Dynamic Systems and Applications 24 (2015) 399-418

THE NARROWING SET-VALUED STOCHASTIC INTEGRAL EQUATIONS

MAREK T. MALINOWSKI

Institute of Mathematics, Cracow University of Technology
ul. Warszawska 24, 31-155 Kraków, Poland
malinowskimarek@poczta.fm

ABSTRACT. We analyze set-valued stochastic integral equations whose solutions are mappings with values in the hyperspace of subsets of square integrable random vectors space. In this paper we give a new formulation of these equations resulting in a new property of solutions. Namely, the diameter of the solution values will be a nonincreasing function. Hence we call these equations "narrowing". We prove a result on existence and uniqueness of the solution to the narrowing set-valued stochastic integral equations. We establish a boundedness type result for the solution and an error of an approximate solution. Also the continuous dependence of the solution with respect to data of the equation is shown.

Keywords and phrases. Set-valued mapping, set-valued stochastic process, set-valued stochastic integral equation, set-valued stochastic differential equation.

AMS (MOS) Subject Classification. 93E03, 93C41, 26E25, 28B20, 60H10, 60G20.

1. INTRODUCTION

Set-valued mappings appear naturally in many branches of science, such as economics, biomathematics, physics, game theory, artificial intelligence (see e.g. [7, 13] and references therein). There is a huge interest in this area due to many applications in control theory and optimization (see e.g. [4]–[6] and references therein). Also, thinking about mathematical models of dynamical systems with incomplete information or systems with velocities that are not uniquely determined, one often focuses on set-valued differential equations [25].

Set-valued differential equations with solutions taking on values in compact and convex subsets of the Euclidean space were introduced in [9, 10, 12] and studied since then by many authors. Their range includes, for instance, existence of solutions [1], stability [2, 3, 8, 16, 17, 19, 41, 43], equations involving causal operators [14, 16, 23], monotone iterative technique [15], variation of constants formula [18], equations on time scales [20, 28, 43], periodic solutions [21], monotone flows [27], equations with second type Hukuhara derivative [29]–[32], quasilinearization [42].

Received April 20, 2015 1056-2176 $15.00 ©Dynamic Publishers, Inc.

On the other hand a new mathematical apparatus in a form of set-valued stochastic integral equations [33]–[40] generalizes the deterministic set-valued differential equations to a stochastic extent. The papers we have been doing so far use a formulation of the set-valued stochastic integral equations in such a form, nevertheless - very natural, which causes that the diameter of the solution values is a nondecreasing function. We show this in Theorem 3.1 of the current paper, where we consider the set-valued stochastic integral equations with solutions being set-valued mappings taking on values in the hyperspace of nonempty closed bounded and convex subsets of square integrable random vectors space. This paper presents a study of set-valued stochastic equations with solutions having a nonincreasing diameter of their values. This is shown in Theorem 3.2 later on. Accordingly to the new property of solutions we call these equations "narrowing". Consequently the equations studied earlier are called the "widening" set-valued stochastic integral equations. We show that the theory of the narrowing set-valued stochastic integral equations is well-posed. Existence of unique solution is shown together with the stabilities of the solution under small changes of the equation parameters. As distict from the usual analysis, to obtain existence of solutions we impose existence of some Hukuhara differences, see condition (H4). This condition is necessary and cannot be omitted. It is not used in the previously studied widening equations, because in their analysis there is no need to use the Hukuhara differences. The existence of solutions is obtained under Lipschitz condition with an integrable stochastic process instead of the Lipschitz constant and a condition of boundedness by an integrable stochastic process.

The paper is organized as follows. In Section 2 we summarize some preliminary facts and properties on the Hukuhara difference, set-valued stochastic processes and set-valued stochastic trajectory integrals. The main results are presented in Section 3. We introduce the notion of the narrowing set-valued stochastic integral equation. Then, the existence and uniqueness of solutions is proven. Also, the continuous dependence of solutions with respect to data of the equation is shown. We indicate that under conditions (H1)–(H3), considered in Section 3, the similar results can be obtained for the widening set-valued stochastic integral equations.

2. PRELIMINARIES

Let $\mathcal{X}$ be a separable Banach space, and let $\mathcal{K}_c^b(\mathcal{X})$ be the hyperspace of all nonempty closed bounded and convex subsets of $\mathcal{X}$. The Hausdorff metric $H_{\mathcal{X}}$ in $\mathcal{K}_c^b(\mathcal{X})$ is defined by

$$H_{\mathcal{X}}(A,B) := \max\left\{ \sup_{a\in A} \operatorname{dist}_{\mathcal{X}}(a,B), \sup_{b\in B} \operatorname{dist}_{\mathcal{X}}(b,A) \right\},$$

where $\operatorname{dist}_{\mathcal{X}}(a,B) := \inf_{b\in B} \|a-b\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$ denotes a norm in $\mathcal{X}$. It is known (see [22]) that $(\mathcal{K}_c^b(\mathcal{X}), H_{\mathcal{X}})$ is a complete metric space. The addition and the multiplication by reals are defined as usual, i.e. for $A, B \in \mathcal{K}_c^b(\mathcal{X})$ and $\mu \in \mathbb{R}$ we have $A+B := \{a+b : a \in A, b \in B\}$, $\mu A := \{\mu a : a \in A\}$. The Hukuhara difference of $A, B \in \mathcal{K}_c^b(\mathcal{X})$ is defined as the set $A \ominus B \in \mathcal{K}_c^b(\mathcal{X})$ such that $(A \ominus B) + B = A$. If $A \ominus B$ exists, it is unique.

For the metric $H_{\mathcal{X}}$ and $A, B, C, D \in \mathcal{K}_c^b(\mathcal{X})$ and $\mu \in \mathbb{R}$ one has

(P1) $H_{\mathcal{X}}(A+B, C+D) \leqslant H_{\mathcal{X}}(A, C) + H_{\mathcal{X}}(B, D)$,
(P2) $H_{\mathcal{X}}(\mu A, \mu B) = |\mu| H_{\mathcal{X}}(A, B)$,
(P3) $H_{\mathcal{X}}(A+C, B+C) = H_{\mathcal{X}}(A, B)$,
(P4) if $A \ominus B$ exists then $H_{\mathcal{X}}(A \ominus B, \{0\}) = H_{\mathcal{X}}(A, B)$,
(P5) if $A \ominus B$ and $A \ominus C$ exist then $H_{\mathcal{X}}(A \ominus B, A \ominus C) = H_{\mathcal{X}}(B, C)$,
(P6) if $A \ominus B$ and $C \ominus D$ exist then $H_{\mathcal{X}}(A \ominus B, C \ominus D) \leqslant H_{\mathcal{X}}(A, C) + H_{\mathcal{X}}(B, D)$.

Also, it is known [22] that the family of nonempty, closed and convex subsets of a separable and reflexive Banach space $\mathcal{X}$ supplied with the Mosco topology $\tau_{M_{\mathcal{X}}}$ is a Polish topological space. The Mosco topology is metrizable and weaker than the topology $\tau_{H_{\mathcal{X}}}$ generated by the Hausdorff metric $H_{\mathcal{X}}$.

Let $(U, \mathcal{U}, \mu)$ be a measure space. Recall that a set-valued mapping $F\colon U \to \mathcal{K}_c^b(\mathcal{X})$ is said to be $\mathcal{U}$-measurable (or set-valued random variable) if it satisfies:

$$\{u \in U : F(u) \cap O \neq \emptyset\} \in \mathcal{U} \text{ for every open set } O \subset \mathcal{X}.$$

A set-valued random variable F is said to be L^p-integrally bounded ($p \geqslant 1$), if $u \mapsto H_{\mathcal{X}}(F(u), \{0\})$ belongs to $L^p(U, \mathcal{U}, \mu; \mathbb{R})$.

