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ON q-STURM LIOUVILLE OPERATORS WITH EIGENVALUE PARAMETER CONTAINED IN THE BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study dissipative q-Sturm-Liouville operators with eigenvalue parameter contained in the boundary conditions by using Krein's theorem. We proved a theorem on completeness of the system of eigenvectors and associated vectors of the dissipative q-Sturm-Liouville operators.

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1. INTRODUCTION

Studying q-difference calculus (or quantum calculus) has been an area of a great interest since the beginning of the 19th century. Quantum calculus is ordinary calculus without limits. By doing so, we enlarge the class of functions that we deal with, by considering non-differentiable functions and apply quantum derivatives on them [43]. The subject of q-difference equations have evolved into a multidisciplinary subject [9]. There are several physical models involving q-difference and their related problems (see [2], [8], [10], [11], [14], [23]). In [1], Adıvar and Bohner investigated the eigenvalues and the spectral singularities of non-selfadjoint q-difference equations of second order with spectral singularities. Eryılmaz [6] studied q-Sturm-Liouville boundary value problem in the Hilbert space with a spectral parameter in the boundary condition and he proved theorems on the completeness of the system of eigenvalues and eigenvectors of operator by Pavlov's method. In [24], Tuna and Eryılmaz proved a theorem on completeness of the system of eigenfunctions and associated functions of dissipative q-Sturm-Liouville operators by using the Lidskii's theorem. Eryılmaz and Tuna [7] proved theorems on the completeness of the system of eigenvalues and eigenvectors of the maximal dissipative q-Sturm-Liouville difference operator by using a

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functional model of a nonself-adjoint operator. In [18], Huseynov and Bairamov examined the properties of eigenvalues and eigenvectors of a quadratic pencil of q-difference equations. In [2], [3], Annaby and Mansour studied a q-analogue of Sturm-Liouville eigenvalue problems and formulated a self-adjoint q-difference operator in a Hilbert space. They also discussed properties of the eigenvalues and the eigenfunctions. We refer the reader to consult the reference [3], [19].

The study of problems consisting of parameter dependent systems is of great interest to a lot of numerous problems in physics and engineering. A boundary value problem with a spectral parameter in the boundary condition appears commonly in mathematical models of mechanic. It occurs whenever the separation of variables method (Fourier method) is used to solve the proper partial differential equation with boundary conditions having a directional derivative [31]. There are a lot of studies about parameter dependent problems [27]–[42].

In this paper, we construct a dissipative q-difference operator in the Hilbert space $L_q^2(0,a)$ with a spectral parameter in the boundary condition. Later, we prove that the system of all eigenvectors and associated vectors of the dissipative q-difference operator is complete in the space $L_q^2(0,a)$ by using Krein's theorems.

The organization of this document is as follows: In Section 2, some preliminary concepts related to q-calculus and essentials of Krein's theorems are presented for the convenience of the reader. In Section 3, we construct the q-Sturm-Liouville operator. Finally, we prove a theorem on the completeness of the system of eigenvectors and associated vectors of dissipative operators in Section 4.

2. PRELIMINARIES

In this section, we provide some basic definitions and theorems on q-derivative, q-integration, q-exponential function, q-trigonometric function, q-Taylor formula, q-Beta and Gamma functions, Euler-Maclaurin formula, etc which are useful in the following discussion. These definitions and theorems can be found in [2], [3], [5], [6], [7], [19] and the references therein.

Definition 2.1. Let q be a positive number with $0 < q < 1$, $A \subset \mathbb{R}$and $a \in \mathbb{C}$. A q-difference equation is an equation that contains q-derivatives of a function defined on A. Let $y(x)$ be a complex-valued function on $x \in A$. The q-difference operator D_q is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{\mu(x)}, \text{ for all } x \in A,$$

where $\mu(x) = (q-1)x$.

Definition 2.2. The q-derivative at zero is defined by

$$D_q y(0) = \lim_{n\to\infty} \frac{y(q^n x) - y(0)}{q^n x}, \ x \in A,$$

if the limit exists and does not depend on x.

