(\frac{\Gamma((\delta+\beta)(q-2)+1)}{\Gamma(\alpha+(\delta+\beta)(q-2)+1)}+\theta\Big[\frac{\Gamma((\delta+\beta)(q-2)+1)}{\Gamma(\alpha-\delta+(\delta+\beta)(q-2)+1)} + \sum_{i=1}^{m-1}\alpha_i(\eta_i-\eta_{i-1})\frac{\Gamma((\delta+\beta)(q-2)+1)}{\Gamma(\alpha-\delta+(\delta+\beta)(q-2)+1)}\Big]\Big) < 1,$$

where

$$\Delta=\frac{(q-1)L}{\Gamma(\beta+1)}\frac{\lambda^{q-2}\Gamma(\delta+1)^{q-2}}{\Gamma(\delta+\beta+1)^{q-2}}$$

then the problem (1.1) *has a unique solution.*

Proof. The inequality $f(t,x)\le-\lambda t^{\delta}$ implies that $\lambda t^{\delta}\le-f(t,x(t))$. Therefore, replace $f(t,x(t))$ by $-f(t,x(t))$ in the proof of Theorem 3.2. □

Example 3.1 Consider the following fractional boundary value problem

$$\begin{cases} -D^{1/3}(\phi_{\frac{9}{5}}(D^{3/2}x(t)))=f(t,x(t)), \quad 1<\alpha\le 2, \quad t\in[0,1],\\ D^{1/4}x(0)=0, \quad D^{1/4}x(1)=\frac{1}{2}\int_0^{\frac{1}{4}}D^{1/4}x(s)ds+\frac{1}{2}\int_{\frac{1}{3}}^{1}D^{1/4}x(s)ds,\\ D^{\alpha}x(0)=0, \end{cases} \tag{3.2}$$

Here $\alpha=3/2$, $\delta=1/4$, $\beta=1/3$, $a_1=1/2$, $a_2=0$, $a_3=1/2$, $\eta_0=0$, $\eta_1=1/4$, $\eta_2=1/3$, $\eta_3=1$, and $f(t,x)=\dfrac{t}{7(t+1)}sinx$ and $w(t)=1$. As $|f(t,x)|\le w(t)=1$ and $|f(t,x)-f(t,y)|\le\dfrac{1}{7}|x-y|$, then (A_1) is satisfied with $L=\dfrac{1}{7}$ and $\Lambda_1\simeq 0.30<1$. Hence, by the conclusion of Theorem 3.1, the boundary value problem (3.2) has a unique solution on $[0,1]$.

REFERENCES

[1] R. P. Agarwal, S. K Ntouyas, B. Ahmad, M. S. Alhothuali, Existence of solutions for integro-differential equations of fractional order with nonlocal three-point fractional boundary conditions, Advances in Difference Equations, 2013, 2013:128.

[2] B. Ahmad, S. K. Ntouyas, and A. Alsaedi, New Existence Results for Nonlinear Fractional Differential Equations with Three-Point Integral Boundary Conditions, Advances in Difference Equations, Volume 2011, Article ID 107384, 11 p.

[3] A. Boucherif, S. K. Ntouyas, Nonlinear initial value problems for first order fractional differential equations, Dynamic Systems and Applications, 20 (2011) 247–260.

[4] W. X. Zhou, Y. D. Chu, Existence of solutions for fractional differential equations with multi-point boundary conditions, Commun Nonlinear Sci Numer Simulat, 17 (2012) 1142–1148.

[5] A. Guezane-Lakouda, R. Khaldi, Solvability of a fractional boundary value problem with fractional integral condition, Nonlinear Analysis, 75 (2012) 2692–2700.

[6] N. Nyamoradi, T. Bashiri, S. M. Vaezpour, D. Baleanu, Uniqueness and existence of positive solutions for singular fractional differential equations, Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 130, pp. 1–13.

[7] W. X. Zhou, Y. D. Chu, Existence of solutions for fractional differential equations with multi-point boundary conditions, Commun Nonlinear Sci Numer Simulat, 17 (2012) 1142–1148.

[8] X. Liu, M. Jia, X. Xiang, On the solvability of a fractional differential equation model involving the p-Laplacian operator, Computers and Mathematics with Applications, 64 (2012) 3267–3275.

[9] Z. Liu, L. Lu. I. Szanto, Existence of solutions for fractional impulsive differential equations with p-Laplacian operator, Acta Mathematica Hungaria, 141 (3) (2013) 203–219.

[10] B.Ahmad, J.J. Nieto, Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, Boundary Value Problems, 2011, 2011:36.

[11] W. Sudsutad, J. Tariboon, Boundary value problems for fractional differential equations with three-point fractional integral boundary conditions, Advances in Difference Equations, 2012, 2012:93.

[12] W. Sudsutad, J. Tariboon, Existence results of fractional integro-differential equations with m-point multi-term fractional order integral boundary conditions, Boundary Value Problems, 2012, 2012:94.

[13] I. Podlubny, Fractional Differential Equations. Academic Press, San Diego (1999).

[14] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V, Amsterdam, 2006.

[15] M. A. Krasnoselskii, 11Two remarks on the method of successive approximations," Uspekhi Matematicheskikh Nauk, vol. 10, pp. 123–127, 1955.

[16] R. P. Agarwal, M. Meehan, D. Oregan, Fixed Point Theory and Applications, Cambridge University Press, Cambridge, (2001).