Dynamic Systems and Applications 24 (2015) 327-340

GENERAL STABILITY IN MEMORY-TYPE THERMOELASTICITY WITH SECOND SOUND

MUHAMMAD I. MUSTAFA

Mathematics Department, University of Sharjah, P. O. Box 2727
Sharjah, United Arab Emirates

ABSTRACT. In this paper we consider an n-dimentional thermoelastic system of second sound with viscoelastic damping. We establish an explicit and general decay rate result without imposing restrictive assumptions on the behavior of the relaxation function at infinity. Our result allows a larger class of ralxation functions and generalizes previous results existing in the literature.

Keywords and phrases: thermoelasticity with second sound, viscoelastic damping, general decay, convexity

AMS Classification: 35B37, 35L55, 74D05, 93D15, 93D20

1. INTRODUCTION

In this paper we are concerned with the following problem

$$
(1.1)\quad \begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \int_0^t g(t-s)\Delta u(s)ds + \beta\nabla\theta = 0, & \text{in } \Omega\times(0,\infty) \\ b\theta_t + k\operatorname{div} q + \beta\operatorname{div} u_t = 0, & \text{in } \Omega\times(0,\infty) \\ \tau q_t + q + k\nabla\theta = 0, & \text{in } \Omega\times(0,\infty) \\ u(x,t) = \theta(x,t) = 0, & \text{on } \partial\Omega\times(0,\infty) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x),\ q(x,0) = q_0(x), & x\in\Omega, \end{cases}
$$

a memory-type thermoelastic system of second sound associated with homogeneous Dirichlet boundary conditions and initial data in suitable function spaces. Here Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ with a smooth boundary $\partial\Omega$, $u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, $\theta = \theta(x,t)$ is the difference temperature, $q = q(x,t)$ is the heat flux vector, and the relaxation function g is a positive nonincreasing function. The coefficients $b, k, \beta, \mu, \lambda, \tau$ are positive constants, where τ is the thermal relaxation time and μ, λ are Lame moduli. In this work, we study the decay properties of the solutions of (1.1) for functions g of general-type decay.

In classical thermoelasticity, the heat conduction is governed by the Fourier's law, which means that the heat flux is proportional to the gradient of temperature. This theory predicts an infinite speed of heat propagation; that is any thermal disturbance

Received April 2, 2015

at one point has an instantaneous effect elsewhere in the body. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances, which are almost entirely thermal, propagate in a finite speed. Also, for some applications like working with very short laser pulses in laser cleaning of computer chips, it is worthwhile thinking of another model removing this paradox, but still keeping the essentials of a heat conduction process. To overcome this physical paradox, many theories have merged. One of which, called thermoelasticity with second sound, suggests the replacement of Fourier's law by so called Cattaneo's law. The third equation of system (1.1) represents Cattaneo's law of heat conduction modeling thermal disturbances as wave-like pulses traveling at finite speed. Here, if $\tau = 0$, we obtain a classical thermoelastic system. For a discussion of the model of thermoelasticity with second sound, we refer to [2, 3, 7, 19], and for classical thermoelasticity, we refer to books by Jiang and Racke [6] and Zheng [21].

Results concerning existence and asymptotic behavior of smooth as well as weak solutions in thermoelasticity with second sound have been established by many mathematicians. Tarabek [20] treated problems related to

$$\left\{\begin{array}{l} u_{tt} - a(u_x, \theta, q)u_{xx} + b(u_x, \theta, q)\theta_x = \alpha_1(u_x, \theta)qq_x \\ \theta_t + g(u_x, \theta, q)q_x + d(u_x, \theta, q)u_{tx} = \alpha_2(u_x, \theta)qq_t \\ \tau(u_x, \theta)q_t + q + k(u_x, \theta)\theta_x = 0 \end{array}\right. \tag{1.2}$$

in both bounded and unbounded situations and established global existence results for small initial data. He also showed that the classical solutions tend to equilibrium as t tends to infinity; however, no rate of decay has been discussed. In his work, Tarabek used the usual energy argument and exploited some relations from the second law of thermodynamics to overcome the difficulty arising from the lack of Poincare's inequality in the unbounded domains.

Racke [16] discussed lately (1.2) and established uniform decay results for several linear and nonlinear initial boundary value problems. In particular, he studied (1.2), with $\alpha_1 = \alpha_2 = 0$, for a rigidly clamped medium with temperature hold constant on the boundary, and showed that, for small enough initial data, classical solutions decay exponentially to the equilibrium state. Messaoudi and Said-Houari [13] extended the decay result of [16] to the case when $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Irmscher and Racke [5] obtained explicit sharp exponential decay rates for solutions of the system of classical thermoelasticity as well as for that of thermoelasticity with second sound in one dimension and compared the results of both models with respect to the asymptotic behavior of solutions. Recently, Qin *et al.* [15] considered a one-dimensional nonlinear system of thermoelasticity with thermal memory and second sound and proved global existence and exponential decay of solution provided that the initial data are close to equilibrium and the relaxation function decays exponentially. Also, Racke and Wang

[18] considered a nonlinear one-dimensional Cauchy problem of thermoelasticity with second sound, discussed the well-posedness and described the long-time behavior of the global small solutions, obtaining a polynomial decay rate.