Definition 2.3. A right inverse to D_q, the Jackson q-integration is given by

$$\int_0^x f(t)\, d_q t = x(1-q)\sum_{n=0}^{\infty} q^n f(q^n x), \ x \in A,$$

provided that the series converges, and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t, \quad a, b \in A.$$

Definition 2.4. An operator A is called dissipative if $Im(Ax, x) \geq 0$, $(\forall x \in D(A))$. A bounded operator is dissipative if and only if

$$ImA = \frac{1}{2i}(A - A^*) \geq 0.$$

Definition 2.5. Let $L_q^2(0, a)$ be the space of all complex-valued functions defined on $[0, a]$ such that

$$\|f\| := \left(\int_0^a |f(x)|\, d_q x\right)^{1/2} < \infty.$$

The space $L_q^2(0, a)$ is a separable Hilbert space with the inner product

$$(f, g) := \int_0^a f(x)\, \overline{g(x)} d_q x, \quad f, g \in L_q^2(0, a).$$

Now, we present some definitions and facts from the theory of operators (for more complete information see [12]). Let A denote a linear non-selfadjoint operator in the Hilbert space with domain $D(A)$. A complex number λ_0 is called an eigenvalue of the operator A if there exists a non-zero element $y_0 \in D(A)$ such that $Ay_0 = \lambda_0 y_0$; in this case, y_0 is called the eigenvector of A for λ_0. The eigenvectors for λ_0 span a subspace of $D(A)$, called the eigenspace for λ_0.

The element $y \in D(A)$, $y \neq 0$ is called a root vector of A corresponding to the eigenvalue λ_0 if $(T - \lambda_0 I)^n y = 0$ for some $n \in \mathbb{N}$. The root vectors for λ_0 span a linear subspace of $D(A)$, is called the root lineal for λ_0. The algebraic multiplicity of λ_0 is the dimension of its root lineal. A root vector is called an associated vector if it is not an eigenvector. The completeness of the system of all eigenvectors and associated vectors of A is equivalent to the completeness of the system of all root vectors of this operator.

An operator A is called dissipative if $Im(Ax, x) \geq 0$,for all $x \in D(A)$. A bounded operator is dissipative if and only if

$$ImA = \frac{1}{2i}(A - A^*) \geq 0.$$

Let A be an arbitrary compact operator acting in the Hilbert space H. Let $\{\mu_j(A)\}_{j\in\mathbb{N}}$ be a sequence of all nonzero eigenvalues of A arranged by considering algebraic multiplicity and with decreasing modulus, and $\nu(A)$ $(\leq \infty)$ is a sum of algebraic multiplicities of all nonzero eigenvalues of A. If A is a nuclear operator, then $\sum_{j=1}^{\nu(A)} |\mu_j(A)| < +\infty$and if A is a Hilbert-Schmidt operator, then $\sum_{j=1}^{\nu(A)} |\mu_j(A)|^2 < +\infty$. We will denote the class of all nuclear and Hilbert-Schmidt operators in H by σ_1 and σ_2, respectively.

Theorem 2.6 ([12]). *The system of root vectors of a compact dissipative operator B with nuclear imaginary component is complete in the Hilbert space H so long as at least one of the following two conditions is fulfilled:*

$$\lim_{\rho\to\infty} \frac{n_+(\rho, B_R)}{\rho} = 0, \ \ or \ \lim_{\rho\to\infty} \frac{n_-(\rho, B_R)}{\rho} = 0,$$

where $n_+(\rho, B_R)$ and $n_-(\rho, B_R)$ denote the numbers of the characteristic values of the real component B_R of the operator B in the intervals $[0,\rho]$ and $[-\rho, 0]$, respectively.

Definition 2.7 ([12]). Let f be an entire function. If for each $\varepsilon > 0$ there exists a finite constant $C_\varepsilon > 0$, such that

$$|f(\lambda)| \leq C_\varepsilon e^{\varepsilon|\lambda|}, \quad \lambda \in \mathbb{C} \tag{2.1}$$

then f is called an entire function of order ≤ 1 of growth and minimal type.

Theorem 2.8 ([20]). *If the entire function f satisfies the condition* (2.1)*, then*

$$\lim_{\rho\to\infty} \frac{n_+(\rho, f)}{\rho} = \lim_{\rho\to\infty} \frac{n_-(\rho, f)}{\rho} = 0$$

where $n_+(\rho, f)$ and $n_-(\rho, f)$ denote the numbers of the zeros of the function f in the intervals $[0,\rho]$ and $[-\rho, 0]$, respectively.