Define $I := [0, T]$, where $T < \infty$. Let $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t \in I}, P)$ be a complete filtered probability space satisfying usual hypotheses, i.e. $\{\mathcal{A}_t\}_{t \in I}$ is an increasing and right continuous family of sub-σ-algebras of $\mathcal{A}$ and $\mathcal{A}_0$ contains all P-null sets. Let $\{B(t)\}_{t \in I}$ be an $\{\mathcal{A}_t\}$-Brownian motion. Let $\mathcal{N}$ denote the σ-algebra of the nonanticipating elements in $I \times \Omega$, i.e.

$$\mathcal{N} = \{A \in \beta_I \otimes \mathcal{A} : A^t \in \mathcal{A}_t \text{ for every } t \in I\},$$

where β_I is the Borel σ-algebra of subsets of I and $A^t = \{\omega : (t, \omega) \in A\}$. A d-dimensional stochastic process $f\colon I \times \Omega \to \mathbb{R}^d$ is called nonanticipating if $f(\cdot, \cdot)$ is $\mathcal{N}$-measurable.

By λ we denote the Lebesgue measure on (I, β_I). Consider the space

$$L^2_{\mathcal{N}}(\lambda \times P) := L^2(I \times \Omega, \mathcal{N}, \lambda \times P; \mathbb{R}^d).$$

Then for every $f \in L^2_{\mathcal{N}}(\lambda \times P)$ and $\tau, t \in I$, $\tau < t$ the Itô stochastic integral $\int_\tau^t f(s) dB(s)$ exists and one has $\int_\tau^t f(s) dB(s) \in L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d) \subset L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$.

Let $F\colon I\times\Omega\to\mathcal{K}_c^b(\mathbb{R}^d)$ be a set-valued stochastic process, i.e. a family of $\mathcal{A}$-measurable set-valued mappings $F(t,\cdot)\colon\Omega\to\mathcal{K}_c^b(\mathbb{R}^d)$, $t\in I$. We call F nonanticipating if $F(\cdot,\cdot)$ is an $\mathcal{N}$-measurable set-valued mapping. Let us define the set

$$S^2_{\mathcal{N}}(F,\lambda\times P):=\{f\in L^2_{\mathcal{N}}(\lambda\times P): f\in F,\ \ \lambda\times P\text{-a.e.}\}.$$

If F is $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded, then by the Kuratowski and Ryll-Nardzewski Selection Theorem (see [24]) it follows that $S^2_{\mathcal{N}}(F,\lambda\times P)\neq\emptyset$. Hence for every $\tau,t\in I$, $\tau<t$ we can define the set-valued stochastic Aumann trajectory integral $(S)\int_\tau^t F(s)ds$ as a subset of $L^2(\Omega,\mathcal{A}_t,P;\mathbb{R}^d)$ in the following way:

$$(S)\int_\tau^t F(s)ds:=\left\{\int_\tau^t f(s)ds: f\in S^2_{\mathcal{N}}(F,\lambda\times P)\right\}.$$

Now we consider the set-valued stochastic Itô trajectory integral. Like in the preceding considerations, let $F\colon I\times\Omega\to\mathcal{K}_c^b(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded set-valued stochastic process. Then for $\tau,t\in I$, $\tau<t$ we can define the set-valued trajectory Itô stochastic integral

$$(S)\int_\tau^t F(s)dB(s):=\left\{\int_\tau^t f(s)dB(s): f\in S^2_{\mathcal{N}}(F,\lambda\times P)\right\}.$$

By this definition we have $\int_\tau^t F(s)dB(s)\subset L^2(\Omega,\mathcal{A}_t,P;\mathbb{R}^d)$.

Lemma 2.1. *Let $r\in\mathbb{R}$ and let $F\colon I\times\Omega\to\mathcal{K}_c^b(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded set-valued stochastic process. Then for $\tau<t$ $(\tau,t\in I)$*

$$(S)\int_\tau^t rF(s)ds=r(S)\int_\tau^t F(s)ds\quad\textit{and}\quad(S)\int_\tau^t rF(s)dB(s)=r(S)\int_\tau^t F(s)dB(s).$$

In the rest of the paper, for the sake of convenience, we will write L^2 instead of $L^2(\Omega,\mathcal{A},P;\mathbb{R}^d)$ and L^2_t instead of $L^2(\Omega,\mathcal{A}_t,P;\mathbb{R}^d)$ where $t\in I$. From now on we assume that the σ-algebra $\mathcal{A}$ is separable with respect to the probability measure P.

We have the following properties for the stochastic trajectory integrals (see e.g. [37, 38]).

Lemma 2.2. *For a nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded set-valued stochastic process $F\colon I\times\Omega\to\mathcal{K}_c^b(\mathbb{R}^d)$ and for every $\tau,a,t\in I$, $\tau\leqslant a\leqslant t$ it holds that*

$$(S)\int_\tau^t F(s)ds=(S)\int_\tau^a F(s)ds+(S)\int_a^t F(s)ds,\ \textit{and}$$

$$(S)\int_\tau^t F(s)dB(s)=(S)\int_\tau^a F(s)dB(s)+(S)\int_a^t F(s)dB(s).$$

Lemma 2.3. *Let $F, G\colon I\times\Omega\to\mathcal{K}_c^b(\mathbb{R}^d)$ be nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded set-valued stochastic processes. Then for every $\tau, t\in I$, $\tau<t$*

$$H^2_{L^2}\left((S)\int_\tau^t F(s)ds, (S)\int_\tau^t G(s)ds\right)\leqslant (t-\tau)\int\limits_{[\tau,t]\times\Omega} H^2_{\mathbb{R}^d}(F,G)ds\times dP, \text{ and}$$

$$H^2_{L^2}\left((S)\int_\tau^t F(s)dB(s), (S)\int_\tau^t G(s)dB(s)\right)\leqslant \int\limits_{[\tau,t]\times\Omega} H^2_{\mathbb{R}^d}(F,G)ds\times dP.$$

Lemma 2.4. *Let $F\colon I\times\Omega\to\mathcal{K}_c^b(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded set-valued stochastic process. Then the mappings*

$$[\tau,T]\ni t\mapsto (S)\int_\tau^t F(s)ds\in\mathcal{K}_c^b(L^2),\quad [\tau,T]\ni t\mapsto (S)\int_\tau^t F(s)dB(s)\in\mathcal{K}_c^b(L^2)$$

are H_{L^2}-continuous.

3. SET-VALUED STOCHASTIC INTEGRAL EQUATIONS

Let $F, G\colon I\times\Omega\times\mathcal{K}_c^b(L^2)\to\mathcal{K}_c^b(\mathbb{R}^d)$ and $X_0\in\mathcal{K}_c^b(L^2_0)$ be given. By a set-valued stochastic integral equation we mean the following relation in the metric space $(\mathcal{K}_c^b(L^2), H_{L^2})$:

$$X(t) = X_0 + (S)\int_0^t F(s,X(s))ds + (S)\int_0^t G(s,X(s))dB(s) \text{ for } t\in I. \tag{3.1}$$

By a global solution to (3.1) we mean an H_{L^2}-continuous set-valued mapping $X\colon I\to\mathcal{K}_c^b(L^2)$ that satisfies (3.1). A global solution $X\colon I\to\mathcal{K}_c^b(L^2)$ to (3.1) is unique if $X(t)=Y(t)$ for every $t\in I$, where $Y\colon I\to\mathcal{K}_c^b(L^2)$ is any solution of (3.1). Let $J:=[0,\tilde{T}]\subset I=[0,T]$, where $\tilde{T}<T$. A set-valued mapping $X\colon J\to\mathcal{K}_c^b(L^2)$ is said to be a local solution to (3.1) if it is H_{L^2}-continuous and satisfies (3.1) for $t\in J$. The uniqueness of the local solution is defined in an obvious way.

Theorem 3.1. *Suppose that $X\colon I\to\mathcal{K}_c^b(L^2)$ is a global solution to* (3.1). *Then the function $t\mapsto \operatorname{diam}X(t)$ is nondecreasing.*

Proof. Since X is a solution to (3.1) and Lemma 2.2 holds, we can write for $\tau<t$ ($\tau,t\in I$) that

$$X(t) = X(\tau) + (S)\int_\tau^t F(s,X(s))ds + (S)\int_\tau^t G(s,X(s))dB(s).$$

Let x be a fixed point from the set $(S)\int_\tau^t F(s,X(s))ds + (S)\int_\tau^t G(s,X(s))dB(s)$. Then we have $X(t)\supset X(\tau)+\{x\}$ which implies that $\operatorname{diam}X(t)\geqslant\operatorname{diam}X(\tau)$. ☐

The same property is true for local solutions to (3.1) and it convinces that we can call equations of type (3.1) as the widening set-valued stochastic integral equations.

