Dynamic Systems and Applications 24 (2015) 503-522

THE MOTSCH-TADMOR MODEL WITH MULTIPLICATIVE WHITE NOISES IN FLOCKS

YI LIU AND XIAOPING XUE

Department of Mathematics, Harbin Institute of Technology
Harbin, 150001, China; and Department of Mathematics, Suihua University,
Suihua, 152000, China
Department of Mathematics, Harbin Institute of Technology
Harbin, 150001, China

ABSTRACT. We enlarge the range of critical exponent β of the communication rate for unconditional flocking in the model proposed by Motsch, S. and Tadmor, E. [*J. Statist. Phys.,* **141** (2011), 923–947], and describe an asymmetric stochastic model which emphasizes the asymmetric interaction between agents and employs the multiplicative white noise for the stochastic forces acting on ith-agent. For the case of asymmetric communication rate, we present sufficient conditions to guarantee the strong stochastic flocking to occur and show the almost sure exponential convergence toward constant equilibrium state through the control of the parameters and initial data. Our results are illustrated through the numerical simulations.

AMS (MOS) Subject Classification. 39A10.

1. INTRODUCTION

Emergent collection behaviors such as flocking, consensus, and synchronization are ubiquitous phenomena that often found in a group of autonomous agents such as species of fishes, birds, bacteria, etc. [1, 3, 4, 5, 7, 11]. The terminology 11flocking" represents phenomena in which self-propelled particles using only limited environmental information and simple rules organize into an ordered motion. In recent years, many models have been introduced to appraise the emergent behavior of self-organized systems. The starting point for our discussion is the pioneering work of F. Cucker, S. Smale [5, 6], S. Y. Ha [7, 8], S. Motsch and E. Tadmor [11], which led to many subsequent studies.

1.1. **Motsch-Tadmor model.** In the recent paper, Motsch and Tadmor [11] introduced a new model for self-organized dynamics which addressed several drawbacks of the celebrated CS model. They argued that if a small group of individuals are located far away from a much larger group of individuals, the internal dynamics in the small

Received March 30, 2015

group is almost halted since the number of individuals is large. Their model can be written as

$$\begin{cases} dx_i = v_i dt \\ dv_i = \alpha \sum_{j\neq i}^{N} a_{ij}(v_j - v_i)dt \end{cases} \tag{1.1}$$

where α is a positive constant and the coefficients a_{ij}, given by,

$$a_{ij} = \frac{\phi(|x_j - x_i|)}{\sum_{k=1}^{N} \phi(|x_k - x_i|)} > 0,$$

while $\phi(r) = (1 + r)^{-2\beta}, \beta > 0$ is the influence function. The system (1.1) can be rewritten in the form

$$\begin{cases} dx_i = v_i dt \\ dv_i = \alpha(\overline{v}_i - v_i)dt \end{cases} \tag{1.2}$$

where $\overline{v}_i = \sum_{j=1}^{N} a_{ij}v_j$, $\sum_{j=1}^{N} a_{ij} = 1$.

They developed a new framework to analyze the phenomenon of flocking for a rather general class of dynamical systems of the form. The paper utilized the concept of active sets which enable to define the notion of a neighborhood of the an agent and were able to find explicit criteria for the unconditional emergence of a flock. In particular, they derived a sufficient condition for flocking of their proposed model: flocking occurs independent of the initial configuration, when the interaction function ϕ decayed sufficiently slowly so that its tail is not square integrable (i.e. $\int^{\infty} \phi^2(r)dr = \infty$).

1.2. **The stochastic CS models.** Recently several mathematical models for flocking were introduced and analyzed [5, 6, 20, 21, 22]. Among them, their main interest in their papers is the work of Cucker and Smale [5, 6]. But the general CS model does not take into account any interactions between the particles system and the environment. One possible way of modeling such interactions is to add noises terms to the deterministic dynamical system. Therefore after Cucker and Smale's seminal works, several extension of the CS model with general communication rates and external forces have been addressed in many papers [4, 7, 8]. In [7] and [8] the main interest is to investigate how additive white noises (1.3) and multiplicative white noises (1.4) affect the long-time dynamics of the CS flocking model. These two kinds of stochastic CS models read as:

$$\begin{cases} dx_i = v_i dt, \ t > 0 \\ dv_i = \frac{\lambda}{N} \sum_{j=1}^{N} \psi_{ji}(v_j - v_i)dt + \sqrt{D}dW_i, \ 1 \leq i \leq N. \end{cases} \tag{1.3}$$

and

$$(1.4)\qquad \begin{cases} dx_i = v_i dt, \ t > 0 \\ dv_i = \frac{\lambda}{N}\sum_{j=1}^{N} \psi_{ji}(v_j - v_i)dt + Dg_i(v)dW_t, \ 1 \leq i \leq N. \end{cases}$$

where λ is a positive coupling strength and D is a non-negative noise strength. In system (1.3), the noise term dW_i is i.i.d. and d-dimensional white noise characterized by mean zero and the following covariance relation: for $1 \leq \alpha, \beta \leq d$, $1 \leq i, j \leq N$,

$$E(dW_i^{\alpha}(t)) = 0, \ V(dW_i^{\alpha}(t)dW_j^{\beta}(t_*)) = \delta_{\alpha\beta}\delta_{ij}\delta(t - t_*).$$

While in system (1.4), $W(t)$ is the one-dimensional Brownian motion. The white noise $dW(t)$ is characterized by mean zero and its covariance relation,

$$E(dW(t)) = 0, \ V(dW(t)dW(t_*)) = \delta(t - t_*).$$

where δ_{ij} is the Dirac delta function, and

$$g_i(v) = v_i - v_e, \ v_e : a \text{ constant state in } \mathcal{R}^d.$$

Flocking in above two systems were studied in two setups: a constant communication rate $\psi = 1$ and a radially symmetric rate $\psi(|x_i - x_j|^2)$. For additive white noises in the system (1.3), when the communicate rate between the particles was assumed to be constant, the system exhibited a flocking behavior that is independent of the initial configuration and in the radially symmetric communication rate case the system showed that the relative fluctuations of the particle velocity around the mean velocity have a uniformly bounded variance in time by adding a lower bound assumption, but there existed an unconditional strong flocking in the above two cases of communication rates for multiplicative white noises in the system (1.4).

1.3. **An asymmetric stochastic model.** However for the biological groups such as birds, fishes, ants, etc., the asymmetric communication weight is more realistic. Inspired by the above papers, we present an asymmetric stochastic model. Our model emphasizes the asymmetric interaction between agents and employs multiplicative white noises for the stochastic forces acting on ith-agent. We not only modify drawback produced by symmetric interaction between agents but also consider the interactions with their neighboring environment such as fluids, external forcing, stochastic noises, etc. Let $(x_i(t), v_i(t)) \in \mathcal{R}^{2d}$ represent the position and velocity of particle i, so we present model which can be written as follow:

$$(1.5)\qquad \begin{cases} dx_i = v_i dt, \\ dv_i = \alpha \sum_{j=1}^{N} a_{ij}(v_j - v_i)dt + Dv_i dW_t, \end{cases}$$

subject to deterministic initial data

$$(x_i, v_i)(0) = (x_{i0}, v_{i0}), \ i = 1, 2, \ldots, N.$$

Here $a_{ij} \neq a_{ji}$, lack the symmetry property, and $Dv_i dW_t$ show the influence between agents and environment with multiplicative white noises.