3. CONSTRUCTION OF THE q-STURM-LIOUVILLE OPERATOR

In this section, we construct the q-Sturm-Liouville operator. Let us consider the q-Sturm-Liouville equation

$$l(y) := -\frac{1}{q} D_{q^{-1}} D_q y(x) + v(x) y(x), \quad 0 \leq x \leq a < +\infty, \tag{3.1}$$

where $v(x)$ is defined on $[0,a]$ and it is continuous at zero. The q-Wronskian of $y_1(x)$, $y_2(x)$ is defined to be

$$W_q(y_1, y_2)(x) := y_1(x) D_q y_2(x) - y_2(x) D_q y_1(x), \quad x \in [0,a].$$

Let L_0 denote the closure of the minimal operator generated by (3.1) and by D_0 its domain. We also denote by D the set of all functions $y(x)$ from $L_q^2(0,a)$ such that $y(x)$ and $D_q y(x)$ are continuous in $[0,a)$ and $l(y) \in L_q^2(0,a)$; D is the domain of the maximal operator L. Furthermore $L = L_0^*$ (see [22]). Suppose that the operator L_0 has defect index $(2,2)$.

For every $y, z \in D$ we have q-Lagrange's identity ([2])

$$(Ly, z) - (y, Lz) = [y, \overline{z}]\,(a) - [y, \overline{z}]\,(0)$$

where $[y, \overline{z}] := y\,(x)\,\overline{D_{q^{-1}} z\,(x)} - D_{q^{-1}} y\,(x)\,\overline{z\,(x)}$.

Let us denote by $u\,(x, \lambda)$, $v\,(x, \lambda)$ the solutions of the equation $l\,(y) = \lambda y$ satisfying the initial conditions

$$\begin{aligned} u\,(0, \lambda) &= \cos\alpha, \quad D_{q^{-1}} u\,(0, \lambda) = \sin\alpha, \\ v\,(0, \lambda) &= -\sin\alpha, \quad D_{q^{-1}} v\,(0, \lambda) = \cos\alpha, \end{aligned}$$

where $\alpha \in \mathbb{R}$. The solutions $u\,(x, \lambda)$ and $v\,(x, \lambda)$ form a fundamental system of solutions of $l\,(y) = \lambda y$ and they are entire functions of λ (see [2]). Let $u\,(x) = u\,(x, 0)$ and $v\,(x) = v\,(x, 0)$ the solutions of the equation $l\,(y) = 0$ satisfying the initial conditions

$$\begin{aligned} u\,(0) &= \cos\alpha, \quad D_{q^{-1}} u\,(0) = \sin\alpha, \\ v\,(0) &= -\sin\alpha, \quad D_{q^{-1}} v\,(0) = \cos\alpha. \end{aligned}$$

Now consider boundary value problem governed by

$$l(y) = \lambda y, \quad y \in D, \tag{3.2}$$

subject to the boundary conditions

$$[y, u]_a - h[y, v]_a = 0, \ Imh > 0 \tag{3.3}$$

$$\alpha_1 y\,(0) - \alpha_2 D_{q^{-1}} y\,(0) = \lambda(\alpha_1' y\,(0) - \alpha_2' D_{q^{-1}} y\,(0)), \tag{3.4}$$

where λ is spectral parameter and $\alpha_1, \alpha_2, \alpha_1', \alpha_2' \in \mathbb{R}$ and α is defined by

$$\alpha := \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} = \alpha_1' \alpha_2 - \alpha_1 \alpha_2' > 0. \tag{3.5}$$

For simplicity we assume that

$$\begin{aligned} R_0(y) &: = \alpha_1 y\,(0) - \alpha_2 D_{q^{-1}} y\,(0), \\ R_0'(y) &: = \alpha_1' y\,(0) - \alpha_2' D_{q^{-1}} y\,(0), \\ B_1^a(y) &: = [y, u]_a, \\ B_2^a(y) &: = [y, v]_a, \\ B_1^0(y) &: = y\,(0), \\ N_2^0(y) &: = D_{q^{-1}} y\,(0)_\infty, \end{aligned}$$

$$R_a(y) \ : \ = B_2^a(y) - hB_1^a(y).$$

Lemma 3.1. *For arbitrary $y, z \in D$, we have*

$$R_0(\overline{z}) = \overline{R_0(z)}, \ R_0^{'}(\overline{z}) = \overline{R_0^{'}(z)}$$

and

$$[y,z]_0 = \frac{1}{\alpha}\left[R_0(y)\overline{R_0^{'}(z)} - R_0^{'}(y)\overline{R_0(z)}\right]. \tag{3.6}$$

Proof.