In order to consider the mappings $X\colon I \to \mathcal{K}_c^b(L^2)$ with the nonincreasing diameter function $\operatorname{diam} X(\cdot)$ as some solutions to the set-valued stochastic integral equations, it is necessary to change the form of the equation. Below we present such a variation of formulation of the set-valued stochastic integral equations. Namely, we will consider the equations of the following form:

$$X(t)+(S)\int_0^t(-1)F(s,X(s))ds+(S)\int_0^t(-1)G(s,X(s))dB(s)=X_0 \ \text{ for } \ t\in I. \tag{3.2}$$

It is easy to see that in the case of singleton-valued X_0, F and G the equations (3.1) and (3.2) coincide. Both of them generalize the classical single-valued stochastic differential equations.

Note that the equation (3.2) can be rewritten as

$$X(t)=X_0\ominus\left[(S)\int_0^t(-1)F(s,X(s))ds+(S)\int_0^t(-1)G(s,X(s))dB(s)\right] \tag{3.3}$$

for $t\in I$. We will call the equations (3.2) as the narrowing set-valued stochastic integral equations because, as we prove below, their solutions X possess property that $\operatorname{diam} X(\cdot)$ is nonincreasing. The global and local solutions to (3.2) and their uniqueness are defined like for the global and local solutions to (3.1).

Theorem 3.2. *Let $X\colon I\to\mathcal{K}_c^b(L^2)$ be a global solution to* (3.2)*. Then the function $t\mapsto \operatorname{diam} X(t)$ is nonincreasing.*

Proof. Note that due to Lemma 2.2 we have

$$\begin{aligned} &X(t)+(S)\int_\tau^t(-1)F(s,X(s))ds+(S)\int_\tau^t(-1)G(s,X(s))dB(s)\\ &\quad= X_0\ominus\left[(S)\int_0^\tau(-1)F(s,X(s))ds+(S)\int_0^\tau(-1)G(s,X(s))dB(s)\right]\end{aligned}$$

for $\tau<t$. The Hukuhara difference above exists because X is a solution to (3.2) and it equals $X(\tau)$. Choosing any point x that belongs to the set $(S)\int_\tau^t(-1)F(s,X(s))ds+(S)\int_\tau^t(-1)G(s,X(s))dB(s)$ we get

$$X(t)+\{x\}\subset X(\tau).$$

As a consequence, the inequality $\operatorname{diam} X(t)\leqslant \operatorname{diam} X(\tau)$ follows easily. □

A similar assertion holds true for the local solutions to (3.2).

Below we present an existence and uniqueness theorem. It will be achieved with the following conditions imposed on the data of the equation.

Assume that $X_0\in\mathcal{K}_c^b(L_0^2)$ and $F,G\colon I\times\Omega\times\mathcal{K}_c^b(L^2)\to\mathcal{K}_c^b(\mathbb{R}^d)$ satisfy

(H1) the set-valued mappings $F(\cdot,\cdot,\cdot),G(\cdot,\cdot,\cdot)\colon I\times\Omega\times\mathcal{K}_c^b(L^2)\to\mathcal{K}_c^b(\mathbb{R}^d)$ are $\mathcal{N}\times\beta(\tau_{M_{L^2}})$-measurable, where $\beta(\tau_{M_{L^2}})$ is the Borel σ-algebra induced by the Mosco topology $\tau_{M_{L^2}}$, (H2) there exists $K_F \in L^2(I \times \Omega, \beta_I \otimes \mathcal{A}, \lambda \times P; \mathbb{R})$ such that $\lambda \times P$-a.e. for every $A, B \in \mathcal{K}_c^b(L^2)$

$$H^2_{\mathbb{R}^d}\big(F(t,\omega,A), F(t,\omega,B)\big) \leqslant K_F(t,\omega) H^2_{L^2}(A,B),$$

and there exists $K_G \in L^2(I \times \Omega, \beta_I \otimes \mathcal{A}, \lambda \times P; \mathbb{R})$ such that $\lambda \times P$-a.e. for every $A, B \in \mathcal{K}_c^b(L^2)$

$$H^2_{\mathbb{R}^d}\big(G(t,\omega,A), G(t,\omega,B)\big) \leqslant K_G(t,\omega) H^2_{L^2}(A,B),$$

(H3) there exists $C_F \in L^1(I \times \Omega, \beta_I \otimes \mathcal{A}, \lambda \times P; \mathbb{R})$ such that $\lambda \times P$-a.e.

$$H^2_{\mathbb{R}^d}\big(F(t,\omega,\{\Theta\}), \{\theta\}\big) \leqslant C_F(t,\omega),$$

and there exists $C_G \in L^1(I \times \Omega, \beta_I \otimes \mathcal{A}, \lambda \times P; \mathbb{R})$ such that $\lambda \times P$-a.e.

$$H^2_{\mathbb{R}^d}\big(G(t,\omega,\{\Theta\}), \{\theta\}\big) \leqslant C_G(t,\omega),$$

where Θ, θ denote the zero elements in L^2 and $\mathbb{R}^d$, respectively,

(H4) there exists $\tilde{T} \in (0, T]$ such that the sequence $\{X_n\}_{n=0}^\infty$ described by

$$X_0(t) = X_0, \;\; t \in J = [0, \tilde{T}],$$

and for $n = 1, 2, \ldots$ and for $t \in J$

$$X_n(t) = X_0 \ominus \left[(S)\int_0^t (-1)F(s, X_{n-1}(s))ds + (S)\int_0^t (-1)G(s, X_{n-1}(s))dB(s)\right]$$

can be defined, i.e. the Hukuhara differences do exist.

Notice that the conditions (H2) and (H3) formulated with integrable stochastic processes K_F, K_G, C_F, C_G are weaker than the Lipschitz assumptions with constants and boundedness conditions with constants. The condition (H4) cannot be omitted in the studies of equation (3.2). It is motivated by the representation (3.3).

In what follows we present a result on existence of unique solution to (3.2). Due to (H4) this solution can be local or global depending on whether $\tilde{T} < T$ or $\tilde{T} = T$. In its proof, the sequence $\{X_n\}_{n=0}^\infty$ will be exploited. Below we make a discussion that under conditions (H1)–(H4) each X_n is a well-defined H_{L^2}-continuous set-valued mapping.

Indeed, starting with $X_0(\cdot)$ we see easily that $X_0(\cdot)$ is well-defined. Now, using (H1), we observe that the set-valued mappings $F(\cdot,\cdot,X_0), G(\cdot,\cdot,X_0)\colon I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ are nonanticipating. Due to (H2) and (H3) the following inequalities hold $\lambda \times P$-a.e.

$$H^2_{\mathbb{R}^d}(F(t,\omega,X_0), \{\theta\}) \leqslant 2K_F(t,\omega) H^2_{L^2}(X_0, \{\Theta\}) + 2C_F(t,\omega),$$

$$H^2_{\mathbb{R}^d}(G(t,\omega,X_0), \{\theta\}) \leqslant 2K_G(t,\omega) H^2_{L^2}(X_0, \{\Theta\}) + 2C_G(t,\omega).$$