In the multi-dimensional case the situation is much different where the dissipation given by heat conduction is not in general strong enough to produce uniform rate of decay to the solution as in the one-dimensional case. The exponential rate of decay in two or three dimensional space was obtained by Racke [17] under the conditions $rotu = rotq = 0$. This applies automatically to the radially symmetric solution, since it is only a special case. Messaoudi [8] investigated (1.1), in the absence of the viscoelastic term and the presence of a source term in the first equation, and proved a local existence, as well as, a blow up result for solutions with negative initial energy. This result was later extended to certain solutions with positive energy by Messaoudi and Said-Houari [12].

Although the dissipative effects of heat conduction induced by Cattaneo's law are usually weaker than those induced by Fourier's law, but the results mentioned above nourish the expectation that always both models lead to exponential stability (or both do not). In general, this is not true, where it was shown by Fernández Sare and Racke [4] that, for a Timoshenko system, the coupling via Cattaneo's law causes loss of the exponential decay usually obtained in the case of coupling via Fourier's law [14]. They also proved that this surprising result holds even for Timoshenko systems with history. In the presence of an extra frictional damping, Messaoudi *et al.* [11] established exponential decay results for several linear and nonlinear Timoshenko systems of thermoelasticity with second sound.

Regarding viscoelastic damping, we mention the works of Messaoudi and Al-Shehri who treated an n-dimensional classical thermoelasticity in [9] and thermoelasticity with second sound in [10] both subject to boundary conditions of memory type. If g is the relaxation function and f is the resolvent kernel of $\frac{-g'}{g(0)}$, they showed in [9] that the energy decays at the same rate as of $(-f')$, while, in [10], when $(-f')$ decays exponentially, the energy decays at a polynomial rate.

Our aim in this work is to investigate (1.1) and obtain a general relation between the decay rate for the energy and that of the relaxation function g without imposing restrictive assumptions on the behavior of g at infinity. In this paper, we provide an explicit energy decay formula that allows a wider class of functions g which are not necessarily of exponential or polynomial-type decay. The proof is based on the multiplier method and makes use of some properties of convex functions including the use of the general Young's inequality and Jensen's inequality. The paper is organized as follows. In section 2, we present some notation and material needed for our work. Some technical lemmas and the proof of our main result will be given in section 3.

2. PRELIMINARIES

In the sequel we consider (u, θ, q) to be a solution of system (1.1) with the regularity needed to justify the calculations in this paper. Throughout this paper, c is used to denote a generic positive constant. We also consider the following assumption

(A) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0, \qquad \mu - \int_0^{+\infty} g(s)ds = l > 0 \tag{2.1}$$

and there exists a positive function $H \in C^1(\mathbb{R}_+)$, with $H(0) = 0$, and H is linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r < 1$, such that

$$g'(t) \leq -H(g(t)), \quad \forall t > 0.$$

Now, we introduce the first and second order energy functionals

$$\begin{aligned} E_1(t) := &\frac{1}{2} \int_\Omega \left(|u_t|^2 + (\mu - \int_0^t g(s)ds) \left|\nabla u\right|^2 + (\mu + \lambda)(\operatorname{div} u)^2 + b\theta^2 + \tau \left|q\right|^2 \right) dx \\ &+ \frac{1}{2}(g \circ \nabla u)(t) \end{aligned}$$

and

$$\begin{aligned} E_2(t) := &\frac{1}{2} \int_\Omega \left(|u_{tt}|^2 + (\mu - \int_0^t g(s)ds) \left|\nabla u_t\right|^2 + (\mu + \lambda)(\operatorname{div} u_t)^2 + b\theta_t^2 + \tau \left|q_t\right|^2 \right) dx \\ &+ \frac{1}{2}(g \circ \nabla u_t)(t) \end{aligned}$$

where $|\nabla u|^2 = \sum_{i=1}^n |\nabla u_i|^2$ and

$$(g \circ v)(t) = \int_\Omega \int_0^t g(t-s) \left|v(t) - v(s)\right|^2 ds\, dx.$$

By multiplying the first equation in (1.1) by u_t, the second equation by θ, and the third equation by q, adding the resulting equations, and integrating over Ω, we obtain

$$E_1'(t) = -\int_\Omega |q|^2 dx + \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2} g(t) \int_\Omega |\nabla u|^2 dx. \tag{2.2}$$

Then, hypothesis (A) implies that E_1 is a nonincreasing function of t. Differentiating the first three equations in (1.1) with respect to t and assuming throughout the paper that $u_0 \equiv 0$, we get by similar calculations

$$E_2'(t) \leq -\int_\Omega |q_t|^2 dx \leq 0. \tag{2.3}$$

Our main stability result is the following

Theorem 2.1. *Assume that (A) holds. Then there exist positive constants C, K, t_1 and ε_0 such that*

(I) In the special case $H(t) = ct^p$, where $1 \le p < \frac{3}{2}$, the solution of (1.1) *satisfies*

$$E_1(t) \le \frac{C}{t^{\frac{1}{2p-1}}} \qquad \forall t \ge t_1.$$

(II) In the general case, the solution of (1.1) *satisfies*

$$E_1(t) \le H_1^{-1}\left(\frac{K}{t}\right) \qquad \forall t \ge t_1, \tag{2.4}$$

where

$$H_1(t) = tH_0'(\varepsilon_0 t) \qquad \text{and} \qquad H_0(t) = H(D(t))$$

provided that D is a positive C^1 function, with $D(0) = 0$, for which H_0 is strictly increasing and strictly convex C^2 function on $(0, r]$ and

$$\int_0^{+\infty} \frac{g(s)}{H_0^{-1}(-g'(s))} ds < +\infty. \tag{2.5}$$

Remarks.

