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STATISTICAL CONVERGENCE AND SOME QUESTIONS OF OPERATOR THEORY

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ABSTRACT. We use the concepts of statistical convergence and Berezin symbols for solving of some problems of operator theory. Namely, we prove that under some conditions the weak statistical limit of compact operators is compact. We also use statistical convergence for the solving of similar problem for the sequence of operators from Schatten-Neuman class. Some related questions are also discussed.

Key Words. Statistical convergence, Compact operator, Schatten-Neumann class, Berezin symbol.

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1. Introduction and Background

In this paper, we use the concept of statistical convergence for solving of some problems of operator theory.

Recall that the concept of statistical convergence was firstly introduced by Fast in [4], see also Steinhaus [14]. In what follows statistical convergence studied in many further papers (see, for instance, Fridy [5, 6], Kolk [8], Pehlivan and Karaev [11], Connor et al. [2]). Following [9], note that if S is a subset of the positive integers $\mathbb{N}$, then S_n denotes the set $\{s \in S : s \leq n\}$ and $|S_n|$ denotes the number of elements in S_n. The natural density of S by Niven and Zuckerman [9] is given by

$$\delta(S) = \lim_{n\to\infty} \frac{|S_n|}{n}.$$

Definition 1.1. A sequence $(x_k)_{k\geq 1}$ of real (or complex) numbers is said to be statistically convergent to some number L, if for each $\varepsilon > 0$ the set

$$S_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

has natural density zero; in this case we abbreviate $st\text{-}\lim_k x_k = L$.

We recall that (see Fridy [5]) for two sequences $x = (x_k)_{k\geq 1}$ and $y = (y_k)_{k\geq 1}$ the notion "$x_k = y_k$ for almost all k" means that $\delta(\{k : x_k \neq y_k\}) = 0$. The following is the classical result of Fridy [5].

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Lemma 1.2. *The following statements are equivalent:*

(i) $(x_k)_{k\geq 1}$ *is a statistically convergent sequence;*

(ii) $(x_k)_{k\geq 1}$ *is a sequence for which there is a convergent sequence* $(y_k)_{k\geq 1}$ *such that* $x_k = y_k$ *for almost all* k.

An immediate and useful corollary of this lemma is the following.

Corollary 1.3. *If* $(x_k)_{k\geq 1}$ *is a sequence such that* $st\text{-}\lim_k x_k = L$, *then* $(x_k)_{k\geq 1}$ *has a subsequence* $(y_k)_{k\geq 1}$ *such that* $\lim_k y_k = L$ *(in the usual sense).*

The notion of statistically convergence was extended to the sequences of Banach spaces by Connor, Ganichev and Kadets in their paper [2] as follows.

Definition 1.4 ([2])**.** Let X be a Banach space, $(x_k)_{k\geq 1}$ be a X-valued sequence, and $x \in X$ be an element.

(i) The sequence $(x_k)_{k\geq 1}$ is norm statistically convergent to x provided that

$$\delta\left(\{k : \|x_k - x\|_X > \varepsilon\}\right) = 0 \text{ for all } \varepsilon > 0.$$

(ii) The sequence $(x_k)_{k\geq 1}$ is weakly statistically convergent to x provided that, for any continuous linear functional f on X, the sequence $(f(x_k - x))_{k\geq 1}$ is statistically convergent to 0.

Note that similar to the sequences of numbers, if a Banach space valued sequence $x = (x_k)_{k\geq 1}$ is norm statistically convergent, then there exists a usual convergent sequence $y = (y_k)_{k\geq 1}$ such that $x_k = y_k$ for almost all k, i.e., $\delta\left(\{k : x_k \neq y_k\}\right) = 0$ (see [2]). As a consequence, many of the results for real statistically convergent sequences carry over to norm statistically convergent sequences (see Kolk [8]). It is also natural to define a series $\sum_k x_k$ to be norm statistically convergent to x by requiring the sequence of partial sums $(\sum_{k=1}^{n} x_k)$ to be norm statistically convergent to x.

