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QUENCHING FOR DEGENERATE SEMILINEAR PARABOLIC PROBLEMS

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ABSTRACT. Suppose that c is a positive constant. Let $f(u)$ be a $C^2([0,c))$ function such that $f>0$, $f'>0$, $f''>0$, $f(u)\to\infty$ when $u\to c^-$, and $\int_0^c f(u)\,du<\infty$, we investigate the quenching problem $\xi^q u_\tau - (\xi^\gamma u_\xi)_\xi = f(u)$ for $0<\xi<a$, $0<\tau<\Gamma\le\infty$, $u(\xi,0)=0$ for $0\le\xi\le a$, and $u(0,\tau)=0=u(a,\tau)$for $0<\tau<\Gamma$. It is assumed that $q\ge 0$ and $\gamma\in[0,1)$. In this paper, we study the existence and uniqueness of the classical solution u to the problem. Furthermore, the quenching set of the solution is discussed.

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1. INTRODUCTION

Let a be a positive constant, Γ be a positive real number, q be a nonnegative constant, γ be a nonnegative constant less than 1, and f be a twice differentiable function such that $f>0$, $f'>0$, $f''>0$, $f(u)\to\infty$ when $u\to c^-$, and $\int_0^c f(u)\,du<\infty$. In this paper, we study the following degenerate semilinear parabolic initial-boundary value problem

$$\xi^q u_\tau - (\xi^\gamma u_\xi)_\xi = f(u) \text{ in } (0,a)\times(0,\Gamma), \tag{1.1}$$

$$u(\xi,0)=0 \text{ on } [0,a] \text{ and } u(0,\tau)=u(a,\tau)=0 \text{ in } (0,\Gamma). \tag{1.2}$$

The above problem is motivated by the paper of Chen, Liu, and Xie [3]. They discussed the problem (1.1)–(1.2) with $f(u)=\int_0^a u^p d\xi$. They showed that u blows up in a finite time and the blow-up set is $[0,a]$. If $q=0$ and $u(\xi,0)=u_0$ where u_0 is a smooth function such that $u_0\in[0,c)$, Ke and Ning [6] discussed the equation, $u_t-(p(\xi)\,u_\xi)_\xi = f(u)$ where $p(0)=0$, $p(\xi)\in C^1(0,\infty)$, and $p(\xi)>0$ in $(0,\infty)$ with $1/p\in L^1([0,a])$ and $\int_0^\infty 1/p(\xi)\,d\xi=\infty$. They investigated the critical length of u and proved that all possible quenching points of u must lie in a compact subset of $(0,a)$. When $\gamma=0$, $u(\xi,0)=u_0(\xi)$, and $f(u)=u^p$, Floater [5] showed that if the solution blows up in a finite time, then u blows up at $\xi=0$ when $1<p\le q+1$. Chan and Liu [1] examined the reverse case. They showed that $\xi=0$ is not a blow-up point of u and the blow-up set is a compact subset of $(0,a)$ when $p>q+1$. When

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$f(u) = u^p$, Chan [2] studied the existence and uniqueness of the classical solution to the problem. He also proved that u blows up at $\xi = 0$ when $1 < p \leq q + 1$.

When $f(u) = 0$, Day [4] used (1.1) to describe the heat conduction in a rigid slab. The boundary $\xi = 0$ and $\xi = a$ are in contact with a heat reservoir. ξ^q and ξ^γ are representing the heat capacity and the thermal conductivity of the slab, respectively.

In (1.1), the terms ξ^q and ξ^γ tend to zero when ξ approaches zero if q and γ are positive, thus the coefficient of u_τ and $u_{\xi\xi}$ is degenerate. In Section 2, we shall prove that there exists a unique classical solution to the problem (1.1)–(1.2). We firstly show that problem (1.1)–(1.2) shall have a unique solution over the domain $[\delta, a] \times [0, \Gamma)$ where δ is a positive real number less than a. The classical solution to the problem (1.1)–(1.2) is the limiting solution of δ equation when $\delta \to 0$. In Section 3, a sufficient condition for u quenching at a finite time shall be given through constructing a lower solution. Then, we shall prove that the quenching set for u is a compact subset of $[0, a]$.