Hence we can infer that $F(\cdot,\cdot,X_0)$ and $G(\cdot,\cdot,X_0)$ are $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded. This allows us to claim that the set-valued stochastic trajectory integrals in formulation of $X_1(t)$ are well-defined and belong to $\mathcal{K}_c^b(L^2_t)$. Since $X_0 \in \mathcal{K}_c^b(L^2_0) \subset \mathcal{K}_c^b(L^2_t)$

and it is assumed that the Hukuhara differences in (H4) exist, we obtain that $X_1(t) \in \mathcal{K}_c^b(L_t^2)$ for every $t \in J$. Lemma 2.4 allows us to infer that the mapping $t \mapsto X_1(t)$ is H_{L^2}-continuous. Since the Mosco topology $\tau_{M_{L^2}}$ is is weaker than the topology generated by the Hausdorff metric H_{L^2}, the mapping $t \mapsto X_1(t)$ is continuous with respect to topology $\tau_{M_{L^2}}$ as well. Hence the set-valued mappings $(t,\omega) \mapsto F(t,\omega,X_1(t))$ and $(t,\omega) \mapsto G(t,\omega,X_1(t))$ are nonanticipating. Since

$$H^2_{\mathbb{R}^d}(F(t,\omega,X_1(t)),\{\theta\}) \leqslant 2K_F(t,\omega)\sup_{t\in J} H^2_{L^2}(X_1(t),\{\Theta\}) + 2C_F(t,\omega),$$

$$H^2_{\mathbb{R}^d}(G(t,\omega,X_1(t)),\{\theta\}) \leqslant 2K_G(t,\omega)\sup_{t\in J} H^2_{L^2}(X_1(t),\{\Theta\}) + 2C_G(t,\omega)$$

and $\sup_{t\in J} H^2_{L^2}\big(X_1(t),\{\Theta\}\big) < \infty$, we get that $(t,\omega) \mapsto F(t,\omega,X_1(t))$ and $(t,\omega) \mapsto G(t,\omega,X_1(t))$ are $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded. Now we are able to state that X_2 is well-defined and H_{L^2}-continuous. Proceeding recursively one can see that every X_n is well-defined and H_{L^2}-continuous.

Theorem 3.3. *Let $X_0 \in \mathcal{K}_c^b(L_0^2)$ and $F, G\colon I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d)$ satisfy conditions* (H1)–(H4). *Then equation* (3.2) *has a unique local (or global) solution.*

Proof. It is clear that using Lemma 2.4 we obtain that each X_n is continuous with respect to the metric H_{L^2}. We shall show that $\{X_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the space $C(J,\mathcal{K}_c^b(L^2))$ endowed with a supremum metric.

Due to property (P6), Lemma 2.1, (P3) and (P1) we have for $t \in J$

$$\begin{aligned}
&H^2_{L^2}(X_1(t),X_0(t))\\
&\quad= H^2_{L^2}\left((S)\int_0^t F(s,X_0)ds + (S)\int_0^t G(s,X_0)dB(s),\{\Theta\}\right)\\
&\quad\leqslant 2H^2_{L^2}\left((S)\int_0^t F(s,X_0)ds,\{\Theta\}\right) + 2H^2_{L^2}\left((S)\int_0^t G(s,X_0)dB(s),\{\Theta\}\right).
\end{aligned}$$

Now by Lemma 2.3

$$\begin{aligned}
&H^2_{L^2}(X_1(t),X_0(t))\\
&\quad\leqslant 2t\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F(s,X_0),\{\theta\})ds\times dP + 2\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G(s,X_0),\{\theta\})ds\times dP\\
&\quad\leqslant 4t\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F(s,X_0),F(s,\{\Theta\}))ds\times dP\\
&\qquad + 4t\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F(s,\{\Theta\}),\{\theta\})ds\times dP\\
&\qquad + 4\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G(s,X_0),G(s,\{\Theta\}))ds\times dP\\
&\qquad + 4\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G(s,\{\Theta\}),\{\theta\})ds\times dP
\end{aligned}$$

and by the assumptions (H2) and (H3)

$$\begin{aligned} H^2_{L^2}(X_1(t), X_0(t)) &\leqslant (4t+4)H^2_{L^2}(X_0, \{\Theta\}) \int_{[0,t]\times\Omega} (K_F(s)+K_G(s))ds \times dP \\ &\quad + (4t+4)\int_{[0,t]\times\Omega} (C_F(s)+C_G(s))ds \times dP \\ &\leqslant M_1, \end{aligned}$$

where

$$\begin{aligned} M_1 &= (4\tilde{T}+4)\Big[H^2_{L^2}(X_0, \{\Theta\}) \int_{J\times\Omega} (K_F(s)+K_G(s))ds \times dP \\ &\quad + \int_{J\times\Omega} (C_F(s)+C_G(s))ds \times dP\Big] < \infty. \end{aligned} \tag{3.4}$$

For $n \geqslant 2$ we have

$$\begin{aligned} &H^2_{L^2}(X_n(t), X_{n-1}(t)) \\ &\leqslant 2t\int_{[0,t]\times\Omega} K_F(s) H^2_{L^2}(X_{n-1}(s), X_{n-2}(s))ds \times dP \\ &\quad + 2\int_{[0,t]\times\Omega} K_G(s) H^2_{L^2}(X_{n-1}(s), X_{n-2}(s))ds \times dP \\ &\leqslant \left[2t\left(\int_{[0,t]\times\Omega} K^2_F(s)ds \times dP\right)^{1/2} + 2\left(\int_{[0,t]\times\Omega} K^2_G(s)ds \times dP\right)^{1/2}\right] \\ &\quad \times \left(\int_{[0,t]\times\Omega} H^4_{L^2}(X_{n-1}(s), X_{n-2}(s))ds \times dP\right)^{1/2}. \end{aligned}$$

Hence

$$H^4_{L^2}(X_n(t), X_{n-1}(t)) \leqslant M_2 \int_0^t H^4_{L^2}(X_{n-1}(s), X_{n-2}(s))ds,$$

where

$$M_2 = 8\tilde{T}^2 \int_{J\times\Omega} K^2_F(s)ds \times dP + 8\int_{J\times\Omega} K^2_G(s)ds \times dP. \tag{3.5}$$

This allows us to infer that

$$H_{L^2}(X_n(t), X_{n-1}(t)) \leqslant \left(M_1 \frac{(M_2 t)^{n-1}}{(n-1)!}\right)^{1/4} \tag{3.6}$$

and

$$\sup_{t\in J} H_{L^2}(X_n(t), X_{n-1}(t)) \leqslant \left(M_1 \frac{(M_2 \tilde{T})^{n-1}}{(n-1)!}\right)^{1/4}.$$

Consequently for $m < n$

$$\sup_{t\in J} H_{L^2}(X_n(t), X_m(t)) \leqslant \sum_{k=m}^{n-1} \left(M_1 \frac{(M_2 \tilde{T})^{k}}{k!}\right)^{1/4}.$$

Now it is clear that $\{X_n\}$ is a Cauchy sequence in $C(J,\mathcal{K}_c^b(L^2))$. Thus there exists $X \in C(J,\mathcal{K}_c^b(L^2))$ such that

$$\sup_{t\in J} H_{L^2}(X_n(t),X(t)) \to 0 \text{ as } n\to\infty.$$

In the sequel we shall show that X is a solution to (3.2). Observe that for every fixed $t\in J$ we have

$$\begin{aligned}&H_{L^2}^2\left(X(t), X_0\ominus\left[(S)\int_0^t(-1)F(s,X(s))ds+(S)\int_0^t(-1)G(s,X(s))dB(s)\right]\right)\\&\quad\leqslant\ 2H_{L^2}^2(X(t),X_n(t))+2R_n(t),\end{aligned}$$

where

$$\begin{aligned}R_n(t)\ &=\ H_{L^2}^2\Big(X_0\ominus\Big[(S)\int_0^t(-1)F(s,X_{n-1}(s))ds+(S)\int_0^t(-1)G(s,X_{n-1}(s))dB(s)\Big],\\&\qquad X_0\ominus\Big[(S)\int_0^t(-1)F(s,X(s))ds+(S)\int_0^t(-1)G(s,X(s))dB(s)\Big]\Big).\end{aligned}$$

Note that

$$\begin{aligned}R_n(t)\ &\leqslant\ 2H_{L^2}^2\left((S)\int_0^t F(s,X_{n-1}(s))ds,(S)\int_0^t F(s,X(s))ds\right)\\&\quad+2H_{L^2}^2\left((S)\int_0^t G(s,X_{n-1}(s))dB(s),(S)\int_0^t G(s,X(s))dB(s)\right)\\&\leqslant\ \left[2t\left(\int_{[0,t]\times\Omega}K_F^2(s)ds\times dP\right)^{1/2}+2\left(\int_{[0,t]\times\Omega}K_G^2(s)ds\times dP\right)^{1/2}\right]\\&\quad\times\left(\int_{[0,t]\times\Omega}H_{L^2}^4(X_{n-1}(s),X(s))ds\times dP\right)^{1/2}.\end{aligned}$$

Hence

$$\begin{aligned}R_n^2(t)\ &\leqslant\ M_2\int_0^t H_{L^2}^4(X_{n-1}(s),X(s))ds\\&\leqslant\ M_2\tilde{T}\left(\sup_{t\in J}H_{L^2}(X_{n-1}(t),X(t))\right)^4.\end{aligned}$$

Since $\sup_{t\in J}H_{L^2}(X_{n-1}(t),X(t))\overset{n\to\infty}{\longrightarrow}0$, we have $R_n(t)\overset{n\to\infty}{\longrightarrow}0$ for every $t\in J$. Now it is easy to see that

$$H_{L^2}\left(X(t),X_0\ominus\left[(S)\int_0^t(-1)F(s,X(s))ds+(S)\int_0^t(-1)G(s,X(s))dB(s)\right]\right)=0$$

for every $t\in J$. This means that X is a solution to (3.2).