1. Theorem 2.1 ensures

$$\lim_{t\to+\infty} E_1(t) = 0$$

with an explicit formula for the decay rate of the energy. Our result is obtained under very general hypotheses on the relaxation function g that allow to deal with a much wider class of functions g.

2. The usual case of treating a relaxation function g that satisfies (2.1) and $g' \le -kg^p$, $1 \le p < 3/2$, is a special case of our more general result. We will provide a proof for this special case.

3. The condition $g' \le -kg^p$, $1 \le p < 3/2$ assumes $g(t) \le \omega e^{-kt}$ when $p = 1$ and $g(t) \le \frac{\omega}{t^{\frac{1}{p-1}}}$ when $1 < p < 3/2$. Our result allows relaxation functions which are not necessarily of exponential or polynomial decay. For instance, if

$$g(t) = a\exp(-\sqrt{t})$$

and a is chosen so that g satisfies (2.1), then $g'(t) = -H(g(t))$ where

$$H(t) = \frac{t}{2\ln(a/t)}.$$

Since

$$H'(t) = \frac{\ln\left(\frac{a}{t}\right) + 1}{2\left[\ln\left(\frac{a}{t}\right)\right]^2} \qquad \text{and} \qquad H''(t) = \frac{\ln\left(\frac{a}{t}\right) + 2}{2t\left[\ln\left(\frac{a}{t}\right)\right]^3},$$

then the function H satisfies hypothesis (A) on the interval $(0, r]$ for any $0 < r < a$. Also, by taking $D(t) = t^\alpha$, (2.5) is satisfied for any $\alpha > 1$. Therefore, an explicit rate of decay can be obtained by Theorem 2.1. The function $H_0(t) = H(t^\alpha)$ has derivative

$$H_0'(t) = \frac{\alpha t^{\alpha-1}\left[1 + \ln\left(\frac{a}{t^\alpha}\right)\right]}{2\left[\ln\left(\frac{a}{t^\alpha}\right)\right]^2}$$

Then, we do some direct calculations and use (2.4) to deduce that $E_1(t) \leq C_\alpha / t^{\frac{1}{2\alpha}}$, for any $\alpha > 1$. Therefore, taking $\alpha \to 1$, the energy decays at the following rate

$$E_1(t) \leq \frac{C}{t^{\frac{1}{2}}}.$$

4. The well-known Jensen's inequality will be of essential use in establishing our main result. If F is a convex function on $[a, b]$, $f : \Omega \to [a, b]$ and h are integrable functions on Ω, $h(x) \geq 0$, and $\int_\Omega h(x)dx = k > 0$, then Jensen's inequality states that

$$F\left[\frac{1}{k}\int_\Omega f(x)h(x)dx\right] \leq \frac{1}{k}\int_\Omega F[f(x)]h(x)dx.$$

5. By (A), we easily deduce that $\lim_{t\to+\infty} g(t) = 0$. This implies that $\lim\limits_{t\to+\infty}(-g'(t))$ cannot be equal to a positive number, and so it is natural to assume that $\lim\limits_{t\to+\infty}(-g'(t)) = 0$. Hence, there is $t_1 > 0$ large enough such that $g(t_1) > 0$ and

$$\max\{g(t), -g'(t)\} < \min\{r, H(r), H_0(r)\}, \qquad \forall\, t \geq t_1. \tag{2.6}$$

As g is nonincreasing, $g(0) > 0$ and $g(t_1) > 0$, then $g(t) > 0$ for any $t \in [0, t_1]$ and

$$0 < g(t_1) \leq g(t) \leq g(0), \qquad \forall\, t \in [0, t_1].$$

Therefore, since H is a positive continuous function, then

$$a \leq H(g(t)) \leq b, \qquad \forall\, t \in [0, t_1]$$

for some positive constants a and b. Consequently, for all $t \in [0, t_1]$,

$$g'(t) \leq -H(g(t)) \leq -a = -\frac{a}{g(0)}g(0) \leq -\frac{a}{g(0)}g(t)$$

which gives, for some positive constant d,

$$g'(t) \leq -dg(t), \qquad \forall\, t \in [0, t_1]. \tag{2.7}$$

3. PROOF OF THE MAIN RESULT

In this section we prove Theorem 2.1. For this purpose, we establish several lemmas.