The purpose of this paper is to study the time-asymptotic flocking for this model. Our starting point is to revisit the definition of strong flocking which was established in the previous context once. The rest of the paper is organized as follows. In section 2 we first improve the flocking estimates for general symmetric communication rates in [11] and then introduce an asymmetric stochastic model. For the asymmetric communication rate between agents, we explicitly obtain all statistical quantities about the random dynamical system which leads to time-asymptotic *strong* flocking. [See Definition 2.7]. The main result is summarized in Theorem 2.13. In the end of Sec. 2, we prove that the mean velocity of each agent is convergent when $t \to +\infty$. Numerical simulations for the cases studied in this work are shown and discussed in section 3. We end section 4 with concluding remarks. For reader's convenience, we shall briefly introduce the mathematical definition of Brownian motions and the relation between white noise and Brownian motions.

Definition 1.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A (standard) one-dimensional Brownian motions is a real-valued continuous $\{\mathcal{F}_t\}$-adapted process $\{W_t(\omega)\}_{t\geq 0}$ with the following properties:

1. $W_0 = 0$ a.s.;
2. for $0 \leq s < t < \infty$, the increment $W_t(\omega) - W_s(\omega)$ is normally distributed with mean zero and variance $t - s$;
3. for $0 \leq s < t < \infty$, the increment $W_t(\omega) - W_s(\omega)$ is independent of $\{\mathcal{F}_s\}$.

Definition 1.2. The stochastic process $X_t(\omega)$ is white noise process to satisfy the following properties:

1. $X_{t_1}(\omega)$ and $X_{t_2}(\omega)$ are independent if $t_1 \neq t_2$;
2. $\{X_t(\omega)\}$ is stationary, i.e. the (joint) distribution of $\{X_{t_1+t}(\omega), \ldots, X_{t_k+t}(\omega)\}$ does not depend on t;
3. $E(X_t(\omega)) = 0$ for all t.

From above definitions, we known that so-called *white noise* is formally regard as the derivative of a Brownian motion $W_t(\omega)$, i.e. $X_t(\omega) = dW_t(\omega)$. For notational convenience, we omit variable "ω". For any random variable, $X_t(\omega)$ means X_t as long as there is no confusion. In this paper, $\mathcal{R}^d$ represents Euclidean spaces, $\langle \cdot, \cdot \rangle$ represents inner product, and $|\cdot|$ represents Euclidean norm.

2. THE MOTSCH-TADMOR MODEL WITH NOISES

2.1. **Critical exponent for unconditional flocking without noise.** In this section, we want to improve the flocking estimate for general symmetric communication rates in [11] and prove that there will exist unconditional flocking for slowly decaying influence function as long as $\int^\infty \phi(r)dr = \infty$ compared to $\int^\infty \phi^2(r)dr = \infty$ in [11]. In other words we amplify communication rate $\phi(\cdot)$ with parameter β from $\frac{1}{4}$ to $\frac{1}{2}$.

Definition 2.1. Let $(x_i(t), v_i(t))_{i=1,2,\dots,N}$ be a given particles system, and let $A_x(t)$ and $A_v(t)$ denote its diameters in position and velocity phase spaces, that is

$$A_x(t) = \max_{1\le i\ne j\le N} |x_j(t) - x_i(t)|,$$

$$A_v(t) = \max_{1\le i\ne j\le N} |v_j(t) - v_i(t)|.$$

Lemma 2.2. *Let (x_i, v_i) be a solution of the dynamical system* (1.1)*. Then,*

$$\langle v_i - v_j, \overline{v}_i - \overline{v}_j\rangle \le \left(1 - \frac{\phi(A_x(t))}{N}\right) A_v^2(t).$$

Proof. Using system (1.1) we have

$$\begin{aligned}
\overline{v}_i - \overline{v}_j &= \sum_{p=1}^N a_{ip} v_p - \overline{v}_j = \sum_{p=1}^N a_{ip}(v_p - \overline{v}_j) \\
&= \sum_{p=1}^N a_{ip}(v_p - \sum_{q=1}^N a_{jq} v_q) = \sum_{p=1}^N a_{ip}\left(\sum_{q=1}^N a_{jq} v_p - \sum_{q=1}^N a_{jq} v_q\right) \\
&= \sum_{p=1}^N a_{ip} \sum_{q=1}^N a_{jq}(v_p - v_q).
\end{aligned}$$

Hence

$$\begin{aligned}
\langle v_i - v_j, \overline{v}_i - \overline{v}_j\rangle &= \langle v_i - v_j, \sum_{p,q=1}^N a_{ip} a_{jq}(v_p - v_q)\rangle \\
&= \sum_{p,q=1}^N a_{ip} a_{jq} \langle v_i - v_j, v_p - v_q\rangle \\
&= \sum_{p=1}^N a_{ip} \sum_{q\ne p}^N a_{jq} \langle v_i - v_j, v_p - v_q\rangle.
\end{aligned}$$

Since

$$\sum_{j=1}^N a_{ij} = 1,\ a_{ij} = \frac{\phi(|x_j - x_i|)}{\sum_{k=1}^N \phi(|x_k - x_i|)} \ge \frac{\phi(A_x(t))}{N}.$$

We get

$$\begin{aligned}
\langle v_i - v_j, \overline{v}_i - \overline{v}_j\rangle &= \sum_{p=1}^N a_{ip} \sum_{q\ne p}^N a_{jq} \langle v_i - v_j, v_p - v_q\rangle \\
&= \sum_{p=1}^N a_{ip}(1 - a_{jp}) \langle v_i - v_j, v_p - v_q\rangle
\end{aligned}$$

$$\leq \left(1 - \frac{\phi(A_x(t))}{N}\right) A_v^2(t).$$

□

Lemma 2.3. *Let* (x_i, v_i) *be a solution of the dynamical system* (1.1). *Then the diameters of this solution* $A_x(t)$ *and* $A_v(t)$ *satisfy,*

$$\begin{cases} \frac{d}{dt} A_x(t) \leq A_v(t), \\ \frac{d}{dt} A_v(t) \leq -\frac{\alpha}{N}\phi(A_x(t)) A_v(t). \end{cases} \tag{2.1}$$

Proof. We fix our attention to two trajectories $x_p(t)$ and $x_q(t)$, where p and q will be determined later.

$$\frac{d}{dt}|x_p - x_q|^2 = 2\langle x_p - x_q, \dot{x}_p - \dot{x}_q\rangle \leq 2|x_p - x_q| \cdot |v_p - v_q|,$$

which implies

$$\frac{d}{dt}|x_p - x_q| \leq A_v(t).$$

Let $|x_p - x_q| = A_x(t)$, therefore we get

$$\frac{d}{dt} A_x(t) \leq A_v(t).$$

Next, we turn to study the corresponding to relative speeds in velocity phase space, and use Lemma 2.2 we have

$$\begin{aligned} \frac{d}{dt}|v_p - v_q|^2 &= 2\alpha\langle v_p - v_q, \overline{v}_p - \overline{v}_q\rangle - 2\alpha|v_p - v_q|^2 \\ &= 2\alpha \sum_{j=1}^{N} \sum_{i=1}^{N} a_{pj} a_{qi} \langle v_p - v_q, v_j - v_i\rangle - 2\alpha|v_p - v_q|^2 \\ &\leq 2\alpha A_v^2(t) \sum_{j=1}^{N} a_{pj}(1 - a_{qj}) - 2\alpha|v_p - v_q|^2 \\ &\leq 2\alpha A_v^2(t) \left(1 - \frac{\phi(A_x(t))}{N}\right) - 2\alpha|v_p - v_q|^2. \end{aligned}$$

In particular, if we choose p and q such that $|v_p - v_q| = A_v(t)$, that is

$$\frac{d}{dt} A_v(t) \leq -\frac{\alpha}{N}\phi(A_x(t)) A_v(t).$$

□

Theorem 2.4. *Consider the diameters* $A_x(t)$ *and* $A_v(t)$ *governed by the inequalities* (2.1) *in the system* (1.1), *where* $\phi(r) = (1 + r)^{-2\beta}$, $\beta > 0$ *is its influence function such that*

$$A_v(0) < \frac{\alpha}{N} \int_{A_x(0)}^{\infty} \phi(r) dr. \tag{2.2}$$

Then the underlying dynamical system (1.1) *exhibits a flocking behavior, that is*

$$\sup_{0\le t<+\infty} A_x(t) < +\infty \textit{ and } \lim_{t\to+\infty} A_v(t) = 0.$$

In particular, if $\phi(r)$ has a diverging tail,

$$\int_{A_x(0)}^{\infty} \phi(r)dr = \infty.$$

Then there is unconditional flocking.