2. **Statistical convergence and compactness of operators**

The following definition is well known (see, for example, in Pehlivan and Karaev [11]).

Definition 2.1. The sequence $(T_n)_{n\geq 1} \in \mathcal{B}(H)$ (Banach algebra of bounded linear operators on the Hilbert space H) is called weakly statistically convergent to $T \in \mathcal{B}(H)$ if $\langle T_n x, y\rangle$ statistically converges to $\langle Tx, y\rangle$ for any $x, y \in H$.

Here we will interested with the following question: if $(T_n)_{n\geq 1} \in \mathcal{B}(H)$ is a sequence of compact operators weakly statistically converging to the operator T, then under which additional conditions T is also compact?

Note that for the detail of this question for the usual weakly convergent sequences of compact operators, the reader can be consult in the paper by Karaev [7].

In this section, we will prove a positive result under some assumption on the so-called Berezin symbols of compact operators T_n. So, let us first introduce some necessary notations and preliminaries.

Recall that a Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H} = \mathcal{H}(\Omega)$ is the Hilbert space of complex-valued functions on some set Ω with nonempty boundary $\partial\Omega$ such that the evaluation functionals $f \to f(\lambda)$, $\lambda \in \Omega$, are continuous $\mathcal{H}$. Then by the classical Riesz theorem about representation of linear continuous functionals on the Hilbert space, for each $\lambda \in \Omega$ there exists a unique function $k_{\mathcal{H},\lambda} \in \mathcal{H}$ such that

$$\langle f, k_{\mathcal{H},\lambda} \rangle = f(\lambda)$$

for all $f \in \mathcal{H}$. The functions $k_{\mathcal{H},\lambda}(z)$, $\lambda \in \Omega$, are called the reproducing kernels of the space $\mathcal{H}$. It is well-known that (see Aronzajn [1] and Saitoh [12, 13]) the reproducing kernel $k_{\mathcal{H},\lambda}$ of $\mathcal{H}$ is represented by

$$k_{\mathcal{H},\lambda}(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $(e_n(z))_{n\geq 1}$ of the space (separable) $\mathcal{H}(\Omega)$.

For example, the classical Hardy, Bergman and Fock spaces are RKHSs.

Following Nordgren and Rosenthal [10], we say that RKHS $\mathcal{H}(\Omega)$ is standard if the underlying set Ω is a subset of a topological space and the boundary $\partial\Omega$ is non-empty and has the property that the normalized reproducing kernel $\left(\widehat{k}_{\lambda_n,\mathcal{H}}\right)_{n\geq 1} = \left(k_{\lambda_n,\mathcal{H}} / \|k_{\lambda_n,\mathcal{H}}\|_{\mathcal{H}}\right)_{n\geq 1}$ converges weakly to 0 whenever $(\lambda_n)_{n\geq 1} \in \Omega$ converges to any point in $\partial\Omega$.

Note that the most of RKHS, including Hardy, Bergman and Fock Hilbert spaces, are standard in this sense. (Also note that every finite dimensional Hilbert space is non-standard, because in the finite dimensional space the weak and strong convergence coincide.)

For any bounded operator $T : \mathcal{H} \to \mathcal{H}$ its Berezin symbol (see, Nordgren and Rosenthal [10] and Zhu [15]) is defined by

$$\widetilde{T}(\lambda) := \left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle, \ \lambda \in \Omega.$$

It is clear that $\left|\widetilde{T}(\lambda)\right| \leq \|T\|$ for all $\lambda \in \Omega$, and hence $\widetilde{T}$ is a bounded function on Ω. It is also easy to see that for every compact operator T on the standard RKHS $\mathcal{H}$ its Berezin symbol $\widetilde{T}$ vanish on the boundary.

The following result of Nordgren and Rosenthal [10, Corollary 2.8] characterizes compact operator on the standard RKHS $\mathcal{H}$, in terms of boundary behavior of Berezin symbols its unitary orbits $U^{-1}TU$, $U \in \mathcal{B}(\mathcal{H})$.