2. EXISTENCE AND UNIQUENESS OF THE CLASSICAL SOLUTION

Let L be a degenerate semilinear parabolic operator such that

$$Lu = \xi^q u_\tau - \left(\xi^\gamma u_\xi\right)_\xi .$$

In the beginning of this section, let us recall the following comparison lemma (c.f. [2]).

Lemma 2.1. *For any $s \in (0, \Gamma)$ and a bounded nonnegative function $B(\xi, \tau)$ on $[0, a] \times [0, s]$, if u and $v \in C([0, a] \times [0, s]) \cap C^{2,1}((0, a) \times (0, s])$, and*

$$(L - B)\, u \geq (L - B)\, v \ \text{ in } \ (0, a) \times (0, s],$$

$$u \geq v \ \text{ on the parabolic boundary } \ ([0, a] \times \{0\}) \cup (\{0, a\} \times (0, s]),$$

then $u \geq v$ on $[0, a] \times [0, s]$.

Let $\theta = \xi^\nu (a - \xi)^\nu$ where $\nu \in (0, 1)$ and $\nu + \gamma < 1$. Suppose that h_0 is a positive constant such that $h_0 \theta(\xi) < c$ on $[0, a]$. We also assume that $\tilde{\delta}$ is a positive constant with $\tilde{\delta} < a/2$ and $h(\tau)$ is a positive increasing solution to the following initial value problem:

$$h'(\tau) = \frac{f\left(\left(\frac{a^2}{4}\right)^\nu h(\tau)\right)}{\tilde{\delta}^{q+\nu}\left(a - \tilde{\delta}\right)^\nu} \ \text{ for } \tau \in (0, t_0] \ \text{ and } h(0) = h_0,$$

where t_0 is a positive constant satisfying $(a^2/4)^\nu h(t_0) < c$. Furthermore, we let

$$\zeta = \min\left\{\nu(1 - \gamma - \nu)\, \tilde{\delta}^{\gamma+\nu-2}\left(a - \tilde{\delta}\right)^\nu,\ \nu(1 - \nu)\left(a - \tilde{\delta}\right)^{\gamma+\nu} \tilde{\delta}^{\nu-2}\right\},$$

which satisfies

$$\zeta \geq \frac{f\left(\left(\frac{a^2}{4}\right)^{\nu} h(\tau)\right)}{h_0} \text{ for } \tau \in (0, t_0].$$

The above inequality holds in some interval $(0, t_0]$ because $\tilde{\delta}^{\gamma+\nu-2}$and $\tilde{\delta}^{\nu-2}$ tend to large numbers when $\tilde{\delta}$ is close to 0. Suppose that $\psi(\xi, \tau) = \theta(\xi) h(\tau)$, the next lemma shows that u is bounded above by ψ on $[0, a] \times [0, t_0]$.

Lemma 2.2. *$\psi \geq u$ on $[0, a] \times [0, t_0]$.*

Proof. The function ψ satisfies the following expression:

$$\begin{aligned}
&L\psi - f(\psi) \\
&\quad = \xi^{q+\nu} (a-\xi)^{\nu} h'(\tau) - \nu h(\tau) \frac{d}{d\xi} \left[\xi^{\gamma+\nu-1} (a-\xi)^{\nu} - \xi^{\gamma+\nu} (a-\xi)^{\nu-1}\right] \\
&\qquad - f\left(\xi^{\nu} (a-\xi)^{\nu} h(\tau)\right) \\
&\quad = \xi^{q+\nu} (a-\xi)^{\nu} h'(\tau) + \nu (1-\gamma-\nu) \xi^{\gamma+\nu-2} (a-\xi)^{\nu} h(\tau) \\
&\qquad + \nu (\gamma+2\nu) \xi^{\gamma+\nu-1} (a-\xi)^{\nu-1} h(\tau) + \nu (1-\nu) \xi^{\gamma+\nu} (a-\xi)^{\nu-2} h(\tau) \\
&\qquad - f\left(\xi^{\nu} (a-\xi)^{\nu} h(\tau)\right).
\end{aligned}$$