Suppose that $X\colon J\to\mathcal{K}_c^b(L^2)$ and $Y\colon J\to\mathcal{K}_c^b(L^2)$ are two solutions to (3.2). Then it can be verified that for $t\in J$

$$H_{L^2}^4(X(t),Y(t))\leqslant M_2\int_0^t H_{L^2}^4(X(s),Y(s))ds.$$

Thus, after application of the Gronwall inequality, we can infer that

$$H^4_{L^2}(X(t), Y(t)) = 0 \text{ for every } t \in J$$

which implies that $X(t) = Y(t)$ for every $t \in J$. Hence the uniqueness of X is proven. $\square$

The sequence $\{X_n\}$ defined in (H4) converges to the solution $X\colon J \to \mathcal{K}^b_c(L^2)$ to (3.2). Hence it can derive some approximate solutions to (3.2). An estimation of an error between the nth approximation X_n and the exact solution X is a subject of the next result.

Proposition 3.4. *Let for $X_0 \in \mathcal{K}^b_c(L^2_0)$ and $F, G\colon I \times \Omega \times \mathcal{K}^b_c(L^2) \to \mathcal{K}^b_c(\mathbb{R}^d)$ the conditions* (H1)–(H4) *be satisfied. Then for every $n \in \mathbb{N}$ it holds that*

$$\sup_{t\in J} H_{L^2}(X_n(t), X(t)) \leqslant 2^{3/4}\left(M_1\frac{(M_2\tilde{T})^n}{n!}\right)^{1/4}\exp\{2M_2\tilde{T}\},$$

where the constants M_1 and M_2 are defined as in (3.4) *and* (3.5)*, respectively.*

Proof. Proceeding similarly like in the proof of Theorem 3.3 we get for $t \in J$

$$\begin{aligned}
&H^2_{L^2}(X_n(t), X(t)) \\
&\quad\leqslant\ 2t\left(\int_{[0,t]\times\Omega} K^2_F(s)ds \times dP\int_{[0,t]\times\Omega} H^4_{L^2}(X_{n-1}(s), X(s))ds \times dP\right)^{1/2} \\
&\qquad + 2\left(\int_{[0,t]\times\Omega} K^2_G(s)ds \times dP\int_{[0,t]\times\Omega} H^4_{L^2}(X_{n-1}(s), X(s))ds \times dP\right)^{1/2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&H^4_{L^2}(X_n(t), X(t)) \\
&\quad\leqslant\ M_2\int_0^t H^4_{L^2}(X_{n-1}(s), X(s))ds \\
&\quad\leqslant\ 8M_2\int_0^t H^4_{L^2}(X_{n-1}(s), X_n(s))ds + 8M_2\int_0^t H^4_{L^2}(X_n(s), X(s))ds
\end{aligned}$$

and by (3.6) we can write

$$H^4_{L^2}(X_n(t), X(t)) \leqslant 8M_1\frac{(M_2\tilde{T})^n}{n!} + 8M_2\int_0^t H^4_{L^2}(X_n(s), X(s))ds.$$

Thus by the Gronwall inequality

$$H^4_{L^2}(X_n(t), X(t)) \leqslant 8M_1\frac{(M_2\tilde{T})^n}{n!}\exp\{8M_2 t\} \text{ for every } t \in J$$

and the assertion follows easily. $\square$

Proposition 3.5. *Under assumptions of Theorem* 3.3 *for the solution* $X\colon J \to \mathcal{K}_c^b(L^2)$ *to* (3.2) *it holds that*

$$\sup_{t\in J} H_{L^2}(X(t),\{\Theta\}) \leqslant M_3 \exp\{8M_2\tilde{T}\},$$

where

$$M_3 = \Big[8H^4_{L^2}(X_0,\{\Theta\}) + 2^8\tilde{T}^2\Big(\int_{J\times\Omega} C_F(s)ds\times dP\Big)^2 + 2^8\Big(\int_{J\times\Omega} C_G(s)ds\times dP\Big)^2\Big]^{1/4}$$

and M_2 *is defined like in* (3.5).

Proof. By (P4), the triangle inequality, Lemma 2.1 and (P1) we have for $t\in J$

$$H^4_{L^2}(X(t),\{\Theta\}) \leqslant \Big[H_{L^2}(X_0,\{\Theta\}) + H_{L^2}\left((S)\int_0^t F(s,X(s))ds + (S)\int_0^t G(s,X(s))dB(s),\{\Theta\}\right)\Big]^4.$$

Hence

$$H^4_{L^2}(X(t),\{\Theta\}) \leqslant 8H^4_{L^2}(X_0,\{\Theta\}) + 8H^4_{L^2}\left((S)\int_0^t F(s,X(s))ds + (S)\int_0^t G(s,X(s))dB(s),\{\Theta\}\right).$$

Thus

$$H^4_{L^2}(X(t),\{\Theta\}) \leqslant 8H^4_{L^2}(X_0,\{\Theta\}) + 2^6H^4_{L^2}\left((S)\int_0^t F(s,X(s))ds,\{\Theta\}\right) + 2^6H^4_{L^2}\left((S)\int_0^t G(s,X(s))dB(s),\{\Theta\}\right).$$

Due to Lemma 2.3

$$\begin{aligned}H^4_{L^2}(X(t),\{\Theta\}) \leqslant\ & 8H^4_{L^2}(X_0,\{\Theta\}) + 2^6\Big[2t\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F(s,X(s)),F(s,\{\Theta\}))ds\times dP \\ & + 2t\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F(s,\{\Theta\}),\{\theta\})ds\times dP\Big]^2 \\ & + 2^6\Big[2\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G(s,X(s)),G(s,\{\Theta\}))ds\times dP \\ & + 2\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G(s,\{\Theta\}),\{\theta\})ds\times dP\Big]^2\end{aligned}$$

and due to assumptions (H2) and (H3)

$$\begin{aligned}
&H^4_{L^2}(X(t),\{\Theta\})\\
&\quad\leqslant\ 8H^4_{L^2}(X_0,\{\Theta\})\\
&\qquad + 2^6\Big[2t\int_{[0,t]\times\Omega} K_F(s)H^2_{L^2}(X(s),\{\Theta\})ds\times dP + 2t\int_{[0,t]\times\Omega} C_F(s)ds\times dP\Big]^2\\
&\qquad + 2^6\Big[2\int_{[0,t]\times\Omega} K_G(s)H^2_{L^2}(X(s),\{\Theta\})ds\times dP + 2\int_{[0,t]\times\Omega} C_G(s)ds\times dP\Big]^2.
\end{aligned}$$