Proof. Let

$$\xi(A_x, A_v)(t) = A_v(t) + \frac{\alpha}{N}\int_{A_x(0)}^{A_x(t)} \phi(s)ds.$$

The energy function ξ is decreasing along the trajectory (A_x, A_v),

$$\begin{aligned}\frac{d}{dt}\xi(A_x, A_v) &= \frac{d}{dt}A_v + \frac{\alpha}{N}\phi(A_x(t))\cdot\frac{d}{dt}A_x \\ &\le -\frac{\alpha}{N}\phi(A_x(t))A_v(t) + \frac{\alpha}{N}\phi(A_x(t))A_v(t) = 0,\end{aligned}$$

and we deduce that,

$$A_v(t) - A_v(0) \le -\frac{\alpha}{N}\int_{A_x(0)}^{A_x(t)} \phi(s)ds.$$

By our assumption (2.2), there exists $A_* > A_x(0)$, such that

$$A_v(0) = \frac{\alpha}{N}\int_{A_x(0)}^{A_*} \phi(s)ds,$$

and the above inequality now reads,

$$A_v(t) \le \frac{\alpha}{N}\int_{A_x(0)}^{A_*} \phi(s)ds - \frac{\alpha}{N}\int_{A_x(0)}^{A_x(t)} \phi(s)ds = \frac{\alpha}{N}\int_{A_x(t)}^{A_*} \phi(s)ds.$$

Since $A_v(t) \ge 0$, we conclude that we have a flock with a uniformly bounded diameter,

$$A_x(t) \le A_* \quad \text{for all } t \ge 0.$$

Hence we obtain

$$\frac{d}{dt}A_v(t) \le -\frac{\alpha}{N}\phi_* A_v(t) = -\frac{\alpha}{N}\phi(A_*)A_v(t), \quad \phi_* := \min_{0\le r\le A_*}\phi(r) = \phi(A_*),$$

and Gronwall inequality proves that $A_v(t)$ converges exponentially fast to zero. So if

$$\int_{A_x(0)}^{\infty} \phi(r)dr = \infty.$$

there is unconditional flocking. □

Remark 2.5. There was an example in [6] showed that when $\beta > \frac{1}{2}$ convergence is guaranteed under some condition on initial positions and velocities of agents only. Next we will present a model incorporated with multiplicative white noises and will find that there exists an unconditional flocking for all $\beta \ge 0$.

2.2. **The Motsch-Tadmor model with noises.** Now we present an asymmetric stochastic flocking model in which we assume that particles interact with the environment via stochastic noise and the normalization of pairwise interaction between agents in terms of relative influence has the consequence of loss of symmetry. Let $(x_i(t), v_i(t)) \in \mathcal{R}^{2d}$ represent the position and velocity of particle i. The asymmetric stochastic dynamics is then governed by the following process:

$$\begin{cases} dx_i = v_i dt, \\ dv_i = \alpha \sum_{j=1}^{N} a_{ij}(v_j - v_i)dt + Dv_i dW_t \end{cases} \tag{2.3}$$

where α is a nonnegative constant and the coefficients a_{ij}, given by,

$$a_{ij} = \frac{\phi(|x_j - x_i|)}{\sum_{k=1}^{N} \phi(|x_k - x_i|)},$$

lack the symmetry property, $a_{ij} \neq a_{ji}$. In this model, we employ multiplicative white noises for the stochastic forces acting on ith-agent,

$$Dv_i dW_t,$$

where D is a non-negative constant proportional to noise strength and W_t is the one-dimensional Brownian motion. The white noise dW_t is characterized by mean zero and its covariance relation,

$$E(dW_t) = 0,\ V(dW_t dW_{t_*}) = \delta(t - t_*)$$

where δ is the Dirac delta function, and subject to deterministic initial data

$$(x_i, v_i)(0) = (x_{i0}, v_{i0}),\ i = 1, 2, \ldots, N.$$

Remark 2.6. When noise terms are turned off, i.e. $D = 0$, the above definition exactly coincides with the definition of asymptotic flocking in the deterministic case.

In this section, we discuss the tools to study the flocking behavior for a rather general class of dynamical systems of the form. The model (2.3) can be written as,

$$\begin{cases} dx_i = v_i dt, \\ dv_i = \alpha(\overline{v}_i - v_i)dt + Dv_i dW_t. \end{cases} \tag{2.4}$$

where $\overline{v}_i = \sum_{j=1}^{N} a_{ij} v_j$, $\sum_{j=1}^{N} a_{ij} = 1$.

Definition 2.7. The stochastic system (2.3) has an asymptotic strong stochastic flocking if and only if the position-velocity process (x_i, v_i), $i = 1, 2, \ldots, N$, satisfy the following two conditions: for $1 \leq i, j \leq N$,

(1) the differences of all pairwise velocity processes go to zero asymptotically,

$$\lim_{t \to +\infty} |v_i(t) - v_j(t)| = 0.\ a.s.$$

(2) the diameter of a group is uniformly bounded in time t,

$$\sup_{0\le t<+\infty} |x_i(t)-x_j(t)| < +\infty. \ a.s.$$

Remark 2.8. The system $(x_i(t), v_i(t))_{i=1,2,\dots,N}$ is said to converge to a flock if the following two conditions hold,

$$\sup_{0\le t<+\infty} A_x(t) < +\infty \ and \ \lim_{t\to+\infty} A_v(t) = 0. \ a.s.$$

Next we will give the main results in our paper.

2.2.1. *A two-particle system.* We give here a more detailed analysis of the case of two agents on a line and the system read as

$$\begin{cases} dx_1 = v_1 dt, dx_2 = v_2 dt, \\ dv_1 = \alpha a_{12}(v_2 - v_1)dt + Dv_1 dW_t, \\ dv_2 = \alpha a_{21}(v_1 - v_2)dt + Dv_2 dW_t. \end{cases} \tag{2.5}$$

We define $x = x_1 - x_2$ and $v = v_1 - v_2$. Then x and v satisfy

$$\begin{cases} dx = vdt, \\ dv = -\alpha(a_{12} + a_{21})vdt + DvdW_t. \end{cases} \tag{2.6}$$

Theorem 2.9. *Assume that $\alpha < 0$ (repulsive coupling) and $D^2 > -2\alpha \max\{a_{12}, a_{21}\}$, let (x, v) be the solution to the system* (2.6). *Then*

$$\lim_{t\to+\infty} |v(t)| = 0. \ a.s.$$

Proof. We apply Itô's formula to obtain

$$|v(t)| = |v_0| e^{-\frac{1}{2}\int_0^t (D^2+\alpha(a_{12}+a_{21}))ds + DW_t}.$$

Since $D^2 > -2\alpha \max\{a_{12}, a_{21}\}$, there exists $c_1 > 0$ to satisfy

$$D^2 + \alpha(a_{12} + a_{21}) > D^2 + 2\alpha \max\{a_{12}, a_{21}\} \ge c_1 > 0.$$

We use

$$\limsup_{t\to\infty} \frac{|W(t)|}{\sqrt{2t\log\log t}} = 1. \ a.s.$$

So there exist constants c_2 and $T(D)$ such that

$$-\frac{1}{2}c_1 t + DW_t \le -c_2 t, \ (t \ge T(D)).$$

Thus we have

$$\begin{aligned} |v(t)| &= |v_0| e^{-\frac{1}{2}\int_0^t (D^2+\alpha(a_{12}+a_{21}))ds + DW_t} \\ &\le |v_0| e^{(-\frac{1}{2}c_1 t + DW_t)} \\ &\le |v_0| e^{-c_2 t}, \ (t \ge T(D)). \end{aligned}$$

That is

$$\lim_{t\to+\infty} |v(t)| = 0. \ a.s.$$

□

Remark 2.10. Even if there exists the repulsive coupling($\alpha < 0$), we still have flocking as long as repulsive coupling is sufficiently weak, or the noise strength is sufficiently strong in the stochastic asymmetric model with multiplicative white noises.