We rewrite the interval $(0, a)$ as $\left(0, \tilde{\delta}\right] \cup \left(\tilde{\delta}, a-\tilde{\delta}\right) \cup \left[a-\tilde{\delta}, a\right)$. When $(\xi, \tau) \in \left(0, \tilde{\delta}\right] \times [0, t_0]$,

$$\begin{aligned}
&L\psi - f(\psi) \\
&\quad \geq \nu h(\tau) (1-\gamma-\nu) \xi^{\gamma+\nu-2} (a-\xi)^{\nu} - f\left(\xi^{\nu} (a-\xi)^{\nu} h(\tau)\right) \\
&\quad \geq \nu h_0 (1-\gamma-\nu) \tilde{\delta}^{\gamma+\nu-2} \left(a-\tilde{\delta}\right)^{\nu} - f\left(\left(\frac{a^2}{4}\right)^{\nu} h(\tau)\right) \\
&\quad \geq h_0 \left[\zeta - \frac{f\left(\left(\frac{a^2}{4}\right)^{\nu} h(\tau)\right)}{h_0}\right] \geq 0.
\end{aligned}$$

When $(\xi, \tau) \in \left[a-\tilde{\delta}, a\right) \times [0, t_0]$,

$$\begin{aligned}
&L\psi - f(\psi) \\
&\quad \geq \nu h(\tau) (1-\nu) \xi^{\gamma+\nu} (a-\xi)^{\nu-2} - f\left(\xi^{\nu} (a-\xi)^{\nu} h(\tau)\right) \\
&\quad \geq \nu h_0 (1-\nu) \left(a-\tilde{\delta}\right)^{\gamma+\nu} \tilde{\delta}^{\nu-2} - f\left(\left(\frac{a^2}{4}\right)^{\nu} h(\tau)\right) \\
&\quad \geq h_0 \left[\zeta - \frac{f\left(\left(\frac{a^2}{4}\right)^{\nu} h(\tau)\right)}{h_0}\right] \geq 0.
\end{aligned}$$

When $(\xi,\tau)\in\left(\tilde{\delta},a-\tilde{\delta}\right)\times[0,t_0]$,

$$L\psi-f(\psi)\geq\tilde{\delta}^{q+\nu}\left(a-\tilde{\delta}\right)^{\nu}h'(\tau)-f\left(\left(\frac{a^2}{4}\right)^{\nu}h(\tau)\right)=0.$$

When $\tau=0$, $\psi(\xi,0)=h_0\theta\geq 0$ on $[0,a]$. At $\xi=0$ and $\xi=a$, $\psi(\xi,\tau)=0$ for $\tau\geq 0$. By Lemma 2.1, $\psi(\xi,\tau)\geq u(\xi,\tau)$ on $[0,a]\times[0,t_0]$. □

Since $f>0$, by Lemma 2.1 $u\geq 0$ on $[0,a]\times[0,\Gamma)$. Let δ be a positive constant less than a and u_δ denote the solution of the Dirichlet initial-boundary value problem:

$$\xi^q u_\tau-\left(\xi^\gamma u_\xi\right)_\xi=f(u)\ \text{ for } \delta<\xi<a, 0<\tau<\Gamma, \tag{2.1}$$

$$u(\xi,0)=0 \text{ for } \delta\leq\xi\leq a,\ u(\delta,\tau)=0=u(a,\tau)\ \text{ for } 0<\tau<\Gamma. \tag{2.2}$$

Let $\omega=(0,a)\times(0,t_0]$, $\bar{\omega}=[0,a]\times[0,t_0]$, $D_\delta=(\delta,a)$, $\omega_\delta=D_\delta\times(0,t_0]$, $\bar{D}_\delta=[\delta,a]$, $\bar{\omega}_\delta=\bar{D}_\delta\times[0,t_0]$, and $\partial\omega_\delta=\left(\bar{D}_\delta\times\{0\}\right)\cup(\{\delta,a\}\times(0,t_0])$, we prove the existence and uniqueness of the solution to the problem (1.1)–(1.2).