$$\begin{aligned}
&\frac{1}{\alpha}\left[R_0(y)\overline{R_0^{'}(z)} - R_0^{'}(y)\overline{R_0(z)}\right] \\
&= \frac{1}{\alpha}\left[\begin{array}{c} (\alpha_1 y(0) - \alpha_2 D_{q^{-1}} y(0))\,\overline{\left(\alpha_1^{'} z(0) - \alpha_2^{'} D_{q^{-1}} z(0)\right)} \\ -\left(\alpha_1^{'} y(0) - \alpha_2^{'} D_{q^{-1}} y(0)\right)\overline{(\alpha_1 z(0) - \alpha_2 D_{q^{-1}} z(0))} \end{array}\right] \\
&= \frac{1}{\alpha}\left[\left(\alpha_1^{'}\alpha_2 - \alpha_1\alpha_2^{'}\right)\left(\overline{y(0)} D_{q^{-1}} z(0) - D_{q^{-1}} y(0)\,\overline{z(0)}\right)\right] \\
&= [y,z]_0.
\end{aligned}$$

□

4. COMPLETENESS OF THE SYSTEM OF ROOT VECTORS OF L

Now, we introduce a special inner product in the Hilbert space $H = L_q^2(0,a) \oplus \mathbb{C}$ which is suitable for boundary value problem that has been defined.

Let $\widehat{f} = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}$, $\widehat{g} = \begin{pmatrix} g^{(1)} \\ g^{(2)} \end{pmatrix} \in H$ and $\alpha > 0$. Then the formula

$$\left(\widehat{f}, \widehat{g}\right) = \int_0^a f^{(1)}(x)\,\overline{g}^{(1)}(x)\,d_q x + \frac{1}{\alpha} f^{(2)}\overline{g}^{(2)}$$

defines an inner product in Hilbert space H. We construct the operator $A : H \longrightarrow H$ with domain $D(A)$ consisting of all vectors $\widehat{f} = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix} \in H$ such that $f^{(1)} \in D$, $R_a\left(f^{(1)}\right) = 0$, $f^{(2)} = R_0^{'}\left(f^{(1)}\right)$. We define the operator A on $D(A)$ by the formula

$$A\widehat{f} = \begin{pmatrix} l\left(f^{(1)}\right) \\ R_0\left(f^{(1)}\right) \end{pmatrix}.$$

Theorem 4.1. *The operator A is dissipative in the space H.*

Proof. Let $\widehat{y} \in D(A)$. From Equation (3.6), we have

$$\begin{aligned}
(A\widehat{y}, \widehat{y}) - (\widehat{y}, A\widehat{y}) &= [y_1, \overline{y_1}]_a - [y_1, \overline{y_1}]_0 + \frac{1}{\alpha}[R_0(y_1)\overline{R_0^{'}(y_1)} - R_0^{'}(y_1)\overline{R_0(y_1)}] \\
&= [y_1, \overline{y_1}]_a = B_1^a(y_1)\,B_2^a(\overline{y_1}) - B_2^a(y_1)\,B_1^a(\overline{y_1}) \\
&= 2Imh\,(B_2^a(y_1))^2.
\end{aligned}$$

It follows from that

$$Im(A\widehat{y},\widehat{y}) = Imh\left(B_2^a\left(y_1\right)\right)^2 \geq 0, \tag{4.1}$$

A is a dissipative operator in H. □

It follows from Theorem 3 that all the eigenvalues of A lie in the closed upper halfplane $Im\lambda \geq 0$.

Theorem 4.2. *The operator A has not any real eigenvalue.*

Proof. Suppose that the operator A has a real eigenvalue λ_0. Let $u_0\left(x\right) = u\left(x,\lambda_0\right)$ be the corresponding eigenfunction. Since $Im\left(Au_0,u_0\right)) = Im\left(\lambda_0\left\|u_0\right\|^2\right)$, we get from (4.1) that $[u_0,v]_a = 0$. By the boundary condition (3.3), we have $[u_0,u]_a = 0$. Thus

$$\left[u_0\left(x,\lambda_0\right),u\right]_a = \left[u_0\left(x,\lambda_0\right),v\right]_a = 0. \tag{4.2}$$

By $v_0\left(x\right) = v\left(x,\lambda_0\right)$, we know that

$$[u_0,v_0]_a[u,v]_a = [u_0,u]_a[v_0,v]_a - [u_0,v]_a[v_0,u]_a.$$

By the equality (4.2), the right-hand side is equal to 0. But

$$\begin{aligned} W_q(u,v)\left(\frac{a}{q}\right) &= [u,v]_a = 1, \\ W_q(u_0,v_0)\left(\frac{a}{q}\right) &= [u_0,v_0]_a = 1. \end{aligned}$$