2.2.2. *Multiple particles system.* In the following, we study the asymmetric stochastic system for general multi-agent system and find the dynamics of our proposed model experience unconditional flocking with the influence of multiplicative white noises.

Lemma 2.11. *Let* (x_i, v_i) *be the solution to the system* (2.3)*. Then* $(A_x(t), A_v(t))$ *satisfy*

$$\begin{cases} dA_x^2(t) \le 2 \cdot A_x(t) \cdot A_v(t)dt, \\ dA_v^2(t) \le (-2\alpha \cdot \frac{\phi(A_x(t))}{N} + D^2)A_v^2(t)dt + 2DA_v^2(t)dW_t. \end{cases}$$

Proof. Fix i and j which will be determined later.
(1) We use Itô's formula $dt \cdot dt = 0$, $dt \cdot dW_t = 0$, $dW_t \cdot dW_t = dt$ and Cauchy-Schwartz inequality to obtain

$$\begin{aligned} d|x_i(t) - x_j(t)|^2 &= 2\langle x_i - x_j, dx_i - dx_j\rangle + \frac{1}{2} \cdot 2\langle dx_i - dx_j, dx_i - dx_j\rangle \\ &= 2\langle x_i - x_j, v_i - v_j\rangle dt + \langle v_i - v_j, v_i - v_j\rangle dt \cdot dt \\ &= 2\langle x_i - x_j, v_i - v_j\rangle dt \\ &\le 2 \cdot |x_i - x_j| \cdot |v_i - v_j| dt \\ &\le 2A_x(t) \cdot A_v(t)dt. \end{aligned}$$

Since i and j are arbitrary, we have

$$dA_x^2(t) \le 2 \cdot A_x(t) \cdot A_v(t)dt.$$

(2) We choose i and j randomly,

$$(*) \qquad d|v_i - v_j|^2 = 2\langle v_i - v_j, dv_i - dv_j\rangle + \frac{1}{2} \cdot 2\langle dv_i - dv_j, dv_i - dv_j\rangle.$$

The second term on the right-hand of the formula $(*)$ can be treated, and use Itô's formula $dt \cdot dt = 0$, $dt \cdot dW_t = 0$ and $dW_t \cdot dW_t = dt$ to find

$$\begin{aligned} &\langle dv_i - dv_j, dv_i - dv_j\rangle \\ &= \langle \alpha(\overline{v}_i - v_i)dt + Dv_i dW_t - \alpha(\overline{v}_j - v_j)dt - Dv_j dW_t, \alpha(\overline{v}_i - v_i)dt + Dv_i dW_t \\ &\quad - \alpha(\overline{v}_j - v_j)dt - Dv_j dW_t\rangle \\ &= \langle \alpha(\overline{v}_i - v_i - \overline{v}_j + v_j)dt + D(v_i - v_j)dW_t, \alpha(\overline{v}_i - v_i - \overline{v}_j + v_j)dt + D(v_i - v_j)dW_t\rangle \end{aligned}$$

$$
\begin{aligned}
&= \alpha^2 \langle \overline{v}_i - v_i - \overline{v}_j + v_j, \overline{v}_i - v_i - \overline{v}_j + v_j \rangle dt \cdot dt \\
&\quad + 2\alpha D \langle \overline{v_i} - v_i - \overline{v}_j + v_j, v_i - v_j \rangle dt \cdot dW_t + D^2 \langle v_i - v_j, v_i - v_j \rangle dW_t \cdot dW_t \\
&= D^2 |v_i - v_j|^2 dt.
\end{aligned}
$$

The first term on the right-hand of the formula $(*)$ is indicated below

$$
\begin{aligned}
&2\langle v_i - v_j, dv_i - dv_j \rangle \\
&\quad = 2\langle v_i - v_j, \alpha(\overline{v}_i - v_i)dt + Dv_i dW_t - \alpha(\overline{v}_j - v_j)dt - Dv_j dW_t \rangle \\
&\quad = 2[\langle v_i - v_j, \alpha(\overline{v}_i - \overline{v}_j) \rangle dt - \alpha \langle v_i - v_j, v_i - v_j \rangle dt + \langle v_i - v_j, D(v_i - v_j) dW_t \rangle] \\
&\quad = 2\alpha \langle v_i - v_j, \overline{v}_i - \overline{v}_j \rangle dt - 2\alpha |v_i - v_j|^2 dt + 2D|v_i - v_j|^2 dW_t.
\end{aligned}
$$

Using Lemma 2.2 we get

$$
\begin{aligned}
2\alpha \langle v_i - v_j, \overline{v}_i - \overline{v}_j \rangle dt &= 2\alpha \sum_{p=1}^{N} a_{ip} \sum_{q \neq p}^{N} a_{jq} \langle v_i - v_j, v_p - v_q \rangle dt \\
&= 2\alpha \sum_{p=1}^{N} a_{ip} (1 - a_{jp}) \langle v_i - v_j, v_p - v_q \rangle dt \\
&\leq 2\alpha \left(1 - \frac{\phi(A_x(t))}{N} \right) A_v^2(t) dt.
\end{aligned}
$$

So

$$
\begin{aligned}
d|v_i - v_j|^2 &= 2\langle v_i - v_j, dv_i - dv_j \rangle + \langle dv_i - dv_j, dv_i - dv_j \rangle \\
&\leq 2\alpha A_v^2(t) \left(1 - \frac{\phi(A_x(t))}{N} \right) dt - 2\alpha |v_i - v_j|^2 dt \\
&\quad + 2D|v_i - v_j|^2 dW_t + D^2 |v_i - v_j|^2 dt.
\end{aligned}
$$

In particular, if we choose i and j such that $|v_i - v_j| = A_v(t)$, we obtain

$$
\begin{aligned}
dA_v^2(t) &\leq 2\alpha A_v^2(t) \left(1 - \frac{\phi(A_x(t))}{N} \right) dt - 2\alpha A_v^2(t) dt + 2D A_v^2(t) dW_t + D^2 A_v^2(t) dt \\
&= -2\alpha \frac{\phi(A_x(t))}{N} A_v^2(t) dt + D^2 A_v^2(t) dt + 2D A_v^2(t) dW_t \\
&= \left(-2\alpha \frac{\phi(A_x(t))}{N} + D^2 \right) A_v^2(t) dt + 2D A_v^2(t) dW_t.
\end{aligned}
$$

□

Lemma 2.12. *Let (x_i, v_i) be the solution of the system* (2.3) *with bounded initial data, then*

$$
A_v^2(t) \leq A_v^2(0) e^{\int_0^t -(2\alpha \frac{\phi(A_x(s))}{N} + D^2) ds + 2DW_t}. \tag{2.7}
$$

Proof. Without loss of generality, let i and j be two indices of agents maximizing the distance between velocities at time t, (i.e. such that $|v_i - v_j| = A_v(t)$).