Theorem 2.3. *The problem* (1.1)–(1.2) *has a unique nonnegative solution*

$$u\in C(\bar{\omega})\cap C^{2+\alpha,1+\alpha/2}((0,a]\times[0,t_0]).$$

Proof. We prove $u_{\delta_2}\geq u_{\delta_1}$ over the domain $[\delta_1,a]\times[0,\Gamma)$ when $0<\delta_2<\delta_1<a$. We note that $\xi^{-q+\gamma}$ and $\xi^{-q+\gamma-1}\in C^{\alpha,\alpha/2}(\bar{\omega}_\delta)$ for some $\alpha\in(0,1)$. $\xi^{-q}f(u_\delta)\leq f(\psi)/\delta^q$ for some $(\xi,\tau,u_\delta)\in\bar{\omega}_\delta\times[0,c)$. It follows from Theorem 4.2.2 of Ladde, Lakshmikantham, and Vatsala [8, p. 143] that the problem (2.1)–(2.2) has a unique solution $u_\delta\in C^{2+\alpha,1+\alpha/2}(\bar{\omega}_\delta)$. When $\tau\geq 0$ and $\delta_1>\delta_2>0$, $u_{\delta_2}(\delta_1,\tau)\geq u_{\delta_1}(\delta_1,\tau)=0$. It follows from Lemma 2.1 that $u_{\delta_1}\leq u_{\delta_2}$ on $\bar{\omega}_{\delta_1}$. Therefore, $\lim_{\delta\to 0}u_\delta$ exists for all $(\xi,\tau)\in\bar{\omega}$. Let $u(\xi,\tau)=\lim_{\delta\to 0}u_\delta(\xi,\tau)$. We want to show that u is a solution. Let $\tilde{E}=\left[\tilde{b}_1,\tilde{b}_2\right]\times\left[0,\tilde{t}_1\right]$ and $\hat{E}=\left[\hat{b}_1,\hat{b}_2\right]\times\left[0,\hat{t}_1\right]$ such that $\tilde{E}\subset\hat{E}\subset\bar{\omega}$ (where $\tilde{b}_1>\hat{b}_1>0$, $\tilde{b}_2<\hat{b}_2\leq a$, and $\tilde{t}_1\leq\hat{t}_1\leq t_0$). Since $u_\delta\leq\psi$ in $\hat{E}$, we have for any constant $\tilde{q}>1$, the following three conditions are satisfied:

i. $||u_\delta||_{L^{\tilde{q}}(\hat{E})}\leq||\psi||_{L^{\tilde{q}}(\hat{E})}\leq k_1$ for some positive constant k_1.

ii. For $s>0$,

$$\left|\left|\gamma\xi^{-q+\gamma-1}\right|\right|_{L^{\tilde{q}}\left(\left[\hat{b}_1,\hat{b}_2\right]\times(\tau,\tau+s)\right)}\leq\gamma\hat{b}_1^{-q+\gamma-1}\left(\hat{b}_2-\hat{b}_1\right)^{1/\tilde{q}}s^{1/\tilde{q}}$$

tends to 0 when s approaches 0.

iii. As $f(u)$ is an increasing function, it gives $||\xi^{-q}f(u_\delta)||_{L^{\tilde{q}}(\hat{E})}\leq\hat{b}_1^{-q}||f(\psi)||_{L^{\tilde{q}}(\hat{E})}$.

If we choose $\tilde{q}>3/(2-\alpha)$, by Theorem 4.9.1 of Ladyženskaja, Solonnikov, and Ural′ceva [9, pp. 341–342] $u_\delta\in W_{\tilde{q}}^{2,1}\left(\hat{E}\right)$. By Theorem 2.3.3 there [9, p. 80],

$W_{\tilde{q}}^{2,1}\left(\hat{E}\right) \hookrightarrow H^{\alpha,\alpha/2}\left(\hat{E}\right)$. Thus, $||u_\delta||_{H^{\alpha,\alpha/2}\left(\hat{E}\right)} \leq k_2$ for some positive constant k_2. By the triangular inequality, it yields

$$\begin{aligned}
&\left|\left|\xi^{-q} f\left(u_\delta\right)\right|\right|_{H^{\alpha,\alpha/2}\left(\hat{E}\right)} \\
&\leq \hat{b}_1^{-q}\,||f(\psi)||_\infty + \sup_{\substack{(\xi,\tau)\in\hat{E}\\ \left(\tilde{\xi},\tau\right)\in\hat{E}}} \frac{\xi^{-q}\left|f\left(u_\delta\left(\xi,\tau\right)\right) - f\left(u_\delta\left(\tilde{\xi},\tau\right)\right)\right|}{\left|\xi-\tilde{\xi}\right|^\alpha} \\
&\quad + \sup_{\substack{(\xi,\tau)\in\hat{E}\\ \left(\tilde{\xi},\tau\right)\in\hat{E}}} \frac{\left|f\left(u_\delta\left(\tilde{\xi},\tau\right)\right)\right|\left|\xi^{-q}-\tilde{\xi}^{-q}\right|}{\left|\xi-\tilde{\xi}\right|^\alpha} + \sup_{\substack{(\xi,\tau)\in\hat{E}\\ (\xi,\tilde{\tau})\in\hat{E}}} \frac{\xi^{-q}\left|f\left(u_\delta\left(\xi,\tau\right)\right) - f\left(u_\delta\left(\xi,\tilde{\tau}\right)\right)\right|}{\left|\tau-\tilde{\tau}\right|^{\alpha/2}}.
\end{aligned}$$