Further

$$\begin{aligned}
&H^4_{L^2}(X(t),\{\Theta\})\\
&\quad\leqslant\ 8H^4_{L^2}(X_0,\{\Theta\})\\
&\qquad + 2^6\Big[2t\Big(\int_{[0,t]\times\Omega} K^2_F(s)ds\times dP\int_{[0,t]\times\Omega} H^4_{L^2}(X(s),\{\Theta\})ds\times dP\Big)^{1/2}\\
&\qquad + 2t\int_{[0,t]\times\Omega} C_F(s)ds\times dP\Big]^2\\
&\qquad + 2^6\Big[2\Big(\int_{[0,t]\times\Omega} K^2_G(s)ds\times dP\int_{[0,t]\times\Omega} H^4_{L^2}(X(s),\{\Theta\})ds\times dP\Big)^{1/2}\\
&\qquad + 2\int_{[0,t]\times\Omega} C_G(s)ds\times dP\Big]^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&H^4_{L^2}(X(t),\{\Theta\})\\
&\quad\leqslant\ 8H^4_{L^2}(X_0,\{\Theta\})\\
&\qquad + 2^6\Big[4t^2\int_{[0,t]\times\Omega} K^2_F(s)ds\times dP\int_{[0,t]\times\Omega} H^4_{L^2}(X(s),\{\Theta\})ds\times dP\\
&\qquad + 4t^2\Big(\int_{[0,t]\times\Omega} C_F(s)ds\times dP\Big)^2\Big]\\
&\qquad + 2^6\Big[4\int_{[0,t]\times\Omega} K^2_G(s)ds\times dP\int_{[0,t]\times\Omega} H^4_{L^2}(X(s),\{\Theta\})ds\times dP\\
&\qquad + 4\Big(\int_{[0,t]\times\Omega} C_G(s)ds\times dP\Big)^2\Big]\\
&\quad\leqslant\ M_3^4 + 32M_2\int_0^t H^4_{L^2}(X(s),\{\Theta\})ds.
\end{aligned}$$

Applying the Gronwall inequality we infer that

$$H^4_{L^2}(X(t),\{\Theta\}) \leqslant M_3^4\exp\{32M_2t\} \text{ for every } t\in J,$$

which yields the assertion. □

It is worth mentioning that the theory of narrowing set-valued stochastic integral equations will be well-posed, if we show that the solutions to (3.2) do not change much if the data of the equation have some small changes. Below we present some studies in this direction.

Consider equation (3.2) and the same equation with another initial value $\tilde{X}_0$, i.e.

$$X(t)+(S)\int_0^t(-1)F(s,X(s))ds+(S)\int_0^t(-1)G(s,X(s))dB(s)=\tilde{X}_0 \text{ for } t\in I. \tag{3.7}$$

Let $X\colon J_1\to\mathcal{K}_c^b(L^2)$ and $Y\colon J_2\to\mathcal{K}_c^b(L^2)$ denote the unique solutions (if they exist) to these equations, respectively, $J_1=[0,\tilde{T}_1]$, $J_2=[0,\tilde{T}_2]$ for some $\tilde{T}_1,\tilde{T}_2\in(0,T]$. Let $J=J_1\cap J_2$.

Theorem 3.6. *Let X_0,F,G satisfy the conditions* (H1)–(H4). *Assume also that $\tilde{X}_0,F,G$ satisfy* (H1)–(H4). *Then*

$$\sup_{t\in J}H_{L^2}(X(t),Y(t))\leqslant 2^{3/4}H_{L^2}(X_0,\tilde{X}_0)\exp\{2\tilde{M}_2\min\{\tilde{T}_1,\tilde{T}_2\}\},$$

where $\tilde{M}_2=8(\min\{\tilde{T}_1,\tilde{T}_2\})^2\int_{J\times\Omega}K_F^2(s)ds\times dP+8\int_{J\times\Omega}K_G^2(s)ds\times dP.$

Proof. Observe that for $t\in J$ we have, accordingly to (P6), Lemma 2.1 and (P2),

$$\begin{aligned}H^4_{L^2}(X(t),Y(t)) &\leqslant 8H^4_{L^2}(X_0,\tilde{X}_0)\\ &\quad+8\Big[H_{L^2}\Big(\int_0^tF(s,X(s))ds,\int_0^tF(s,Y(s))ds\Big)\\ &\quad+H_{L^2}\Big(\int_0^tG(s,X(s))dB(s),\int_0^tG(s,Y(s))dB(s)\Big)\Big]^4\\ &\leqslant 8H^4_{L^2}(X_0,\tilde{X}_0)\\ &\quad+64H^4_{L^2}\Big(\int_0^tF(s,X(s))ds,\int_0^tF(s,Y(s))ds\Big)\\ &\quad+64H^4_{L^2}\Big(\int_0^tG(s,X(s))dB(s),\int_0^tG(s,Y(s))dB(s)\Big).\end{aligned}$$

Hence by Lemma 2.3 and the assumptions (H2) and (H3) we get

$$\begin{aligned}H^4_{L^2}(X(t),Y(t)) &\leqslant 8H^4_{L^2}(X_0,\tilde{X}_0)\\ &\quad+64t^2\Big(\int_{[0,t]\times\Omega}K_F(s)H^2_{L^2}(X(s),Y(s))ds\times dP\Big)^2\\ &\quad+64\Big(\int_{[0,t]\times\Omega}K_G(s)H^2_{L^2}(X(s),Y(s))ds\times dP\Big)^2.\end{aligned}$$

Thus

$$H^4_{L^2}(X(t),Y(t))\leqslant 8H^4_{L^2}(X_0,\tilde{X}_0)+8\tilde{M}_2\int_0^tH^4_{L^2}(X(s),Y(s))ds$$

and by the Gronwall inequality

$$H^4_{L^2}(X(t),Y(t))\leqslant 8H^4_{L^2}(X_0,\tilde{X}_0)\exp\{8\tilde{M}_2t\} \text{ for every } t\in J.$$

This leads to

$$\sup_{t\in J} H_{L^2}(X(t),Y(t)) \leqslant 2^{3/4} H_{L^2}(X_0,\tilde{X}_0)\exp\{2\tilde{M}_2\min\{\tilde{T}_1,\tilde{T}_2\}\}.$$

□

By this assertion the stability of solution to (3.2) with respect to small changes of initial value follows.

Now, let us consider equation (3.2) and equations (for $n\in\mathbb{N}$)

$$X(t)+(S)\int_0^t(-1)F_n(s,X(s))ds+(S)\int_0^t(-1)G_n(s,X(s))dB(s)=X_0 \tag{3.8}$$

for $t\in I$, with another coefficients F_n and G_n. Let X, X_n denote the unique solutions (if they exist) to these equations, respectively. Assume that they all are defined on a common interval $J=[0,\tilde{T}]$ with $\tilde{T}\in(0,T]$.

Theorem 3.7. *Let X_0,F,G satisfy the conditions* (H1)–(H4)*. Assume also that $\tilde{X}_0,F_n,G_n$ satisfy* (H1)–(H4)*, in particular the conditions* (H2) *and* (H3) *are satisfied with the processes K_{F_n},K_{G_n} and C_{F_n},C_{G_n}, respectively. Assume that there exist constants $S_F,S_G>0$ such that for every $n\in\mathbb{N}$*

$$\int_{J\times\Omega}K^2_{F_n}(s)ds\times dP\leqslant S_F \text{ and } \int_{J\times\Omega}K^2_{G_n}(s)ds\times dP\leqslant S_G.$$

Suppose that for every $A\in\mathcal{K}^b_c(L^2)$

$$\int_{J\times\Omega}H^2_{\mathbb{R}^d}(F_n(s,A),F(s,A))ds\times dP\to 0 \text{ as } n\to\infty \text{ and}$$
$$\int_{J\times\Omega}H^2_{\mathbb{R}^d}(G_n(s,A),G(s,A))ds\times dP\to 0 \text{ as } n\to\infty.$$

Then for the solution $X\colon J\to\mathcal{K}^b_c(L^2)$ to (3.2) *and the solutions $X_n\colon J\to\mathcal{K}^b_c(L^2)$ to* (3.8) *it holds that*

$$\sup_{t\in J}H_{L^2}(X_n(t),X(t))\to 0 \text{ as } n\to\infty.$$

Proof. Proceeding similarly as in the previous proofs we obtain for $t\in J$

$$\begin{aligned}H^4_{L^2}(X_n(t),X(t)) &\leqslant 8H^4_{L^2}\left(\int_0^t F_n(s,X_n(s))ds,\int_0^t F(s,X(s))ds\right)\\ &\quad+8H^4_{L^2}\left(\int_0^t G_n(s,X_n(s))dB(s),\int_0^t G(s,X(s))dB(s)\right).\end{aligned}$$