$$\begin{aligned} d|v_i - v_j|^2 &= 2\langle v_i - v_j, dv_i - dv_j\rangle + \frac{1}{2}\cdot 2\langle dv_i - dv_j, dv_i - dv_j\rangle \\ &= 2\alpha\langle v_i - v_j, \overline{v}_i - \overline{v}_j\rangle dt - 2\alpha|v_i - v_j|^2 dt \\ &\quad + 2D|v_i - v_j|^2 dW_t + D^2|v_i - v_j|^2 dt. \end{aligned}$$

So we obtain

$$dA_v^2(t)\cdot dA_v^2(t) = d|v_i - v_j|^2 \cdot d|v_i - v_j|^2 = 4D^2|v_i - v_j|^4 = 4D^2 A_v^4(t)dt. \tag{2.8}$$

We use Lemma 2.11 and the above equation (2.8) to find

$$\begin{aligned} dlnA_v^2(t) &= \frac{1}{A_v^2(t)}\cdot dA_v^2(t) - \frac{1}{2A_v^4(t)}\, dA_v^2(t)\cdot dA_v^2(t) \\ &\le \frac{1}{A_v^2(t)}\left[\left(-2\alpha\frac{\phi(A_x(t))}{N} + D^2\right)A_v^2(t)dt + 2DA_v^2(t)dW_t\right] \\ &\quad - \frac{1}{2A_v^4(t)}\cdot 4D^2A_v^4(t)dt \\ &= \left(-2\alpha\frac{\phi(A_x(t))}{N} + D^2\right)dt + 2DdW_t - 2D^2dt \\ &= -\left(2\alpha\frac{\phi(A_x(t))}{N} + D^2\right)dt + 2DdW_t. \end{aligned}$$

We integrate the above relation to find the below desired result

$$A_v^2(t) \le A_v^2(0)\cdot e^{-\int_0^t (2\alpha\frac{\phi(A_x(s))}{N} + D^2)ds + 2DW_t}.$$

□

In the next theorem, we can see that there is an unconditional flocking behavior of particles for the stochastic model with the asymmetric communication rate function and multiplicative white noises.

Theorem 2.13. *Let* (x_i, v_i) *be the solution of the system* (2.3)*. Then strong stochastic flocking occurs asymptotically: for* $1 \le i, j \le N$*, we have*

(1) the differences of all pairwise velocity processes go to zero asymptotically,

$$\lim_{t\to+\infty}|v_i(t) - v_j(t)| = 0.\ a.s.$$

(2) the diameter of a group is uniformly bounded in time t*,*

$$\sup_{0\le t<+\infty}|x_i(t) - x_j(t)| < +\infty.\ a.s.$$

Proof. (1) We use Lemma 2.12 to find

$$\begin{aligned} |v_i - v_j| &\le A_v(t) \\ &\le A_v(0)e^{\int_0^t -(\alpha\frac{\phi(A_x(s))}{N} + \frac{D^2}{2})ds + DW_t} \end{aligned}$$

$$= A_v(0)e^{\int_0^t -\alpha\frac{\phi(A_x(s))}{N}ds-\frac{D^2}{2}t+DW_t}.$$

Since

$$\limsup_{t\to\infty}\frac{|W(t)|}{\sqrt{2t\log\log t}} = 1. \ a.s. \tag{2.9}$$

We choose T sufficiently large so that

$$-\frac{D^2}{2}t + DW_t \le -\frac{D^2}{4}t, \ \ t\ge T. \tag{2.10}$$

So we have

$$\lim_{t\to+\infty}|v_i(t)-v_j(t)| = 0. \ a.s.$$

(2) Since

$$dA_x(t)\le A_v(t)dt.$$

We integrate the above relation for $t\ge T$ to find

$$A_x(t)\le A_x(T)+\int_T^t A_v(\tau)d\tau.$$

We use Lemma 2.12 and the inequality (2.10) to find

$$\begin{aligned}
A_x(t) &\le A_x(T)+\int_T^t A_v(\tau)d\tau\\
&\le A_x(T)+\int_T^t A_v(0)e^{-\int_0^\tau(\alpha\frac{\phi(A_x(s))}{N}+\frac{D^2}{2})ds+DW_\tau}d\tau\\
&= A_x(T)+\int_T^t A_v(0)e^{-\int_0^\tau\alpha\frac{\phi(A_x(s))}{N}ds-\frac{D^2}{2}\tau+DW_\tau}d\tau\\
&\le A_x(T)+\int_T^t A_v(0)e^{-\int_0^\tau\alpha\frac{\phi(A_x(s))}{N}ds}\cdot e^{-\frac{D^2}{4}\tau}d\tau\\
&\le A_x(T)+\int_T^t A_v(0)e^{-\frac{D^2}{4}\tau}d\tau\\
&\le A_x(T)+\frac{4}{D^2}\cdot A_v(0)\cdot e^{-\frac{D^2}{4}T}.
\end{aligned}$$

Hence, we have

$$\max_{0\le t<+\infty}A_x(t)\le\max\{\max_{0\le t\le T}A_x(t),\max_{t\ge T}A_x(t)\}<+\infty. \ a.s.$$

That is

$$\sup_{0\le t<+\infty}|x_i(t)-x_j(t)|<+\infty. \ a.s.$$

□

Remark 2.14. The unconditional flocking behavior is verified due to multiplicative white noises for the asymmetric stochastic system in Theorem 2.13. But the issue that whether there exist an unconditional flocking in the asymmetric stochastic system with additive white noises is not known yet.

Lemma 2.15. *Assume that $2\beta \le 1$. Let the diameter $A_v(t)$ of the system* (2.3) *governed by the inequality* (2.7). *Then*

$$\int_0^{+\infty} E(A_v(s))ds < +\infty.$$

Proof. We now use Theorem 2.13 and the property of stochastic differential equation solution to obtain

$$\sup_{0\le t<+\infty} E(A_x(t)) < +\infty.$$

So there exists a random variable $X(\omega)$ that is independent of t so that

$$\sup_{0\le t<+\infty} A_x(t) < X(\omega) \text{ and } E(X(\omega)) < +\infty.$$

Since $e^{-\frac{D^2}{2}t+DW_t}$ is martingale, it has a constant expectation, i.e.,

$$E(e^{-\frac{D^2}{2}t+DW_t}) = e^{-\frac{D^2}{2}\cdot 0+DW_0} = 1.$$

We use the inequality (2.7) and the independence of $\{W_t, t\ge 0\}$ to obtain

$$\begin{aligned} A_v(t) &\le A_v(0)e^{\int_0^t -(\alpha\frac{\phi(A_x(s))}{N}+\frac{D^2}{2})ds+DW_t} \\ &= A_v(0)e^{-\int_0^t \alpha\frac{\phi(A_x(s))}{N}ds-\frac{D^2}{2}t+DW_t} \\ &= A_v(0)e^{-\int_0^t \alpha\frac{\phi(A_x(s))}{N}ds}\cdot e^{-\frac{D^2}{2}t+DW_t}. \end{aligned}$$

Hence

$$\begin{aligned} E(A_v(t)) &\le A_v(0)E(e^{-\int_0^t \alpha\frac{\phi(A_x(s))}{N}ds})\cdot E(e^{-\frac{D^2}{2}t+DW_t}) \\ &= A_v(0)E(e^{-\int_0^t \alpha\frac{\phi(A_x(s))}{N}ds}) \\ &\le A_v(0)E(e^{-\alpha\frac{\phi(X(\omega))}{N}t}). \qquad (2.11) \end{aligned}$$