By the mean value theorem, we have

$$\begin{aligned}
&\left|\left|\xi^{-q} f\left(u_\delta\right)\right|\right|_{H^{\alpha,\alpha/2}\left(\hat{E}\right)} \\
&\quad\leq \hat{b}_1^{-q}\,||f(\psi)||_\infty + \hat{b}_1^{-q}\,||f'(\psi)||_\infty\,||u_\delta||_{H^{\alpha,\alpha/2}\left(\hat{E}\right)} + ||f(\psi)||_\infty\left|\left|\xi^{-q}\right|\right|_{H^{\alpha,\alpha/2}\left(\hat{E}\right)} \\
&\quad\leq k_3
\end{aligned}$$

for some positive constant k_3 which is independent of δ. In addition, $||\xi^{-q+\gamma}||_{H^{\alpha,\alpha/2}\left(\hat{E}\right)}$ and $||\gamma\xi^{-q+\gamma-1}||_{H^{\alpha,\alpha/2}\left(\hat{E}\right)}$ are bounded. Then, by Theorem 4.10.1 of Ladyženskaja, Solonnikov, and Ural′ceva [9, pp. 351–352], we have

$$||u_\delta||_{H^{2+\alpha,1+\alpha/2}\left(\tilde{E}\right)} \leq k_4$$

for some positive constant k_4 which is independent of δ. This implies that u_δ, $(u_\delta)_\tau$, $(u_\delta)_\xi$, and $(u_\delta)_{\xi\xi}$ are equicontinuous in $\tilde{E}$. By the Ascoli-Arzela theorem, $||u||_{H^{2+\alpha,1+\alpha/2}\left(\tilde{E}\right)} \leq k_4$, and the partial derivatives of u are the limits of the corresponding partial derivatives of u_δ. From Lemma 2.2, $\psi \geq u \geq 0$ on $\bar{\omega}$, by the Sandwich theorem $u(0,\tau) = 0 = u(a,\tau)$ for $\tau \in [0,t_0]$. Thus, $u \in C(\bar{\omega}) \cap C^{2+\alpha,1+\alpha/2}((0,a]\times[0,t_0])$. By Lemma 2.1, there exists a unique nonnegative solution u to the problem (1.1)–(1.2) on $\bar{\omega}$. □

We follow Theorem 3 of Chan and Liu [1] to obtain the following result.

Theorem 2.4. *Let Γ be the supremum over t_0 for which there is a unique nonnegative solution $u \in C(\bar{\omega}) \cap C^{2+\alpha,1+\alpha/2}((0,a]\times[0,t_0])$. Then, there is a unique nonnegative solution $u \in C([0,a]\times[0,\Gamma)) \cap C^{2+\alpha,1+\alpha/2}((0,a]\times[0,\Gamma))$. If $\Gamma < \infty$, then u is unbounded in $(0,a)\times(0,\Gamma)$.*

3. QUENCHING SET OF THE SOLUTION

Let $x = \xi/a$ and $t = a^{\gamma-2-q}\tau$. Problem (1.1)–(1.2) becomes

$$x^q u_t - \left(x^\gamma u_x\right)_x = a^{2-\gamma} f(u)\,,\ (x,t) \in (0,1)\times(0,T)\,, \tag{3.1}$$

$$u(x,0)=0 \text{ on } [0,1], \text{ and } u(0,t)=u(1,t)=0 \text{ in } (0,T), \tag{3.2}$$

where $T=a^{\gamma-2-q}\Gamma$. Then, let b be a positive constant less than or equal to 1 and $\hat{u}(x,t)=a^{2-\gamma}f(0)\,m(t)\,x(1-x)/8\geq 0$ in $[0,1]\times(0,T)$ where

$$m(t)=\frac{8b}{3}\left(1-e^{-12t}\right).$$

In the following lemma, we prove that u quenches in a finite time if a is sufficient large through showing $\hat{u}(x,t)\leq u(x,t)$ in $[0,1]\times[0,T)$.