Lemma 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator on the standard RKHS $\mathcal{H} = \mathcal{H}(\Omega)$. Then T is compact if and only if for every unitary operator $U \in \mathcal{B}(\mathcal{H})$, $\widetilde{U^{-1}TU}(\lambda)$ tends to 0 whenever λ tends to the boundary points of $\partial\Omega$.*

The main result of this section is the following, which generalizes a result of the paper [7, Theorem 4.1].

Theorem 2.3. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a standard RKHS on some set Ω and $(T_n)_{n\geq 1}$ be a sequence of compact operators on $\mathcal{H}$ weakly statistically converging to an operator $T \in \mathcal{B}(\mathcal{H})$. If the double statistical limit*

$$st\text{-}\lim_{\substack{m\to\infty \\ n\to\infty}} \widetilde{U^{-1}T_nU}(\lambda_m) = l \tag{1}$$

exists for every unitary operator $U \in \mathcal{B}(\mathcal{H})$, then T is compact and $l = 0$ holds.

Proof. By condition, $T_n \to T$ $(n \to \infty)$ weakly statistically, which means that

$$st\text{-}\lim_{n\to\infty} \langle T_n f, g\rangle = \langle Tf, g\rangle$$

for all $f, g \in \mathcal{H}$. In particular, for $f = \widehat{k}_{\lambda_m,\mathcal{H}}$ and $g = \widehat{k}_{\lambda_m,\mathcal{H}}$, where $(\lambda_m)_{m\geq 1}$ is any sequence tending to a point in $\partial\Omega$, we have

$$st\text{-}\lim_{n\to\infty} \left\langle T_n\widehat{k}_{\lambda_m,\mathcal{H}}, \widehat{k}_{\lambda_m,\mathcal{H}}\right\rangle = \left\langle T\widehat{k}_{\lambda_m,\mathcal{H}}, \widehat{k}_{\lambda_m,\mathcal{H}}\right\rangle,$$

thus

$$st\text{-}\lim_{n\to\infty} \widetilde{T_n}(\lambda_m) = \widetilde{T}(\lambda_m)$$

for any $m \geq 1$. On the other hand, since

$$\begin{aligned}
\widetilde{U^{-1}T_nU}(\lambda_m) &= \left\langle U^{-1}T_nU\widehat{k}_{\lambda_m,\mathcal{H}}, \widehat{k}_{\lambda_m,\mathcal{H}}\right\rangle \\
&= \left\langle U^{*}T_nU\widehat{k}_{\lambda_m,\mathcal{H}}, \widehat{k}_{\lambda_m,\mathcal{H}}\right\rangle \\
&= \left\langle T_nU\widehat{k}_{\lambda_m,\mathcal{H}}, U\widehat{k}_{\lambda_m,\mathcal{H}}\right\rangle
\end{aligned}$$

and

$$st\text{-}\lim_{n\to\infty} \left\langle T_nU\widehat{k}_{\lambda_m,\mathcal{H}}, U\widehat{k}_{\lambda_m,\mathcal{H}}\right\rangle = \left\langle TU\widehat{k}_{\lambda_m,\mathcal{H}}, U\widehat{k}_{\lambda_m,\mathcal{H}}\right\rangle,$$

we obtain that

$$st\text{-}\lim_{n\to\infty} \widetilde{U^{-1}T_nU}(\lambda_m) = \left\langle U^{-1}T_nU(\lambda_m)\widehat{k}_{\lambda_m,\mathcal{H}}, \widehat{k}_{\lambda_m,\mathcal{H}}\right\rangle$$

for every unitary operator $U \in \mathcal{B}(\mathcal{H})$ and $m \geq 1$. Since T_n, $n \geq 1$, are compact operators and $\mathcal{H}$ is standard, by Lemma 2.2 we have $\lim_{m\to\infty} \widetilde{U^{-1}T_nU}(\lambda_m) = 0$, and hence $st\text{-}\lim_{m\to\infty} \widetilde{U^{-1}T_nU}(\lambda_m) = 0$ for any unitary operator $U \in \mathcal{B}(\mathcal{H})$.