Hence

$$\begin{aligned}H^4_{L^2}(X_n(t),X(t)) &\leqslant 32\left[t\int_{[0,t]\times\Omega}H^2_{\mathbb{R}^d}(F_n(s,X_n(s)),F_n(s,X(s)))ds\times dP\right]^2\\ &\quad+32\left[t\int_{[0,t]\times\Omega}H^2_{\mathbb{R}^d}(F_n(s,X(s)),F(s,X(s)))ds\times dP\right]^2\end{aligned}$$

$$+ 32\left[\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X_n(s)),G_n(s,X(s)))ds\times dP\right]^2$$
$$+ 32\left[\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X(s)),G(s,X(s)))ds\times dP\right]^2.$$

Since F_n, G_n satisfy (H2) with the processes K_{F_n} and K_{G_n}, respectively, we can write

$$\begin{aligned} H^4_{L^2}(X_n(t),X(t)) \leqslant\ & 32\tilde{T}^2\left[\int_{[0,t]\times\Omega} K_{F_n}(s)H^2_{L^2}(X_n(s),X(s))ds\times dP\right]^2 \\ &+ 32\tilde{T}^2\left[\int_{J\times\Omega} H^2_{\mathbb{R}^d}(F_n(s,X(s)),F(s,X(s)))ds\times dP\right]^2 \\ &+ 32\left[\int_{[0,t]\times\Omega} K_{G_n}(s)H^2_{L^2}(X_n(s),X(s))ds\times dP\right]^2 \\ &+ 32\left[\int_{J\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X(s)),G(s,X(s)))ds\times dP\right]^2. \end{aligned}$$

Thus

$$\begin{aligned} H^4_{L^2}(X_n(t),X(t)) \leqslant\ & 32\tilde{T}^2\left[\int_{J\times\Omega} H^2_{\mathbb{R}^d}(F_n(s,X(s)),F(s,X(s)))ds\times dP\right]^2 \\ &+ 32\left[\int_{J\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X(s)),G(s,X(s)))ds\times dP\right]^2 \\ &+ \left(32\tilde{T}^2\int_{[0,t]\times\Omega} K^2_{F_n}(s)ds\times dP + 32\int_{[0,t]\times\Omega} K^2_{G_n}(s)ds\times dP\right) \\ &\times \int_0^t H^4_{L^2}(X_n(s),X(s))ds \end{aligned}$$

and

$$\begin{aligned} H^4_{L^2}(X_n(t),X(t)) \leqslant\ & 32\tilde{T}^2\left[\int_{J\times\Omega} H^2_{\mathbb{R}^d}(F_n(s,X(s)),F(s,X(s)))ds\times dP\right]^2 \\ &+ 32\left[\int_{J\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X(s)),G(s,X(s)))ds\times dP\right]^2 \\ &+ \left(32\tilde{T}^2 S_F + 32 S_G\right)\int_0^t H^4_{L^2}(X_n(s),X(s))ds. \end{aligned}$$

Thus by the Gronwall inequality for every $t\in J$

$$\begin{aligned} H^4_{L^2}(X_n(t),X(t)) \leqslant\ & \Bigg(32\tilde{T}^2\left[\int_{J\times\Omega} H^2_{\mathbb{R}^d}(F_n(s,X(s)),F(s,X(s)))ds\times dP\right]^2 \\ &+ 32\left[\int_{J\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X(s)),G(s,X(s)))ds\times dP\right]^2\Bigg) \\ &\times\ \exp\left\{32t(\tilde{T}^2 S_F + S_G)\right\}. \end{aligned}$$

Consequently

$$\begin{aligned}\sup_{t\in J} H_{L^2}(X_n(t),X(t)) &\leqslant \Bigg(32\tilde{T}^2\left[\int_{J\times\Omega} H^2_{\mathbb{R}^d}(F_n(s,X(s)),F(s,X(s)))ds\times dP\right]^2\\ &\quad + 32\left[\int_{J\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X(s)),G(s,X(s)))ds\times dP\right]^2\Bigg)^{1/4}\\ &\quad\times\ \exp\left\{8\tilde{T}(\tilde{T}^2S_F+S_G)\right\}.\end{aligned}$$

Since the both the sequences of numbers $\int_{J\times\Omega} H^2_{\mathbb{R}^d}(F_n(s,X(s)),F(s,X(s)))ds\times dP$ and $\int_{J\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X(s)),G(s,X(s)))ds\times dP$ converge to zero by assumptions, we infer that $\sup_{t\in J} H_{L^2}(X_n(t),X(t))$ converges to zero as well. ☐

Although the condition (H4) can be thought as a constricting one, it is crucial and indispensable in the studies of narrowing set-valued stochastic integral equations. The difficulties are caused by a requirement of existence of the Hukuhara differences. This condition is satisfied immediately in the case of single-valued and singleton defined data of equation (3.2). In this special case, the Hukuhara differences in (3.3) and (H4) reduce to the usual differences in the space L^2. But this is not the only case when (H4) is seen to be fulfilled. Notice that $\mathbb{R}$ can be embedded into $L^2(\Omega,\mathcal{A},P;\mathbb{R})$. Hence the following deterministic, narrowing set-valued integral equation

$$X(t)+\int_0^t(-1)\Psi(s,X(s))ds = X_0 \quad \text{for } t\in I, \tag{3.9}$$

where $\Psi\colon I\times\mathcal{K}^b_c(\mathbb{R})\to\mathcal{K}^b_c(\mathbb{R})$, $X_0\in\mathcal{K}^b_c(\mathbb{R})$ and the integral is the set-valued Aumann integral, is a particular case of equation (3.2). The theory of deterministic, widening set-valued equations is examined widely in [25]. We shall show that for equation (3.9) the condition of type (H4) is satisfied. Assume that there exists a positive constant M such that for any $(t,A)\in I\times\mathcal{K}^b_c(\mathbb{R})$ it holds $H_{\mathbb{R}}(\Psi(t,A),\{0\})\leqslant M$. We claim that the sequence $\{X_n\}$ described by $X_0(t)=X_0$ and $X_n(t)=X_0\ominus\int_0^t(-1)\Psi(s,X_{n-1}(s))ds$ (for $n\in\mathbb{N}$) is well defined on the interval $J=[0,\tilde{T}]$, where $\tilde{T}=\operatorname{diam}X_0/(2M)$. Indeed, for $t\in J$ we have

$$\begin{aligned}\operatorname{diam}\left((-1)\int_0^t\Psi(s,X_{n-1}(s))ds\right) &= \operatorname{diam}\left(\int_0^t\Psi(s,X_{n-1}(s))ds\right)\\ &\leqslant 2\int_0^t H_{\mathbb{R}}(\Psi(s,X_{n-1}(s)),\{0\})ds\\ &\leqslant 2Mt\leqslant \operatorname{diam}X_0.\end{aligned}$$

In the hyperspace $\mathcal{K}^b_c(\mathbb{R})$ we have: if $\operatorname{diam}A\geqslant\operatorname{diam}B$ then $A\ominus B$ exists, where $A,B\in\mathcal{K}^b_c(\mathbb{R})$. Hence the Hukuhara difference $X_0\ominus\int_0^t(-1)\Psi(s,X_{n-1}(s))ds$ exists for each $t\in J$ and condition of the type (H4) is fulfilled. The equations of the type (3.9) are well suited (see [30]) in modeling a problem of number of radioactive nuclei in radioactive substances.

The studies presented in this paper treat of the narrowing set-valued stochastic integral equations (3.2) mainly. It is worth mentioning that under conditions (H1)–(H3) (without (H4)), all the above presented assertions can be repeated for the widening set-valued stochastic integral equations (3.1). Moreover, since (3.1) does not involve any condition on existence of Hukuhara differences, each result established for the (local or global) solutions of the narrowing equations can be repeated for global solutions of the widening equations. At this place we rewrite only one and the most important result on the existence and uniqueness of solution to (3.1). All the remaining counterparts of the results can also be proved.