This contradiction proves the theorem. □

Let $\theta_1\left(x,\lambda\right) = v\left(x,\lambda\right)$, $\theta_2\left(x,\lambda\right) = u\left(x,\lambda\right) - hv\left(x,\lambda\right)$ denote the solutions of equation $l\left(y\right) - \lambda y = f\left(x\right)$ satisfying the conditions

$$\begin{aligned} B_1^0(\theta_2) &= \alpha_2 - \alpha_2^{'}\lambda, \quad B_2^0(\theta_2) = \alpha_1 - \alpha_1^{'}\lambda \\ B_1^a(\theta_1) &= h, \quad B_2^a(\theta_1) = 1 \end{aligned}$$

These functions belong to the space $L_q^2\left(0,a\right)$. Their Wronskian $W_q\left(\theta_1,\theta_2\right) = -1$. Then by (3.6), we have

$$\begin{aligned} \Delta\left(\lambda\right) &= [\theta_1,\theta_2]_x = -[\theta_2,\theta_1]_x = -[\theta_2,\theta_1]_0 \\ &= -\frac{1}{\alpha}\left[R_0(\theta_1)\overline{R_0^{'}(\theta_2)} - R_0^{'}(\theta_1)\overline{R_0(\theta_2)}\right] \\ &= R_0(\theta_2) - \lambda R_0^{'}(\theta_2), \end{aligned}$$

and

$$\begin{aligned} \Delta\left(\lambda\right) &= [\theta_1,\theta_2]_x = -[\theta_2,\theta_1]_x = -[\theta_2,\theta_1]_a \\ &= -B_1^a(\theta_2)B_2^a(\theta_1) + B_1^a(\theta_1)B_2^a(\theta_2) \\ &= -\left(B_2^a(\theta_2) - hB_1^a(\theta_2)\right) \end{aligned}$$

$$= \quad -R_a(\theta_2).$$

Now, we define

$$G(x,\xi,\lambda) = \frac{-1}{\Delta(\lambda)} \left\{ \begin{array}{ll} \theta_2(\xi,\lambda)\,\theta_1(x,\lambda), & x<\xi \\ \theta_1(x,\lambda)\,\theta_2(\xi,\lambda), & \xi<x \end{array} \right\}. \tag{4.3}$$

It can be shown that $G(x,\xi,\lambda)$ satisfies equation (3.2) and boundary conditions (3.3)–(3.4). $G(x,\xi,\lambda)$ is a Green function of the boundary value problem (equations (3.2)–(3.4)).

The integral operator K defined by the formula

$$Kf = \left(\begin{array}{c} \left(\widetilde{G}_{x,\lambda}, f\right) \\ R_0' \left[\left(\widetilde{G}_{x,\lambda}, f\right) \right] \end{array} \right), \quad \widetilde{G}_{x,\lambda} = \left(\begin{array}{c} G(x,\xi,\lambda) \\ R_0' \left[G(x,\xi,\lambda) \right] \end{array} \right), \quad \left(f \in L_q^2(0,a)\right) \tag{4.4}$$

is a compact linear operator in the space $L_q^2(0,a)$. K is a Hilbert Schmidt operator. It is evident that $K = (A-\lambda I)^{-1}$. Consequently the root lineals of the operator $A-\lambda I$ and K coincide and, therefore, the completeness in $L_q^2(0,a)$ of the system of all eigenvectors and associated vectors of $A-\lambda I$ is equivalent to the completeness of those for K. Since the algebraic multiplicity of nonzero eigenvalues of a compact operator is finite, each eigenvector of $A-\lambda I$ may have only a finite number of linear independent associated vectors. On the other hand, the root lineals of the operator $A-\lambda I$ and A also coincide (see [26]). Then the completeness in $L_q^2(0,a)$ of the system of all eigenvectors and associated vectors of A is equivalent to the completeness of those for K.

Let

$$\begin{aligned} \tau_1(\lambda) &:= [\varphi(x,\lambda), u(x,\lambda)]_a, \\ \tau_2(\lambda) &:= [\varphi(x,\lambda), v(x,\lambda)]_a, \\ \tau(\lambda) &:= \tau_1(\lambda) - h\tau_2(\lambda). \end{aligned}$$

It is clear that

$$\sigma_p(A) = \{\lambda : \lambda \in \mathbb{C},\ \tau(\lambda) = 0\}$$

where $\sigma_p(A)$ denotes the set of all eigenvalues of A. Since $\varphi(a,\lambda)$ and $D_{q^{-1}}\varphi(a,\lambda)$ are entire functions of λ of order ≤ 1 (see [1]), consequently, $\tau(\lambda)$ have the same property. Then $\tau(\lambda)$ is entire functions of the order ≤ 1 of growth, and of minimal type.