Now by considering condition (1) of the theorem, and also the equality st-$\lim_{n\to\infty}\widetilde{U^{-1}T_nU}(\lambda_m)=\widetilde{U^{-1}TU}(\lambda_m)$, we obtain

$$\begin{aligned}0 &= st\text{-}\lim_{n\to\infty} st\text{-}\lim_{m\to\infty}\widetilde{U^{-1}T_nU}(\lambda_m) = st\text{-}\lim_{m\to\infty} st\text{-}\lim_{n\to\infty}\widetilde{U^{-1}T_nU}(\lambda_m)\\ &= st\text{-}\lim_{m\to\infty}\widetilde{U^{-1}TU}(\lambda_m).\end{aligned}$$

This shows that st-$\lim_{m\to\infty}\widetilde{U^{-1}TU}(\lambda_m)=l=0$ for any unitary operator $U\in\mathcal{B}(\mathcal{H})$. Then, by Corollary 1.3, there exists a subsequence $(\lambda_{m_k})_{k\geq 1}\in\Omega$ tending to a point in $\partial\Omega$ such that

$$\lim_{k\to\infty}\widetilde{U^{-1}TU}(\lambda_{m_k})=0$$

for any unitary operator $U\in\mathcal{B}(\mathcal{H})$. So, by Lemma 2.2 we assert that T is compact, which proves the theorem. □

3. Statistical convergence and Schatten-Neumann class operators

Recall that if T is a compact operator on a separable Hilbert space H, then there exists orthonormal sets $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ in H such that

$$Tx=\sum_{n=0}^{\infty}\lambda_n\langle x,u_n\rangle v_n,\ x\in H,$$

where λ_n is the n^{th} singular value (s-number) of T. Given $p\in(0,\infty)$, we define the Schatten-Neumann p-class of H, denoted by $\mathfrak{S}_p(H)$ or simply $\mathfrak{S}_p$, to be the space of compact operators T on H with its singular value sequence $(\lambda_n)_{n\geq 1}$ belonging to ℓ^p (the p-summable sequences space). We will only consider the case $1\leq p<+\infty$, since in this case $\mathfrak{S}_p$ is a Banach space with the norm $\|T\|_p:=\left(\sum_{n=1}^{\infty}|\lambda_n|^p\right)^{1/p}$. The class $\mathfrak{S}_1$ is also called the trace class of H (or nuclear operator class) and $\mathfrak{S}_2$ is called the Hilbert-Schmidt class. It is easy to show that if T is a compact operator on H and $p\geq 1$, then $T\in\mathfrak{S}_p$ if and only if $|T|^p:=(T^*T)^{p/2}\in\mathfrak{S}_1$ and $\|T\|_p^p=\||T|\|_p^p=\||T|^p\|_1$.

Our following result improves and generalizes a similar result of the paper [3] by using the concept of statistical convergence and by considering any RKHS instead of weighted Bergman space $L_a^2(dA_\alpha)$; see [3, Lemma 5.2].

Lemma 3.1. *Let $p\in[1,+\infty)$, $T\in\mathcal{B}(\mathcal{H})$ and $T_n\in\mathfrak{S}_p$ for all $n\geq 1$, where $\mathcal{H}=\mathcal{H}(\Omega)$ is a RKHS on some suitable set Ω. If T_n weakly statistically converges to T in $\mathcal{B}(\mathcal{H})$ and $\|T_n\|_p\leq C<+\infty$ for all $n\geq 1$ and for some constant $C>0$, then $T\in\mathfrak{S}_p$ and $\|T\|_p\leq C$.*