Lemma 3.1. *Under the assumption (A), the functional*

$$K_1(t) := \int_\Omega u \cdot u_t dx$$

satisfies, along the solution of (1.1), *the estimate*

$$\begin{aligned} K_1'(t) \leq& -\frac{l}{2}\int_\Omega |\nabla u|^2 dx - (\mu+\lambda)\int_\Omega (\operatorname{div} u)^2 dx \\ &+ \int_\Omega |u_t|^2 dx + c\int_\Omega |\nabla\theta|^2 dx + c(g \circ \nabla u)(t). \end{aligned} \tag{3.1}$$

Proof. Direct computations, using (1.1) and (2.1), yield

$$\begin{aligned} K_1'(t) &= \int_\Omega \left(|u_t|^2 + \mu u \cdot \Delta u + (\mu+\lambda) u \cdot \nabla(\operatorname{div} u) - \beta u \cdot \nabla\theta\right) dx \\ &\quad - \int_\Omega \int_0^t g(t-s) u(t) \cdot \Delta u(s) ds\, dx \\ &\le \int_\Omega \left(|u_t|^2 - l\,|\nabla u|^2 - (\mu+\lambda)(\operatorname{div} u)^2 - \beta u \cdot \nabla\theta\right) dx \\ &\quad \int_\Omega \int_0^t g(t-s) \nabla u(t) \cdot (\nabla u(s) - \nabla u(t)) ds\, dx. \end{aligned}$$

By Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} K_1'(t) &\le \int_\Omega \left(|u_t|^2 - l\,|\nabla u|^2 - (\mu+\lambda)(\operatorname{div} u)^2\right) dx + \delta \int_\Omega |u|^2\, dx + \frac{\beta^2}{4\delta} \int_\Omega |\nabla\theta|^2\, dx \\ &\quad + \delta \left(\int_0^t g(s) ds\right) \int_\Omega |\nabla u|^2\, dx + \frac{1}{4\delta} \int_\Omega \int_0^t g(t-s)\, |\nabla u(s) - \nabla u(t)|^2\, ds\, dx \\ &\le \int_\Omega \left(|u_t|^2 - l\,|\nabla u|^2 - (\mu+\lambda)(\operatorname{div} u)^2\right) dx + \delta c \int_\Omega |\nabla u|^2\, dx + \frac{\beta^2}{4\delta} \int_\Omega |\nabla\theta|^2\, dx \\ &\quad + \delta\mu \int_\Omega |\nabla u|^2\, dx + \frac{1}{4\delta} (g \circ \nabla u)(t) \end{aligned}$$

which, by choosing δ small enough, gives (3.1). □

Lemma 3.2. *Under the assumption (A), the functional*

$$K_2(t) := -\int_\Omega u_t(t) \cdot \int_0^t g(t-s)(u(t) - u(s)) ds\, dx$$

satisfies for any $0 < \delta < 1$*, along the solution of* (1.1)*, the estimate*

$$\begin{aligned} K_2'(t) &\le -\left(\int_0^t g(s) ds - \delta\right) \int_\Omega |u_t|^2\, dx + \delta \int_\Omega |\nabla u|^2\, dx \\ &\quad + \frac{c}{\delta} (g \circ \nabla u)(t) - \frac{c}{\delta} (g' \circ \nabla u)(t) + c \int_\Omega |\nabla\theta|^2\, dx. \end{aligned} \tag{3.2}$$

Proof. By exploiting equations (1.1) and integrating by parts, we have

$$\begin{aligned} K_2'(t) &= \mu \int_\Omega \int_0^t g(t-s) \nabla u(t) \cdot (\nabla u(t) - \nabla u(s)) ds\, dx \\ &\quad + (\mu+\lambda) \int_\Omega \int_0^t g(t-s)(div[u(t)])(div[u(t) - u(s)]) ds\, dx \\ &\quad + \beta \int_\Omega \int_0^t g(t-s) \nabla\theta(t) \cdot (u(t) - u(s)) ds\, dx \\ &\quad - \int_\Omega \left(\int_0^t g(t-s) \nabla u(s) ds\right) \cdot \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) ds\right) dx \\ &\quad - \int_\Omega \int_0^t g'(t-s) u_t(t) \cdot (u(t) - u(s)) ds\, dx - \left(\int_0^t g(s) ds\right) \int_\Omega |u_t|^2\, dx. \end{aligned} \tag{3.3}$$

Using Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}
&\mu \int_\Omega \int_0^t g(t-s)\nabla u(t)\cdot(\nabla u(t)-\nabla u(s))ds\,dx \\
&- \int_\Omega \left(\int_0^t g(t-s)\nabla u(s)ds\right)\cdot\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx \\
&= \left(\mu - \int_0^t g(s)ds\right)\int_\Omega \int_0^t g(t-s)\nabla u(t)\cdot(\nabla u(t)-\nabla u(s))ds\,dx \\
&+ \int_\Omega \left|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right|^2 dx \\
&\le \frac{\delta}{2}\int_\Omega |\nabla u|^2\,dx + \frac{c}{\delta}(g\circ\nabla u)(t). \qquad (3.4)
\end{aligned}$$