Lemma 3.1. *If a is sufficient large, u quenches in a finite time.*

Proof. To obtain this result, we use the method of the lower solution. With $\hat{u}(x,t)=a^{2-\gamma}f(0)\,m(t)\,x(1-x)/8$ where $m(t)$ being a positive increasing function for $t>0$ and $m(0)=0$, $\hat{u}(0,t)=\hat{u}(1,t)=0$ for $t\geq 0$ and $\hat{u}(x,0)=0$ for $x\in[0,1]$. Substitute $\hat{u}(x,t)$ into (3.1) to obtain the differential inequality below:

$$\begin{aligned} & x^q\hat{u}_t-x^\gamma\hat{u}_{xx}-\gamma x^{\gamma-1}\hat{u}_x \\ & \quad\leq \frac{m'(t)}{8}a^{2-\gamma}f(0)\,x(1-x)+\frac{2a^{2-\gamma}f(0)}{8}m(t)-\frac{\gamma a^{2-\gamma}f(0)}{8}m(t)(1-2x) \\ & \quad\leq \frac{m'(t)}{8}a^{2-\gamma}f(0)\frac{1}{4}+\frac{a^{2-\gamma}f(0)}{4}m(t)+\frac{a^{2-\gamma}f(0)}{8}m(t). \end{aligned}$$

If the last expression of this inequality is less than or equal to $a^{2-\gamma}f(0)$, it is equivalent that $m(t)$ satisfies the following differential equation

$$\frac{m'(t)}{32}+\frac{3}{8}m(t)=b,\ m(0)=0$$

where b is a positive constant less than or equal to 1. The solution of the above differential equation is

$$m(t)=\frac{8}{3}b\left(1-e^{-12t}\right).$$

Then, from $f'>0$, $\hat{u}\geq 0$, and the above equation, we obtain

$$x^q\hat{u}_t-x^\gamma\hat{u}_{xx}-\gamma x^{\gamma-1}\hat{u}_x\leq a^{2-\gamma}f(0)\leq a^{2-\gamma}f(\hat{u}).$$

By Lemma 2.1,

$$u(x,t)\geq\hat{u}(x,t)=\frac{a^{2-\gamma}f(0)}{8}m(t)\,x(1-x)$$

for $(x,t)\in[0,1]\times[0,T)$. In particular at $x=1/2$ and $t=\tilde{T}$ where $0<\tilde{T}<T$

$$u\left(\frac{1}{2},\tilde{T}\right)\geq\frac{a^{2-\gamma}f(0)}{32}m\left(\tilde{T}\right).$$

Since $\gamma<1$ and $m\left(\tilde{T}\right)>0$, there exists an a such that $a^{2-\gamma}f(0)\,m\left(\tilde{T}\right)/32\geq c$. Hence, u quenches in a finite time $\tilde{T}$ if a is sufficient large. $\square$

In the following, we are going to prove that the quenching set of u is a compact subset of $(0,1)$. Let $z^{1/(1-\gamma)} = x$. Then, $dz/dx = (1-\gamma)\, z^{-\gamma/(1-\gamma)}$,

$$u_x = u_z \frac{1-\gamma}{z^{\gamma/(1-\gamma)}},$$

$$u_{xx} = (1-\gamma)^2\, z^{-2\gamma/(1-\gamma)} u_{zz} - \frac{\gamma\,(1-\gamma)\, u_z}{z^{(1+\gamma)/(1-\gamma)}}.$$

Having the above expressions, $x^\gamma u_{xx} + \gamma x^{\gamma-1} u_x$ is represented by

$$\begin{aligned} &x^\gamma u_{xx} + \gamma x^{\gamma-1} u_x \\ &\quad = z^{\gamma/(1-\gamma)} \left[(1-\gamma)^2\, z^{-2\gamma/(1-\gamma)} u_{zz} - \frac{\gamma\,(1-\gamma)\, u_z}{z^{(1+\gamma)/(1-\gamma)}} \right] + \gamma z^{(\gamma-1)/(1-\gamma)} u_z \frac{1-\gamma}{z^{\gamma/(1-\gamma)}} \\ &\quad = (1-\gamma)^2\, z^{-\gamma/(1-\gamma)} u_{zz} - \gamma\,(1-\gamma)\, z^{-1/(1-\gamma)} u_z + \gamma\,(1-\gamma)\, z^{-1/(1-\gamma)} u_z \\ &\quad = (1-\gamma)^2\, z^{-\gamma/(1-\gamma)} u_{zz}. \end{aligned}$$