Theorem 4.3. *The system of all root vectors of the dissipative operator K is complete in H.*

Proof. It will be sufficient to prove that the system of all root vectors of the operator $K = (A-\lambda I)^{-1}$ in (4.4) is complete in H. Since $\theta(x,\lambda) = u(x,\lambda) - hv(x,\lambda)$, setting

$h = h_1 + ih_2$ $(h_1, h_2 \in \mathbb{R})$, we get from (4.4) in view of (4.3) that $K = K_1 + iK_2$, where

$$K_1 f = \begin{pmatrix} \left(\widetilde{G}_{1x,\lambda}, f\right) \\ R_0' \left[\left(\widetilde{G}_{1x,\lambda}, f\right)\right] \end{pmatrix}, \quad \widetilde{G}_{1x,\lambda} = \begin{pmatrix} G_1(x,\xi,\lambda) \\ R_0'\left[G_1(x,\xi,\lambda)\right] \end{pmatrix},$$

$$K_2 f = \begin{pmatrix} \left(\widetilde{G}_{2x,\lambda}, f\right) \\ R_0' \left[\left(\widetilde{G}_{2x,\lambda}, f\right)\right] \end{pmatrix}, \quad \widetilde{G}_{2x,\lambda} = \begin{pmatrix} G_2(x,\xi,\lambda) \\ R_0'\left[G_2(x,\xi,\lambda)\right] \end{pmatrix},$$

and

$$G_1(x,\xi,\lambda) = \frac{-1}{\Delta(\lambda)} \begin{cases} v(x,\lambda)\left[u(\xi) - h_1 v(\xi)\right], & 0 \le x \le t \le a \\ v(\xi,\lambda)\left[u(x,\lambda) - h_1 v(x,\lambda)\right], & 0 \le t \le x \le a \end{cases},$$

$$G_2(x,\xi,\lambda) = \frac{1}{\Delta(\lambda)} h_2 v(x,\lambda)\, v(\xi,\lambda), \; h_2 = Imh > 0.$$

□

The operator K_1 is the self-adjoint Hilbert–Schmidt operator in H, and K_2 is the self-adjoint one dimensional operator in H.

Let A_1 denote the operator in H generated by the differential expression l and the boundary conditions

$$\begin{aligned} \alpha_1 y(0) - \alpha_2 D_{q^{-1}} y(0) &= \lambda(\alpha_1' y(0) - \alpha_2' D_{q^{-1}} y(0)), \; \alpha_1, \alpha_2, \alpha_1', \alpha_2' \in \mathbb{R}, \\ [y,u]_a - h_1 [y,v]_a &= 0, \; h_1 = Reh. \end{aligned}$$

It is easy to verify that K_1 is the inverse A_1. Further

$$\sigma_p(A_1) = \{\lambda : \lambda \in \mathbb{C}, \; \Psi(\lambda) = 0\} \tag{4.5}$$

where

$$\Psi(\lambda) := \tau_1(\lambda) - h_1 \tau_2(\lambda). \tag{4.6}$$

Then we find that

$$|\Psi(\lambda)| \le C_\varepsilon e^{\varepsilon|\lambda|}, \quad \forall \lambda \in \mathbb{C}. \tag{4.7}$$

Let $T = -K$ and $T = T_1 + iT_2$, where $T_1 = -K_1$, $T_2 = -K_2$. The characteristic values of the operator K_1 coincide with the eigenvalues of the operator A_1. From (4.5), (4.7) and Theorem 2, we have

$$\lim_{\rho \to \infty} \frac{m_+(\rho, T_1)}{\rho} = 0, \text{ or } \lim_{\rho \to \infty} \frac{m_-(\rho, T_1)}{\rho} = 0,$$

where $m_+(\rho, T_1)$ and $m_-(\rho, T_1)$ denote the numbers of the characteristic values of the real component $T_R = T_1$ in the intervals $[0, \rho]$ and $[-\rho, 0]$, respectively. Thus the dissipative operator T (also of K) carries out all the conditions of Krein's thorem on completeness. This completes the proof.

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