However,

$$\begin{aligned} E(e^{-\alpha\frac{\phi(X(\omega))}{N}s}) &= \int_\Omega e^{-\alpha\frac{\phi(X(\omega))}{N}s}dP(\omega) \\ &= \sum_{n=0}^{\infty}\int_{\{\omega:n\le X(\omega)<n+1\}} e^{-\alpha\frac{\phi(X(\omega))}{N}s}dP(\omega) \\ &\le \sum_{n=0}^{\infty} e^{-\frac{\alpha}{N}\cdot\frac{s}{(2+n)^{2\beta}}}P(n\le X(\omega)<n+1). \end{aligned}$$

Then

$$\begin{aligned} \int_0^t E(e^{-\alpha\frac{\phi(X(\omega))}{N}s})ds &\le \sum_{n=0}^{\infty}\int_0^t e^{-\frac{\alpha s}{N(2+n)^{2\beta}}}ds\cdot P(n\le X(\omega)<n+1) \\ &= \sum_{n=0}^{\infty}\frac{N}{\alpha}(2+n)^{2\beta}[1-e^{-\frac{\alpha t}{N(2+n)^{2\beta}}}]\cdot P(n\le X(\omega)<n+1) \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=0}^{\infty} \frac{N}{\alpha}(2+n)^{2\beta} P(n \leq X(\omega) < n+1) \\
&\leq \sum_{n=0}^{\infty} \frac{N}{\alpha}(2+n) P(n \leq X(\omega) < n+1) \\
&= \frac{2N}{\alpha} \sum_{n=0}^{\infty} P(n \leq X(\omega) < n+1) + \frac{N}{\alpha} \sum_{n=0}^{\infty} nP(n \leq X(\omega) < n+1) \\
&\leq \frac{N}{\alpha}(2 + E(X(\omega))).
\end{aligned}
$$

We deduce that

$$\int_0^{+\infty} E(e^{-\alpha \frac{\phi(X(\omega))}{N} s}) ds < +\infty.$$

This implies

$$\int_0^{+\infty} E(A_v(s)) ds \leq A_v(0) \int_0^{+\infty} E(e^{-\alpha \frac{\phi(X(\omega))}{N} s}) ds < +\infty.$$

□

Remark 2.16. By the dominated convergence theorem, we get $\lim_{t \to +\infty} E(e^{-\alpha \frac{\phi(X(\omega))}{N} t}) = 0$ from the formula (2.11), that is $\lim_{t \to +\infty} E(A_v(t)) = 0$. This is true for all $\beta \geq 0$.

Theorem 2.17. *Let $2\beta \leq 1$ and v_i be the solution of the system* (2.3). *Then for all $1 \leq i \leq N$, $E(v_i(t))$ is convergent when $t \to +\infty$.*

Proof. We note that the strong solution to (2.3) satisfy Itô's integral representations:

$$v_i = \int_0^t \alpha \sum_{j=1}^{N} a_{ij}(v_j - v_i) ds + \int_0^t D v_i dW_s.$$

Since $E(\int_0^t f(s) dW_s) = 0$, we obtain

$$
\begin{aligned}
E(v_i) &= E\left(\int_0^t \alpha \sum_{j=1}^{N} a_{ij}(v_j - v_i) ds\right) + E\left(\int_0^t D v_i dW_s\right) \\
&= E\left(\int_0^t \alpha \sum_{j=1}^{N} a_{ij}(v_j - v_i) ds\right) \\
&= \int_0^t E\left(\alpha \sum_{j=1}^{N} a_{ij}(v_j - v_i)\right) ds.
\end{aligned}
$$

Therefore for arbitrary $t_1 < t_2$, we have

$$
\begin{aligned}
|E(v_i(t_1)) - E(v_i(t_2))| &= |\int_{t_1}^{t_2} E(\alpha \sum_{j=1}^{N} a_{ij}(v_j - v_i)) ds| \\
&\leq \int_{t_1}^{t_2} |E(\alpha \sum_{j=1}^{N} a_{ij}(v_j - v_i))| ds
\end{aligned}
$$

$$\leq \alpha \int_{t_1}^{t_2} E(A_v(s))ds.$$

By Lemma 2.15 we get

$$\int_{t_1}^{t_2} E(A_v(s))ds \to 0, \ (t_1, t_2 \to +\infty).$$

So $\{E(v_i(t))\}_{t\geq 0}$ is Cauchy sequence. That is $E(v_i(t))$ is convergent when $t \to +\infty$. □

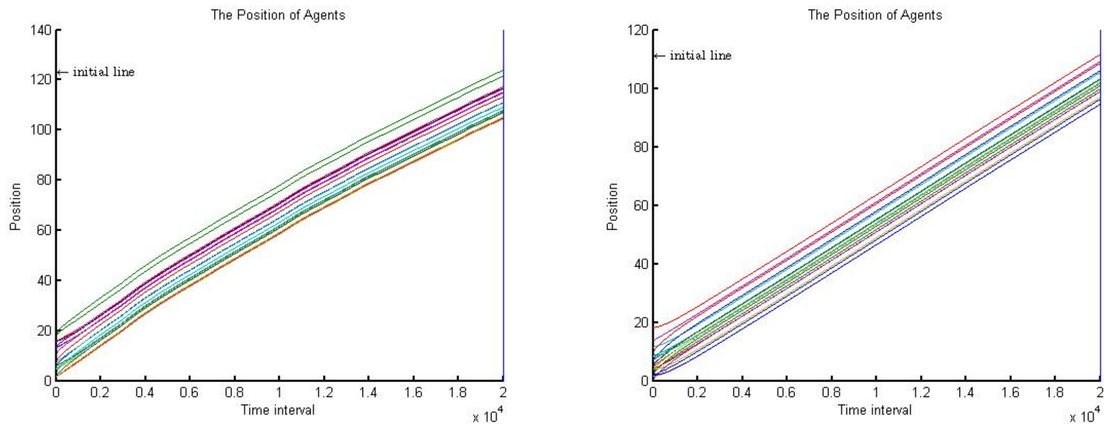
3. **Numerical examples**

In this section we show the result of numerical simulations of the dynamics system (2.3) and compare them with analytical results in Sec. 2. In all the simulations, we take the parameter $N = 20$, $\beta = 0.2$. Let $(x_i, v_i) \in \mathcal{R}^2$ represent the locations and velocities of all agents.

First we choose the different parameter D and the same parameter α: $\alpha = 3.2$; $D_1 = 0.1$, $D_2 = 0.01$. We respectively plot the relation between the time t and the location x_i in Fig. 1 and the relation between the time t and the mean velocity $E(v_i)$ in Fig. 2. We find a clear difference between these two cases of Fig. 2. In the case of $\alpha = 3.2$ and $D = 0.1$, the mean velocities were perturbed strongly compared to the other case of $\alpha = 3.2$ and $D = 0.01$. While in Figs. 3 and 4 we use the different parameter α and same parameter D: $\alpha_1 = 0.8$, $\alpha_2 = 3.2$; $D = 0.01$. In Fig. 3 and Fig. 4 we have plotted the location x_i and the mean velocity $E(v_i)$ versus the time t for the case of $\alpha = 0.8$, $D = 0.01$ and $\alpha = 3.2$, $D = 0.01$ respectively. The curves show that the mean velocity attain consensus faster as the parameter α increases. In particular, Fig. 5 shows the relative velocities of all agents and it is seen that all particles will attain the same velocity after enough time passing. The first case of Fig. 5 shows the influences of the different parameter α on the flocking behavior; we found that the agents flock faster and faster as α increases. In the second case of Fig. 5, our goal is to explore the effect of the noise strength D on the flocking behavior; it shows that the relative velocities were perturbed bigger and bigger as the parameter D increases. All numerical results agree with the analytical results given by Theorem 2.13 and Theorem 2.17.