Also, the use of Young's and Poincaré's inequalities gives

$$\begin{aligned}
&+ (\mu+\lambda)\int_\Omega \int_0^t g(t-s)(\operatorname{div}[u(t)])(\operatorname{div}[u(t)-u(s)])ds\,dx \\
&+ \beta \int_\Omega \int_0^t g(t-s)\nabla\theta(t)\cdot(u(t)-u(s))ds\,dx \\
&- \int_\Omega \int_0^t g'(t-s)u_t(t)\cdot(u(t)-u(s))ds\,dx \\
&\le \frac{\delta}{2}\int_\Omega |\nabla u|^2\,dx + \delta\int_\Omega |u_t|^2\,dx + \frac{c}{\delta}(g\circ\nabla u)(t) \\
&- \frac{c}{\delta}(g'\circ\nabla u)(t) + c\int_\Omega |\nabla\theta|^2\,dx. \qquad (3.5)
\end{aligned}$$

Combining (3.3)–(3.5), (3.2) is established. □

Proof of Theorem 2.1. Taking $E(t) = E_1(t) + E_2(t)$ and $N_1, N_2 > 0$, we define

$$\mathcal{L}(t) := N_1 E(t) + K_1(t) + N_2 K_2(t)$$

and let $g_1 = \int_0^{t_1} g(s)ds > 0$, where t_1 was introduced in (2.6). By combining (2.2), (2.3), (3.1), (3.2) and taking $\delta = l/(4N_2)$, we obtain, for all $t \ge t_1$,

$$\begin{aligned}
\mathcal{L}'(t) \le\, & -N_1\int_\Omega |q|^2\,dx - N_1\int_\Omega |q_t|^2\,dx - \frac{l}{4}\int_\Omega |\nabla u|^2\,dx \\
& - (N_2 g_1 - \frac{l}{4} - 1)\int_\Omega |u_t|^2\,dx - (\mu+\lambda)\int_\Omega (\operatorname{div} u)^2 dx \\
& + (c + cN_2)\int_\Omega |\nabla\theta|^2\,dx \\
& + \left(\frac{4c}{l}N_2^2 + c\right)(g\circ\nabla u)(t) \\
& + \left(\frac{1}{2}N_1 - \frac{4c}{l}N_2^2\right)(g'\circ\nabla u)(t). \qquad (3.6)
\end{aligned}$$

From the third equation of (1.1), we conclude that

$$\int_\Omega |\nabla\theta|^2\,dx \le \frac{2\tau^2}{k^2}\int_\Omega |q_t|^2\,dx + \frac{2}{k^2}\int_\Omega |q|^2\,dx$$

which we use in (3.6) to get

$$\begin{aligned}\mathcal{L}'(t) \le &-\left(N_1 - \frac{2}{k^2}[c+cN_2+1]\right)\int_\Omega |q|^2\,dx - \left(N_1 - \frac{2\tau^2}{k^2}[c+cN_2+1]\right)\int_\Omega |q_t|^2\,dx\\ &-\frac{l}{4}\int_\Omega |\nabla u|^2\,dx - \left(N_2 g_1 - \frac{l}{4} - 1\right)\int_\Omega |u_t|^2\,dx - (\mu+\lambda)\int_\Omega(\operatorname{div} u)^2 dx\\ &-\int_\Omega |\nabla\theta|^2\,dx + \left(\frac{4c}{l}N_2^2 + c\right)(g\circ\nabla u)(t) + \left(\frac{1}{2}N_1 - \frac{4c}{l}N_2^2\right)(g'\circ\nabla u)(t).\end{aligned}$$

At this point, we choose N_2 large enough so that

$$\gamma_1 := N_2 g_1 - \frac{l}{4} - 1 > 0,$$

then N_1 large enough so that

$$\gamma_2 := N_1 - \frac{2}{k^2}[c+cN_2+1] > 0,$$

and

$$N_1 - \frac{2\tau^2}{k^2}[c+cN_2+1] > 0, \qquad \frac{1}{2}N_1 - \frac{4c}{l}N_2^2 > 0.$$

So, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \le &-\int_\Omega\left[\gamma_2 |q|^2 + \frac{l}{4}|\nabla u|^2\,dx + \gamma_1 |u_t|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + \int_\Omega |\nabla\theta|^2\,dx\right]dx\\ &+ c(g\circ\nabla u)(t)\end{aligned}$$

which, using Poincaré's inequality, yields

$$\mathcal{L}'(t) \le -mE_1(t) + c(g\circ\nabla u)(t), \qquad \forall\ t \ge t_1. \tag{3.7}$$

On the other hand, we find that

$$\begin{aligned}|\mathcal{L}(t) - N_1E(t)| &\le |K_1(t)| + N_2\,|K_2(t)|\\ &\le \int_\Omega |u\cdot u_t|\,dx + N_2\int_\Omega\left|u_t(t)\cdot\int_0^t g(t-s)(u(t)-u(s))ds\right|dx\\ &\le \frac{1}{2}\int_\Omega |u|^2\,dx + \frac{1+N_2}{2}\int_\Omega |u_t|^2\,dx + \frac{N_2}{2}\int_\Omega\left|\int_0^t g(t-s)(u(t)-u(s))ds\right|^2 dx\\ &\le c\left[\int_\Omega |\nabla u|^2\,dx + \int_\Omega |u_t|^2\,dx + (g\circ\nabla u)(t)\right]\\ &\le cE_1(t).\end{aligned}$$