Therefore, $x^q u_t = (x^\gamma u_x)_x + a^{2-\gamma} f(u)$ transforms to

$$z^{q/(1-\gamma)} u_t = (1-\gamma)^2\, z^{-\gamma/(1-\gamma)} u_{zz} + a^{2-\gamma} f(u). \tag{3.3}$$

Let

$$E(t) = \frac{1}{2} (1-\gamma)^2 \int_0^1 u_z^2 dz - \int_0^1 \int_0^{u(z,t)} z^{\gamma/(1-\gamma)} a^{2-\gamma} f(u)\, du dz,$$

and $\int_0^{u(z,t)} f(u)\, du = F(u(z,t))$. We modify Theorem 2 of Kong [7] to obtain the following result.

Lemma 3.2. *The quenching points of the solution u are in the interval*

$$\left[\left\{ \frac{c^2 (1-\gamma)^2}{2\,[F(c)(1-\gamma) + E(0)]} \right\}^{1/(1-\gamma)}, \left\{ 1 - \frac{c^2 (1-\gamma)^2}{2\,[F(c)(1-\gamma) + E(0)]} \right\}^{1/(1-\gamma)} \right].$$

Proof: We differentiate $E(t)$ with respect to t,

$$E'(t) = \frac{1}{2} (1-\gamma)^2 \int_0^1 2 u_z u_{zt} dz - \int_0^1 z^{\gamma/(1-\gamma)} a^{2-\gamma} f(u)\, u_t dz.$$

Use integration by parts, $u_t(0,t) = u_t(1,t) = 0$, and (3.3) to obtain

$$\begin{aligned} E'(t) &= (1-\gamma)^2 \left(u_z u_t|_0^1 - \int_0^1 u_{zz} u_t dz \right) - \int_0^1 z^{\gamma/(1-\gamma)} a^{2-\gamma} f(u)\, u_t dz \\ &= -(1-\gamma)^2 \int_0^1 u_{zz} u_t dz - \int_0^1 z^{\gamma/(1-\gamma)} a^{2-\gamma} f(u)\, u_t dz \\ &= -\int_0^1 z^{\gamma/(1-\gamma)} u_t \left[(1-\gamma)^2\, z^{-\gamma/(1-\gamma)} u_{zz} + a^{2-\gamma} f(u) \right] dz \\ &= -\int_0^1 z^{\gamma/(1-\gamma)} u_t z^{q/(1-\gamma)} u_t dz \\ &= -\int_0^1 z^{(\gamma+q)/(1-\gamma)} u_t^2 dz < 0. \end{aligned}$$

Since $E'(t) < 0$, we know that $E(t) < E(0) = \frac{1}{2}(1-\gamma)^2 \int_0^1 u_z^2(z,0)\,dz$. Equivalently,

$$\left(\int_0^1 u_z^2 dz\right)^{1/2} < \frac{2^{1/2}}{(1-\gamma)}\left[\int_0^1 \int_0^{u(z,t)} z^{\gamma/(1-\gamma)} f(u)\,du dz + E(0)\right]^{1/2}.$$

Then, by the Schwarz inequality, it yields

$$\begin{aligned} u(z,t) &= \int_0^z u_z(z,t)\,dz \\ &\leq \left(\int_0^z dz\right)^{1/2}\left(\int_0^z u_z^2 dz\right)^{1/2} \\ &\leq \left(\int_0^z dz\right)^{1/2}\left(\int_0^1 u_z^2 dz\right)^{1/2} \\ &< z^{1/2}\frac{2^{1/2}}{(1-\gamma)}\left[\int_0^1 \int_0^{u(z,t)} z^{\gamma/(1-\gamma)} f(u)\,du dz + E(0)\right]^{1/2}. \end{aligned}$$