Proof. The proof essentially uses the similar arguments from [3], and we present it here only for the sake of completeness. So, for any $n\in\mathbb{N}$, define

$$\xi_n(K)=tr(T_nK).$$

Then we have $\xi_n \in \mathfrak{S}_q^*$, where $\frac{1}{p}+\frac{1}{q}=1$ and $\|\xi_n\| = \|T_n\|_p \leq C < +\infty$. By Banach-Alaoğlu's theorem, there exists a subsequence $\{\xi_{n_k}\}$ such that $\xi_{n_k} \to \xi$ in w^*-topology and $\xi \in \mathfrak{S}_q^*$. Therefore $tr\,(T_{n_k}K) = \xi_{n_k}(K) \to \xi(K)$, for all $K \in \mathfrak{S}_q$ and $|\xi(K)| \leq M\|K\|_q$ for some $M>0$. On the other hand, since T_n weakly statistically converges to T, we deduce that $st\text{-}tr\,(T_nK) \to tr\,(TK)$ for all operators K of finite rank. Thus, the lemma follows since

$$\|T\|_p = \sup\left\{|tr\,(TK)| : rank\,(K) < \infty \text{ and } \|K\|_q \leq 1\right\} < \infty.$$

□

Theorem 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$, and let $T = V|T|$ be its polar decomposition. If $T \in \mathfrak{S}_p$, then $V \in \mathfrak{S}_p$, if $1 \leq p < +\infty$.*

Proof. Put $T_n := T(|T|+\alpha_n)^{-1}$, where $\alpha_n > 0$, $n \geq 1$, and $st\text{-}\lim_{n\to\infty}\alpha_n = 0$. We will first prove that $st\text{-}\lim_{n\to\infty}\|T_nf - Vf\|_{\mathcal{H}} = 0$ for very $f \in \mathcal{H}$, that is T_n tends to T strongly statistically as $n \to \infty$. For this purpose, let (E_λ) be the spectral family for $|T|$. Then, by considering that $st\text{-}\lim_{n\to\infty}\alpha_n = 0$, we have that

$$st - \lim_{n\to\infty}\|T_nf - (I-E_0)f\|_{\mathcal{H}} = 0,\ \forall f \in \mathcal{H}.$$

In fact, notice that $|T| = \int_0^\infty \lambda dE_\lambda$ is the spectral decomposition of $|T|$. We set $A_n := |T|(|T|+\alpha_n)^{-1}$. Then it is clear that

$$A_nE_0f = (|T|+\alpha_n)^{-1}|T|E_0f = 0$$

for any $f \in \mathcal{H}$, and hence

$$\begin{aligned}\|A_nf-(I-E_0)f\|_{\mathcal{H}}^2 &= \|(A_n-I)(I-E_0)f\|_{\mathcal{H}}^2\\ &= \int_0^\infty\left|\frac{\lambda}{\lambda+\alpha_n}-1\right|^2 d\|E_\lambda(I-E_0)f\|^2\\ &= \int_0^\infty\left|\frac{\alpha_n}{\lambda+\alpha_n}\right|^2 d\|E_\lambda(I-E_0)f\|^2.\end{aligned}$$

Since $st\text{-}\lim_{n\to\infty}\alpha_n = 0$, by Corollary 1.3 there exists a subsequence $(\beta_n)_{n\geq1}$ of $(\alpha_n)_{n\geq1}$ such that $\lim_{n\to\infty}\beta_n = 0$. So, from Lebesgue's dominated convergence theorem, Lemma 1.2 and Corollary 1.3, it follows that A_n strongly statistically converges to $I-E_0$ as $n\to\infty$. Thus, we obtain that $T_n \to V(I-E_0)$ strongly statistically as $n\to\infty$. Since E_0 is the projection onto the eigenspace $\{f\in\mathcal{H} : Tf = 0\}$, we have $VE_0 = 0$. Consequently, $T_n \to V$ strongly statistically as $n\to\infty$.

Now suppose $T \in \mathfrak{S}_p$. Then $T_n \in \mathfrak{S}_p$, $\|T_n\|_p \leq C < +\infty$ for some $C>0$ and $T_n \to V$ strongly statistically as $n\to\infty$. By applying Lemma 3.1, we deduce $V \in \mathfrak{S}_p$. The theorem is proved. □

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