Therefore, we can choose N_1 even larger (if needed) so that

$$\mathcal{L}(t) \sim E(t). \tag{3.8}$$

Now, we use (2.2) and (2.7) to conclude that, for any $t \geq t_1$,

$$\begin{aligned}\int_0^{t_1} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds &\leq -\frac{1}{d} \int_0^{t_1} g'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \\ &\leq -cE_1'(t). \end{aligned} \tag{3.9}$$

Next, we take $F(t) = \mathcal{L}(t) + cE_1(t)$, which is clearly equivalent to $E(t)$, and use (3.7) and (3.9), to get, for all $t \geq t_1$,

$$F'(t) \leq -mE_1(t) + c \int_{t_1}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds. \tag{3.10}$$

(I) $H(t) = ct^p$ and $1 \leq p < \frac{3}{2}$:

If $1 < p < \frac{3}{2}$, then one can easily show that $\int_0^{+\infty} g^{1-\delta_0}(s) ds < +\infty$ for any $\delta_0 < 2-p$. Using this fact and (2.2) and choosing t_1 even larger if needed, we deduce that, for all $t \geq t_1$,

$$\begin{aligned}\eta(t) := &\int_{t_1}^{t} g^{1-\delta_0}(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \\ &\leq 2 \int_{t_1}^{t} g^{1-\delta_0}(s) \int_0^1 (|\nabla u(t)|^2 + |\nabla u(t-s)|^2) dx \, ds \\ &\leq cE_1(0) \int_{t_1}^{t} g^{1-\delta_0}(s) ds < 1. \end{aligned} \tag{3.11}$$

Then, Jensen's inequality, (2.2), hypothesis (A), and (3.11) lead to

$$\int_{t_1}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds = \int_{t_1}^{t} g^{\delta_0}(s) g^{1-\delta_0}(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds$$

$$\begin{aligned} &= \int_{t_1}^{t} g^{(p-1+\delta_0)(\frac{\delta_0}{p-1+\delta_0})}(s) g^{1-\delta_0}(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \\ &\leq \eta(t) \left[\frac{1}{\eta(t)} \int_{t_1}^{t} g(s)^{(p-1+\delta_0)} g^{1-\delta_0}(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \right]^{\frac{\delta_0}{p-1+\delta_0}} \\ &\leq \left[\int_{t_1}^{t} g(s)^p \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \right]^{\frac{\delta_0}{p-1+\delta_0}} \\ &\leq c \left[\int_{t_1}^{t} -g'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \right]^{\frac{\delta_0}{p-1+\delta_0}} \leq c \left[-E_1'(t) \right]^{\frac{\delta_0}{p-1+\delta_0}} . \end{aligned}$$

Then, particularly for $\delta_0 = \frac{1}{2}$, we find that (3.10) becomes

$$F'(t) \leq -mE_1(t) + c \left[-E_1'(t) \right]^{\frac{1}{2p-1}} .$$

Now, we multiply by $E_1^{2p-2}(t)$ to get, using the fact that $E_1'(t) \leq 0$,

$$(FE_1^{2p-2})'(t) \leq F'(t) E_1^{2p-2}(t) \leq -mE_1^{2p-1}(t) + cE_1^{2p-2}(t) \left[-E_1'(t) \right]^{\frac{1}{2p-1}} .$$

Then, Young's inequality, with $\sigma = 2p-1$ and $\sigma' = \frac{2p-1}{2p-2}$, gives

$$(FE_1^{2p-2})'(t) \le -mE_1^{2p-1}(t) + \varepsilon E_1^{2p-1}(t) + C_\varepsilon(-E_1'(t)).$$

Consequently, picking $\varepsilon < m$, we obtain

$$F_0'(t) \le -m'E_1^{2p-1}(t)$$

where $F_0 = FE_1^{2p-2} + C_\varepsilon E_1$ and m' is some positive constant. Also, it is easy to show that this inequality is true for $p = 1$. Once again, we use the fact that $E_1'(t) \le 0$ to deduce that

$$(tE_1^{2p-1})'(t) \le E_1^{2p-1}(t) \le -\frac{1}{m'}F_0'(t).$$

A simple integration over (t_1, t) yields

$$tE_1^{2p-1} \le \frac{1}{m'}F_0(t_1) + t_1E_1^{2p-1}(t_1).$$

This gives, for all $t \ge t_1$,

$$E_1(t) \le \frac{C}{t^{\frac{1}{2p-1}}}. \tag{3.12}$$

(II) The general case: We define $I(t)$ by

$$I(t) := \int_{t_1}^{t} \frac{g(s)}{H_0^{-1}(-g'(s))} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds$$

where H_0 is such that (2.5) is satisfied. As in (3.11), we find that $I(t)$ satisfies, for all $t \ge t_1$,