As $\int_0^{u(z,t)} f(u)\,du < F(c)$, this leads to

$$\begin{aligned} u(z,t) &< z^{1/2}\frac{2^{1/2}}{(1-\gamma)}\left[F(c)\int_0^1 z^{\gamma/(1-\gamma)} dz + E(0)\right]^{1/2} \\ &= z^{1/2}\frac{2^{1/2}}{(1-\gamma)}\left[F(c)\frac{1}{\frac{\gamma}{(1-\gamma)}+1} + E(0)\right]^{1/2} \\ &= z^{1/2}\frac{2^{1/2}}{(1-\gamma)}\left[F(c)(1-\gamma) + E(0)\right]^{1/2}. \end{aligned}$$

Set the right side of the above inequality less than c, we obtain

$$z^{1/2}\frac{2^{1/2}}{(1-\gamma)}\left[F(c)(1-\gamma)+E(0)\right]^{1/2} < c.$$

Then, solve for z

$$z < \frac{c^2(1-\gamma)^2}{2\left[F(c)(1-\gamma)+E(0)\right]}.$$

Since $z^{1/(1-\gamma)} = x$, the upper bound of x is given by

$$x < \left\{\frac{c^2(1-\gamma)^2}{2\left[F(c)(1-\gamma)+E(0)\right]}\right\}^{1/(1-\gamma)}.$$

When we integrate $-u_z(z,t)$ with respect to z from z to 1,

$$
\begin{aligned}
u(z,t) &= \int_z^1 -u_z(z,t)\,dz \\
&\le \left(\int_z^1 dz\right)^{1/2}\left(\int_z^1 u_z^2 dz\right)^{1/2} \\
&\le \left(\int_z^1 dz\right)^{1/2}\left(\int_0^1 u_z^2 dz\right)^{1/2} \\
&< (1-z)^{1/2}\frac{2^{1/2}}{(1-\gamma)}\left[\int_0^1\int_0^{u(z,t)} z^{\gamma/(1-\gamma)} f(u)\,du dz + E(0)\right]^{1/2}.
\end{aligned}
$$

Using the similar calculation above, we obtain

$$1 - z < \frac{c^2(1-\gamma)^2}{2\left[F(c)(1-\gamma)+E(0)\right]}.$$

This leads to

$$1 - \frac{c^2(1-\gamma)^2}{2\left[F(c)(1-\gamma)+E(0)\right]} < z = x^{1-\gamma}.$$

Equivalently,

$$\left\{1 - \frac{c^2(1-\gamma)^2}{2\left[F(c)(1-\gamma)+E(0)\right]}\right\}^{1/(1-\gamma)} < x.$$

The proof is complete. □

Theorem 3.3. *If a is sufficiently small, there is a global solution to the problem* (1.1)–(1.2).

Proof. Suppose that $g(x) = k_5 x^\beta$ for some positive constants β and k_5 such that $\beta+\gamma<1$ and $k_5<c$. It is noticed that $g(x)\ge 0$ in $[0,1]$. Then, we compute

$$
\begin{aligned}
&x^q g_t - (x^\gamma g_x)_x - a^{2-\gamma} f(g) \\
&\quad = -\left[k_5 x^\gamma \beta(\beta-1)x^{\beta-2} + \gamma k_5 x^{\gamma-1}\beta x^{\beta-1}\right] - a^{2-\gamma} f\left(k_5 x^\beta\right) \\
&\quad = -k_5 x^{\gamma+\beta-2}\beta(\beta-1+\gamma) - a^{2-\gamma} f\left(k_5 x^\beta\right) \\
&\quad = k_5 x^{\gamma+\beta-2}\beta(1-\beta-\gamma) - a^{2-\gamma} f\left(k_5 x^\beta\right) \\
&\quad \ge k_5\beta(1-\beta-\gamma) - a^{2-\gamma} f\left(k_5 x^\beta\right).
\end{aligned}
$$

We choose a sufficiently small such that $k_5\beta(1-\beta-\gamma) - a^{2-\gamma} f\left(k_5 x^\beta\right) \ge 0$ in $(0,1)$. Hence,

$$x^q g_t - (x^\gamma g_x)_x - a^{2-\gamma} f(g) \ge 0 \text{ in } (0,1).$$

By Lemma 2.1, $g(x) \ge u(x,t)$ on $[0,1]\times[0,\infty)$. Therefore, $u(x,t)$ exists globally. □

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