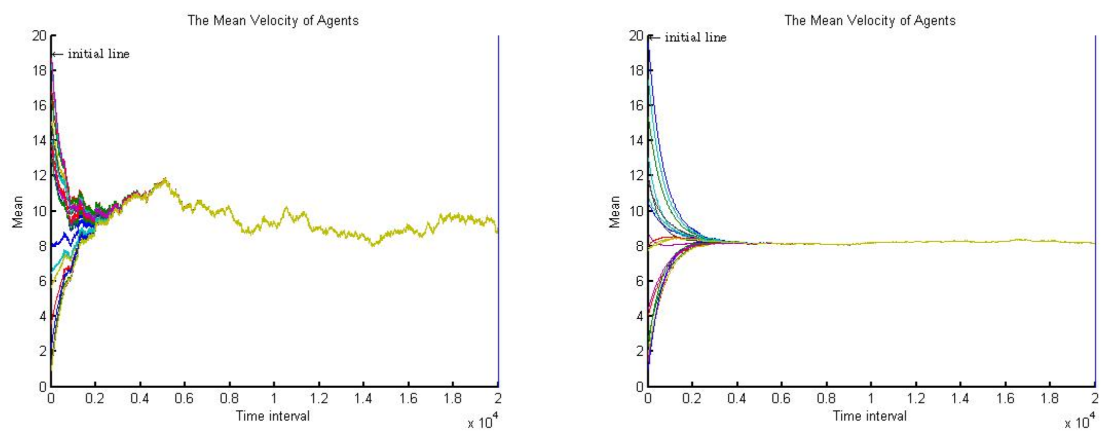
4. **Conclusion**

A mathematical theory on the flocking serves the foundation for several ubiquitous multi-agent in biology, ecology, sensor networks and economics, as well as social behavior like language emergence and evolution. In this paper, We enlarged the range of critical exponent β of the communication rate for unconditional flocking in the model proposed by Motsch, S. and Tadmor, E. [*J. Statist. Phys.*, 141 (2011),



(a) (b)

FIGURE 1. (a) All the realizations of the trajectories of the positions with $\alpha = 3.2$; $D_1 = 0.1$. (b) All the realizations of the trajectories of the positions with $\alpha = 3.2; D_2 = 0.01$.



(a) (b)

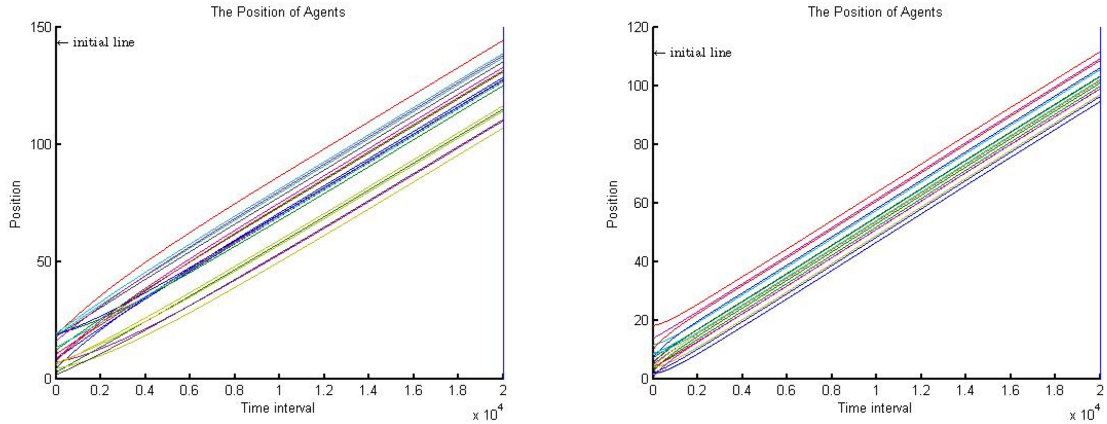
FIGURE 2. (a) The trajectories of the mean speeds of the 20 particles with $\alpha = 3.2$; $D_1 = 0.1$. (b) The trajectories of the mean speeds of the 20 particles with $\alpha = 3.2$; $D_2 = 0.01$.

923–947] and have investigated the emergent flocking behavior of a new asymmetric stochastic model via multiplicative white noises which take the form

$$\begin{cases} dx_i = v_i dt, \\ dv_i = \alpha(\overline{v_i} - v_i)dt + Dv_i dW_t. \end{cases}$$

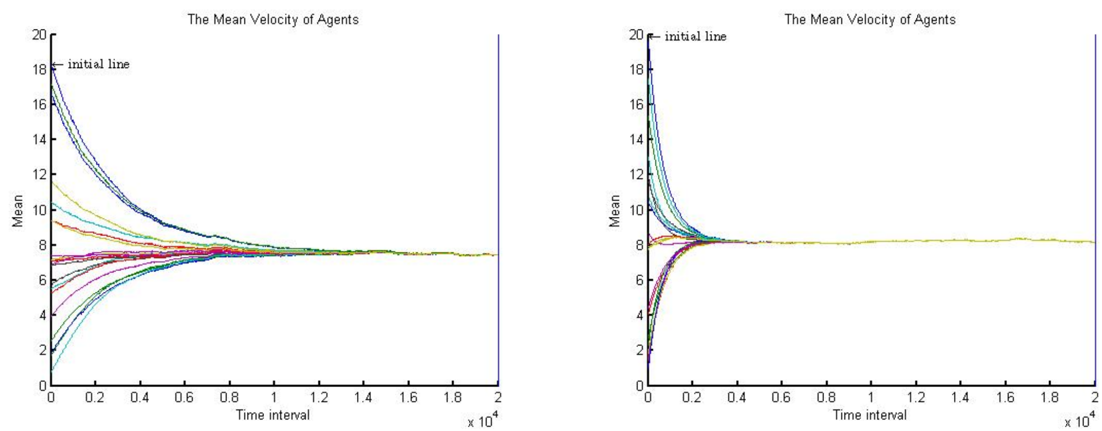
where α is a nonnegative constant and $\overline{v}_i = \sum_{j=1}^{N} a_{ij} v_j$, $\sum_{j=1}^{N} a_{ij} = 1$.

Here we focused our attention on the asymmetry in the improved model where the interactions between agents is governed by the relative distances, which are no longer symmetric and where agents and surroundings interacted with multiplicative white



(a) (b)

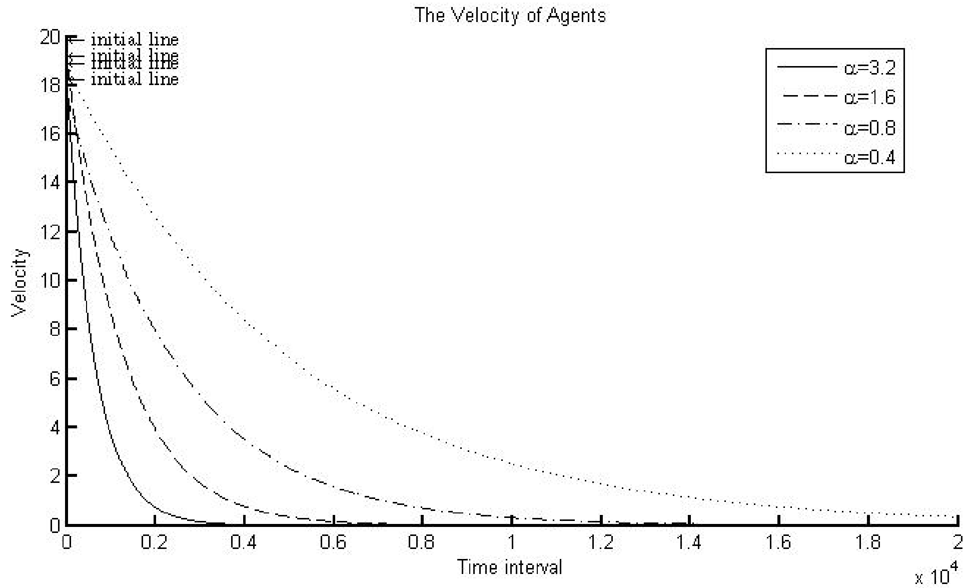
FIGURE 3. (a) All the realizations of the trajectories of the positions with $\alpha_1 = 0.8$; $D = 0.01$. (b) All the realizations of the trajectories of the positions with $\alpha_2 = 3.2$; $D = 0.01$.