$$I(t) < 1. \tag{3.13}$$

We also assume, without loss of generality that $I(t) \ge \beta > 0$, for all $t \ge t_1$; otherwise (3.10) yields an exponential decay. In addition, we define $\xi(t)$ by

$$\xi(t) := -\int_{t_1}^{t} g'(s)\frac{g(s)}{H_0^{-1}(-g'(s))} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds$$

and infer from (A) and the properties of H_0 and D that

$$\frac{g(s)}{H_0^{-1}(-g'(s))} \le \frac{g(s)}{H_0^{-1}(H(g(s)))} = \frac{g(s)}{D^{-1}(g(s))} \le k_0$$

for some positive constant k_0. Then, using (2.2) and choosing t_1 even larger (if needed), one can easily see that $\xi(t)$ satisfies, for all $t \ge t_1$,

$$\begin{aligned} \xi(t) &\le -k_0 \int_{t_1}^{t} g'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \\ &\le -cE_1(0) \int_{t_1}^{t} g'(s) \le cg(t_1)E_1(0) \\ &< \min\{r, H(r), H_0(r)\}. \end{aligned} \tag{3.14}$$

Since H_0 is strictly convex on $(0, r]$ and $H_0(0) = 0$, then

$$H_0(\theta x) \le \theta H_0(x)$$

provided $0 \le \theta \le 1$ and $x \in (0, r]$. The use of this fact, hypothesis (A), (2.6), (3.13), (3.14), and Jensen's inequality leads to

$$\begin{aligned}
\xi(t) &= \frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0[H_0^{-1}(-g'(s))] \frac{g(s)}{H_0^{-1}(-g'(s))} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \\
&\ge \frac{1}{I(t)} \int_{t_1}^{t} H_0[I(t) H_0^{-1}(-g'(s))] \frac{g(s)}{H_0^{-1}(-g'(s))} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \\
&\ge H_0 \left(\frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0^{-1}(-g'(s)) \frac{g(s)}{H_0^{-1}(-g'(s))} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \right) \\
&= H_0 \left(\int_{t_1}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \right)
\end{aligned}$$

This implies that

$$\int_{t_1}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \le H_0^{-1}(\xi(t))$$

and (3.10) becomes

$$F'(t) \le -mE_1(t) + cH_0^{-1}(\xi(t)), \qquad \forall \, t \ge t_1. \tag{3.15}$$

Now, for $\varepsilon_0 < r$ and $c_0 > 0$, using (3.15), and the fact that $E_1' \le 0$, $H_0' > 0, H_0'' > 0$ on $(0, r]$, we find that the functional F_1, defined by

$$F_1(t) := H_0'(\varepsilon_0 \frac{E_1(t)}{E_1(0)}) F(t) + c_0 E_1(t) \tag{3.16}$$

satisfies

$$\begin{aligned}
F_1'(t) &= \varepsilon_0 \frac{E_1'(t)}{E_1(0)} H_0'' \left(\varepsilon_0 \frac{E_1(t)}{E_1(0)} \right) F(t) + H_0' \left(\varepsilon_0 \frac{E_1(t)}{E_1(0)} \right) F'(t) + c_0 E_1'(t) \\
&\le -mE_1(t) H_0' \left(\varepsilon_0 \frac{E_1(t)}{E_1(0)} \right) + cH_0' \left(\varepsilon_0 \frac{E_1(t)}{E_1(0)} \right) H_0^{-1}(\xi(t)) + c_0 E_1'(t).
\end{aligned} \tag{3.17}$$

Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [1] p. 61-64), then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \qquad \text{if } s \in (0, H_0'(r)] \tag{3.18}$$

and H_0^* satisfies the following Young's inequality

$$AB \le H_0^*(A) + H_0(B), \qquad \text{if } A \in (0, H_0'(r)], B \in (0, r] \tag{3.19}$$

With $A = H_0' \left(\varepsilon_0 \frac{E_1(t)}{E_1(0)} \right)$ and $B = H_0^{-1}(\xi(t))$, using (2.2), (3.14) and (3.17)–(3.19), we arrive at

$$\begin{aligned}
F_1'(t) &\le -mE_1(t) H_0' \left(\varepsilon_0 \frac{E_1(t)}{E_1(0)} \right) + cH_1^* \left(H_0' \left(\varepsilon_0 \frac{E_1(t)}{E_1(0)} \right) \right) + c\xi(t) + c_0 E_1'(t) \\
&\le -mE_1(t) H_0' \left(\varepsilon_0 \frac{E_1(t)}{E_1(0)} \right) + c\varepsilon_0 \frac{E_1(t)}{E_1(0)} H_0' \left(\varepsilon_0 \frac{E_1(t)}{E_1(0)} \right) - cE_1'(t) + c_0 E_1'(t).
\end{aligned}$$

Consequently, with a suitable choice of ε_0 and c_0, we obtain, for all $t \ge t_1$,

$$F_1'(t) \le -k_1 \left(\frac{E_1(t)}{E_1(0)} \right) H_0' \left(\varepsilon_0 \frac{E_1(t)}{E_1(0)} \right) = -k_1 H_1 \left(\frac{E_1(t)}{E_1(0)} \right), \tag{3.20}$$

where $H_1(t) = tH_0'(\varepsilon_0 t)$.