Proposition 3.8. *Let* $X_0 \in \mathcal{K}^b_c(L^2_0)$*, and* $F, G\colon I \times \Omega \times \mathcal{K}^b_c(L^2) \to \mathcal{K}^b_c(\mathbb{R}^d)$ *satisfy conditions* (H1)–(H3)*. Then equation* (3.1) *possesses a unique global solution* $X\colon I \to \mathcal{K}^b_c(L^2)$*.*

This assertion can be proved using the sequence of the approximate solutions $\{X_n\}$ defined as

$$X_0(t) = X_0, \ \ t \in I,$$

and for $n = 1, 2, \ldots$

$$X_n(t) = X_0 + (S)\int_0^t F(s, X_{n-1}(s))ds + (S)\int_0^t G(s, X_{n-1}(s))dB(s), \ \ t \in I.$$

The condition (H4) is not needed in Proposition 3.8 and does not apply to the remaining results which can be repeated for the widening set-valued stochastic integral equations with global solutions.

REFERENCES

[1] R. P. Agarwal and D. O'Regan, Existence for set differential equations via multivalued operator equations, *Differential Equations and Applications*, 5:1–5, Nova Sci. Publ., New York, 2007.

[2] B. Ahmad and S. Sivasundaram, Setvalued perturbed hybrid integro-differential equations and stability in terms of two measures, *Dynam. Systems Appl.*, 16:299–310, 2007.

[3] B. Ahmad and S. Sivasundaram, Basic results and stability criteria for set valued differential equations on time scales, *Commun. Appl. Anal.*, 11:419–427, 2007.

[4] N. U. Ahmed, Optimal choice of nonlinear output feedback control law for a class of uncertain parabolic systems, *Dynam. Systems Appl.*, 17:571–582, 2008.

[5] N. U. Ahmed, Existence of optimal output feedback control law for a class of uncertain infinite dimensional stochastic systems: a direct approach, *Evol. Equ. Control Theory*, 1:235–250, 2012.

[6] N. U. Ahmed, Optimal feedback control for uncertain infinite dimensional dynamic systems determined by differential inclusions, *Nonlinear Stud.*, 20:139–147, 2013.

[7] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Basel, 1990.

[8] J. Bao, C. Zhou and C. Gao, Strict stability of impulsive set valued differential equations, *Ann. Differential Equations*, 27:127–131, 2011.

[9] F. S. de Blasi and F. Iervolino, Equazioni differenziali con soluzioni a valore compatto convesso, *Boll. Un. Mat. Ital.*, 2:491–501, 1969.

[10] F. S. de Blasi and F. Iervolino, Euler method for differential equations with set-valued solutions, *Boll. Un. Mat. Ital.* 4:941–949, 1971.

[11] B. Bede and S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets Syst.*, 151:581–599, 2005.

[12] A. J. Brandao Lopes Pinto, F. S. de Blasi and F. Iervolino, Uniqueness and existence theorems for differential equations with compact convex valued solutions, *Boll. Un. Mat. Ital.*, 3:47–54, 1970.

[13] R. S. Burachik and A. N. Iusem, *Set-Valued Mappings and Enlargements of Monotone Operators*, Springer Optimization and Its Applications, Vol. 8, Springer, 2008.

[14] J. V. Devi and Ch. Naidu, Boundedness results for impulsive set differential equations involving causal operators with memory, *Commun. Appl. Anal.*, 17:9–19, 2013.

[15] J. V. Devi and A. S. Vatsala, Monotone iterative technique for impulsive set differential equations, *Nonlinear Stud.*, 11:639–658, 2004.

[16] Z. Drici, F. A. Mcrae and J. V. Devi, Stability results for set differential equations with causal maps, *Dynam. Systems Appl.*, 15:451–463, 2006.

[17] T. Gnana Bhaskar and V. Lakshmikantham, Lyapunov stability for set differential equations, *Dynam. Systems Appl.*, 13:1–10, 2004.

[18] T. Gnana Bhaskar, V. Lakshmikantham and J. V. Devi, Nonlinear variation of parameters formula for set differential equations in a metric space, *Nonlinear Anal.*, 63:735–744, 2005.

[19] T. Gnana Bhaskar and J. V. Devi, Nonuniform stability and boundedness criteria for set differential equations, *Appl. Anal.*, 84:131–143, 2005.

[20] S. Hong, J. Gao and Y. Peng, Solvability and stability of impulsive set dynamic equations on time scales, *Abstr. Appl. Anal.*, Article ID 610365, 19 pages, 2014.

[21] S. Hong and J. Liu, Phase spaces and periodic solutions of set functional dynamic equations with infinite delay, *Nonlinear Anal.*, 74:2966–2984, 2011.

[22] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis, Vol. I: Theory*, Kluwer Academic Publishers, Boston 1997.

[23] J. Jiang, C. F. Li and H. T. Chen, Existence of solutions for set differential equations involving causal operator with memory in Banach space, *J. Appl. Math. Comput.*, 41:183–196, 2013.

[24] K. Kuratowski and Cz. Ryll-Nardzewski, A general theorem on selectors, *Bull. Polon. Acad. Sci.*, 13:397–403, 1965.

[25] V. Lakshmikantham, T. Gnana Bhaskar and J. Vasundhara Devi, *Theory of Set Differential Equations in Metric Spaces*, Cambridge Scientific Publ., Cambridge, 2006.

[26] V. Lakshmikantham and R. N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor & Francis, London 2003.

[27] V. Lakshmikantham and A. S. Vatsala, Set differential equations and monotone flows, *Nonlinear Dyn. Syst. Theory*, 3:151–161, 2003.

[28] V. Lupulescu, Hukuhara differentiability of interval-valued functions and interval differential equations on time scales, *Inform. Sci.*, 248:50–67, 2013.

[29] M. T. Malinowski, Interval differential equations with a second type Hukuhara derivative, *Appl. Math. Lett.*, 24:2118–2123, 2011.

[30] M. T. Malinowski, Interval Cauchy problem with a second type Hukuhara derivative, *Inform. Sci.*, 213:94–105, 2012.

[31] M. T. Malinowski, On set differential equations in Banach spaces - a second type Hukuhara differentiability approach, *Appl. Math. Comput.*, 219:289–305, 2012.

[32] M. T. Malinowski, Second type Hukuhara differentiable solutions to the delay set-valued differential equations, *Appl. Math. Comput.*, 218:9427–9437, 2012.

[33] M. T. Malinowski, Strong solutions to stochastic fuzzy differential equations of Itô type, *Math. Comput. Modelling*, 55:918–928, 2012.

[34] M. T. Malinowski, Itô type stochastic fuzzy differential equations with delay, *Syst. Control Lett.*, 61:692–701, 2012.

[35] M. T. Malinowski, Approximation schemes for fuzzy stochastic integral equations, *Appl. Math. Comput.*, 219:11278–11290, 2013.

[36] M. T. Malinowski, Some properties of strong solutions to stochastic fuzzy differential equations, *Inform. Sci.*, 252:62–80, 2013.

[37] M. T. Malinowski, On a new set-valued stochastic integral with respect to semimartingales and its applications, *J. Math. Anal. Appl.*, 408:669–680, 2013.

[38] M. T. Malinowski, Set-valued and fuzzy stochastic integral equations driven by semimartingales under Osgood condition, *Open Math.*, 13:106–134, 2015.

[39] M. T. Malinowski, Fuzzy and set-valued stochastic differential equations with local Lipschitz condition, *IEEE Trans. Fuzzy Syst.*, DOI: 10.1109/TFUZZ.2014.2372072, in press, 2015.

[40] M. T. Malinowski and R.P. Agarwal, On solutions to set-valued and fuzzy stochastic differential equations, *J. Franklin Inst.* 352:3014–3043, 2015.

[41] I. Tişe, Data dependence of the solutions for set differential equations, *Carpathian J. Math.*, 23:192–195, 2007.

[42] P. Wang and W. Gao, Quasilinearization of an initial value problem for a set valued integro-differential equation, *Comput. Math. Appl.*, 61:2111–2115, 2011.

[43] P. Wang and W. Sun, Practical stability in terms of two measures for set differential equations on time scales, *Scientific World Journal*, Article ID 241034, 2014.