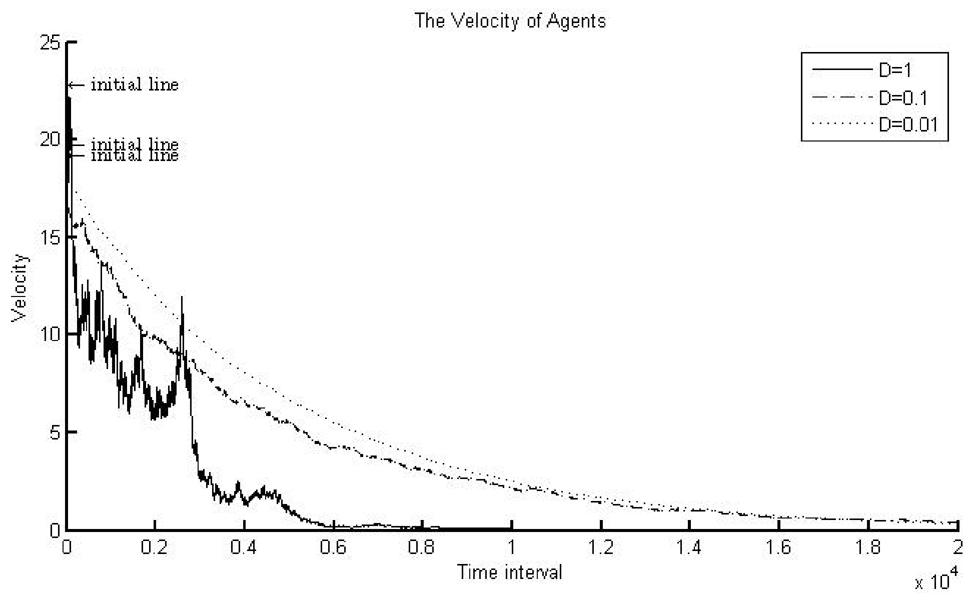
(a) (b)

FIGURE 4. (a) The trajectories of the mean speeds of the 20 particles with $\alpha_1 = 0.8$; $D = 0.01$. (b) The trajectories of the mean speeds of the 20 particles with $\alpha_2 = 3.2$; $D = 0.01$.

noises. In particular, we got a conclusion that the system (2.3) with the asymmetric communication rate between agents satisfied the strong stochastic flocking estimate due to multiplicative white noises. In the end of Sec. 2 we showed that the mean velocity of each agent was convergent when $t \to +\infty$. For numerical simulations our study concluded with several numerical results. In the first case we explored the influence of the different parameters α and D to flocking behavior of the system and found that flocking behavior is faster and faster with the parameter α increasing and the velocities were perturbed bigger and bigger as the parameter D increases. The second case showed that all particles will attain the same velocity as time passing.



(a)



(b)

FIGURE 5. (a) The trajectories of the relative speeds of the 20 particles with $\alpha_1 = 0.4$, $\alpha_2 = 0.8$, $\alpha_3 = 1.6$, $\alpha_4 = 3.2$; $D = 0.01$. (b) The trajectories of the relative speeds of the 20 particles with $D_1 = 0.01$, $D_2 = 0.1$, $D_3 = 1$; $\alpha = 0.4$

All numerical results agree with the analytical results given by Theorem 2.13 and Theorem 2.17.

REFERENCES

[1] J. A. Carrillo, M. Fornasier, J. Rosado, G. Toscani, Asymptotic flocking dynamics for the kinetic Cucker-Smale model, *SIAM J. Math. Anal.*, 42(1):218–236, 2010.

[2] J. A. Carrillo, M. R. D. D'Orsogna, V. Panferov, Double milling in self-propelled swarms from kinetic theory, *Kinet. Relat. Models*, 2(2):363–378, 2009.

[3] F. Cucker, C. Huepe, Flocking with informed agents, *MathS in Action*, 1(1):1–25, 2008.

[4] F. Cucker, E. Mordecki, Flocking in noisy environments, *J. Math. Pures Appl.*, 89(3):278–296, 2008.

[5] F. Cucker, S. Smale, On the mathmatics of emergence, *Jpn. J. Math.*, 2:197–227, 2007.

[6] F. Cucker, S. Smale, Emergent behavior in flocks, *IEEE Trans. Automat. Control*, 52(5):852–862, 2007.

[7] S. -Y. Ha, K. Lee, D. Levy, Emergence of time-asymptotic flocking in a stochastic Cucker-Smale system, *Commun. Math. Sci.*, 7(2):453–469, 2009.

[8] S.-M. Ahn, S. -Y. Ha, Stochastic flocking dynamics of the Cucker-Smale model with multiplicative white noises, *J. Math. Phys.*, 51(10):103301, 2010.

[9] S. -Y. Ha, J. -G. Liu, A simple proof of Cucker-Smale flocking dynamics and mean field limit, *Commun. Math. Sci.*, 7:297–325, 2009.

[10] S. -Y. Ha, M. Slemrod, Flocking dynamics of singularly perturbed oscillator chain and the Cucker-Smale system, *J. Dynam. Differential Equations*, 22(2):325–330, 2010.

[11] S. Motsch, E. Tadmor, A new model for self-organized dynamics and its flocking behavior, *J. Stat. Phys.*, 144(5):923–947, 2011.

[12] I. Aoki, A simulation study on the schooling mechanism in fish, *Jpn. Soc. Sci. Fish.*, 48(8):1081–1088, 1982.

[13] B. Birnir, An ODE model of the motion of pelagic fish, *J. Stat. Phys.*, 128:535–568, 2007.

[14] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, I. Cisbani, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, et al. Interaction ruling animal collective behavior depends on topological rather than metric distance: evidence from a field study, *Proc. Natl. Acad. Sci.*, 105(4):1232–1256, 2008.

[15] S. Camazine, J. J. Deneubourg, N. R. Franks, J. Sneyd, G. Theraulaz, E. Bonabeau, Self-organization in Biological Systems, *Princeton University Press*, Princeton, 2001.

[16] H. Levine, W. -J. Rappel, Self-oranizition in systems of self-propelled particles, *Phys. Rev. E.*, 63:017101, 2000.

[17] P. Degond, S. Motsch, Continuum limit of self-driven particles with orientation interaction, *Math. Models Methods Appl. Sci.*, 18(1):1193–1215, 2008.

[18] P. Degond, S. Motsch, Large scale dynamics of the persistent turning walker model of fish behavior, *J. Stat. Phys.*, 131(6):989–1021, 2008.

[19] V. Grimm, S. F. Railsback, Individual-Based modeling and ecology, *Princeton University Press*, Princeton, 2005.

[20] J. Shen, Cucker-Smale flocking under hierarchical leadership, *SIAM J. Appl. Math.*, 68(3):694–719, 2008.

[21] P. Degond, S. Motsch, Macroscopic limit of self-driven particles with orientation interaction, *C. R. Math. Acad. Sci.*, Paris, 345(10):555–560, 2007.

[22] P. Degond, S. Motsch, Large-scale dynamics of the Persistent Turing Walker model of fish behavior, *J. Stat. Phys.*, 131:989–1022, 2008.

[23] B. Oksendal, Stochastic Differential Equations: A Introduction with Applications, *Spring*, Berlin, 1995.