Since $H_1'(t) = H_0'(\varepsilon_0 t) + \varepsilon_0 t H_0''(\varepsilon_0 t)$, then, using the strict convexity of H_0 on $(0, r]$, we find that $H_1'(t)$, $H_1(t) > 0$ on $(0, 1]$. Thus, taking in account that $E_1' \leq 0$, we have

$$\left[tH_1\left(\frac{E_1(t)}{E_1(0)}\right)\right]'(t) \leq H_1\left(\frac{E_1(t)}{E_1(0)}\right) \leq -\frac{1}{k_1}F_1'(t).$$

A simple integration over (t_1, t) yields

$$tH_1\left(\frac{E_1(t)}{E_1(0)}\right) \leq \frac{1}{k_1}F_1(t_1) + t_1 H_1\left(\frac{E_1(t_1)}{E_1(0)}\right).$$

This gives, for all $t \geq t_1$,

$$E_1(t) \leq H_1^{-1}\left(\frac{K}{t}\right).$$

Therefore, estimate (2.4) is established. □

Acknowledgment. The author thanks University of Sharjah for its continuous support.

REFERENCES

[1] V. I. Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag, New York, 1989.

[2] D. S. Chandrasekharaiah, Hyperbolic thermoelasticity: a review of recent literature, *Appl. Mech. Rev.*, 51:705–729, 1998.

[3] B. D. Coleman, W. J. Hrusa and D. R. Owen, Stability of equilibrium for a nonlinear hyperbolic system describing heat propagation by second sound in solids, *Arch. Rational Mech. Anal.*, 94:267–289, 1986.

[4] H. D. Fernández Sare and R. Racke, On the stability of damped Timoshenko systems: Cattaneo versus Fourier' s law, *Arch. Rational Mech. Anal.*, 194 (1):221–251, 2009.

[5] T. Irmscher and R. Racke, Sharp decay rates in parabolic and hyperbolic thermoelasticity, *IMA J. Appl. Math.*, 71:459–478, 2006.

[6] S. Jiang and R. Racke, *Evolution equations in thermoelasticity*, Monographs Surveys Pure Appl. Math. 112, Chapman & Hall / CRC, Boca Raton, 2000.

[7] H. W. Lord and Y. Shulman, A generalized dynamical theory of thermoelasticity, *J. Mech. Phys. Solids*, 15:299–309, 1967.

[8] S. A. Messaoudi, Local Existence and blow up in thermoelasticity with second sound, *Comm. Partial Diff. Equations*, 26 (8):1681–1693, 2002.

[9] S. A. Messaoudi and A. Al-Shehri, General boundary stabilization of memory-type thermoelasticity, *J. Math. Phys.*, 51:103514, 2010.

[10] S. A. Messaoudi and A. Al-Shehri, General boundary stabilization of memory-type thermoelasticity with second sound, submitted.

[11] S. A. Messaoudi, M. Pokojovy and B. Said-Houari, Nonlinear Damped Timoshenko systems with second sound-Global existence and exponential stability, *Math. Meth. Appl. Sci.*, 32:505–534, 2009.

[12] S. A. Messaoudi and B. Said-Houari, Blow up of solutions with positive energy in nonlinear thermoelasticity with second sound, *J. Appl. Math.*, 2004 (3):201–211, 2004.

[13] S. A. Messaoudi and B. Said-Houari, Exponential Stability in one-dimensional nonlinear thermoelasticity with second sound, *Math. Meth. Appl. Sci.*, 28:205–232, 2005.

[14] J. E. Muñoz Rivera and R. Racke, Mildly dissipative nonlinear Timoshenko systems-global existence and exponential stability, *J. Math. Anal. Appl.*, 276:248–278, 2002.

[15] Y. Qin, Z. Ma and X. Yang, Exponential stability for nonlinear thermoelastic equations with second sound, *Nonlinear Anal.: RWA*, 11 (4):2502–2513, 2010.

[16] R. Racke, Thermoelasticity with second sound - Exponential stability in linear and nonlinear 1-d., *Math. Meth. Appl. Sci.*, 25:409–441, 2002.

[17] R. Racke, Asymptotic behavior of solutions in linear 2- or 3-d thermoelasticity with second sound, *Quart. Appl. Math.*, 61 (2):315–328, 2003.

[18] R. Racke and Y. Wang, Nonlinear well-posedness and rates of decay in thermoelasticity with second sound, *J. Hyperbolic Differ. Equ.*, 5 (1):25–43, 2008.

[19] T. Sabir Öncü and T. Bryant Moodie, On the constitutive relations for second sound in elastic solids, *Arch Rational Mech Anal.*, 121:87–99, 1992.

[20] M. A. Tarabek, On the existence of smooth solutions in one-dimensional thermoelasticity with second sound, *Quart. Appl. Math.*, 50:727–742, 1992.

[21] S. Zheng, *Nonlinear Parabolic Equations and Hyperbolic-Parabolic Coupled Systems*, Pitman Monographs Surveys Pure Applied Mathematics, vol. 76. Longman: Harlow, 1995.