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LOCAL AND NONLOCAL BOUNDARY QUENCHING IN A SUBDIFFUSIVE MEDIUM

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ABSTRACT. A mathematical model for boundary quenching in a subdiffusive medium is analyzed. The quenching effect is simulated by a nonlinear flux condition at the left boundary of a one-dimensional bar. The nonlinearity is allowed to depend upon either the local temperature of the boundary or a global average of temperature. The right boundary of the bar is subjected to either an insulation condition or a zero temperature condition. A separate analysis is carried out for an extension of the model that includes the influence of advection.

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1. **Introduction**

We consider a model for thermal diffusion in a one-dimensional bar ($0 \leq x \leq l$) composed of a material with subdiffusive properties. To simulate a boundary quenching effect, a nonlinear flux condition is imposed at $x = 0$. We treat problems in which the nonlinear flux is determined by either local or nonlocal effects. For the boundary constraint at $x = l$, cases for both a Neumann condition and a Dirichlet condition will be considered.

It is assumed that the temperature $T(x,t)$ of the subdiffusive material is modeled by the fractional differential equation

$$\frac{\partial T}{\partial t} = D_t^{1-\alpha}\frac{\partial^2 T}{\partial x^2}, \quad 0 < x < l, \quad t > 0, \tag{1.1}$$

where

$$D_t^{1-\alpha}(\cdot) \equiv \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-t')^{-1+\alpha}(\cdot)dt', \quad 0 < \alpha < 1. \tag{1.2}$$

The use of (1.1) and (1.2) to describe heat flow in a subdiffusive medium has been discussed in [6], [7], and [10].

The initial temperature distribution in the bar is given by

$$T(x,0) = T_0(x) \geq 0; \quad 0 \leq T_0(x) \leq 1-\delta, \quad 0 < \delta < 1, \tag{1.3}$$

where $T_0(x)$ is continuous for $0 \leq x \leq l$.

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The nonlinear flux condition at $x = 0$ takes the form

$$D_t^{1-\alpha} \frac{\partial T}{\partial x}\bigg|_{x=0} = -F[1 - v(t)], \quad t > 0. \tag{1.4}$$

The specification of $v(t)$ for the local problem is

$$v(t) = u(t) \equiv T(0,t), \tag{1.5}$$

whereas for the nonlocal problem it is

$$v(t) = \bar{u}(t) \equiv \frac{1}{l} \int_0^l |T(x,t)| dx. \tag{1.6}$$

To be consistent with quenching, the nonlinear function $F(1-v)$ is assumed to be twice differentiable and have the properties

$$F(1-v) > 0, \quad F_v(1-v) > 0, \quad F_{vv}(1-v) > 0 \quad \text{for} \quad 0 \le v < 1, \tag{1.7}$$

and

$$F(1-v) \to \infty \quad \text{as} \quad v \to 1^-. \tag{1.8}$$

We note that the upper bound on the initial data in (1.3) ensures that $v(t)$ starts below the quenching value in both the local and nonlocal cases.

At $x = l$, we will impose either an insulation condition

$$\frac{\partial T}{\partial x}\bigg|_{x=l} = 0, \quad t > 0, \tag{1.9}$$

which we will refer to as the Neumann case, or a zero temperature condition

$$T(l,t) = 0, \quad t > 0, \tag{1.10}$$

which we will refer to as the Dirichlet case.

The classical diffusion version of the problem presented here was investigated in [9]. The analysis here represents an extension of those results to materials with subdiffusive properties.

In Section 2, we consider the local problem (1.5) while specifying at $x = l$ either the Neumann case condition (1.9) or the Dirichlet case condition (1.10). This will be followed in Section 3 by similar considerations for the nonlocal problem (1.6).

In Section 4, the basic equation (1.1) will be extended to allow for advection effects. Only the local problem with a Neumann type condition will be considered to examine the effect of advection on quenching.

The goal of the analysis throughout the paper is to determine if there exists a $\hat{t} < \infty$ such that

$$v(t) \to 1, \quad v'(t) \to \infty, \quad \text{as} \quad t \to \hat{t}. \tag{1.11}$$

In view of (1.4), this behavior defines quenching, as discussed in [1], [2], [3], [5], and [8].

2. **Local Problem**

Here we analyze the local problem as described in Section 1 by converting the initial-boundary value problem into a nonlinear Volterra equation for $v(t) = u(t) \equiv T(0,t)$. This conversion is accomplished by using the Green's function $G_\alpha(x,t|x_0,t_0)$ that satisfies the linear problem

$$\frac{\partial G_\alpha}{\partial t} = D_t^{1-\alpha}\frac{\partial^2 G_\alpha}{\partial x^2} + \delta(x-x_0)\delta(t-t_0), \quad 0 < x, x_0 < l, \quad t > t_0^-, \tag{2.1}$$

$$G_\alpha\Big|_{t=t_0^-} = 0, \quad 0 \le x \le l, \tag{2.2}$$

$$\left.\frac{\partial G_\alpha}{\partial x}\right|_{x=0} = 0, \quad t > 0. \tag{2.3}$$

For the Neumann case, the boundary condition at $x = l$ is

$$\left.\frac{\partial G_\alpha}{\partial x}\right|_{x=l} = 0, \quad t > 0, \tag{2.4}$$

while for the Dirichlet case, it is

$$G_\alpha\Big|_{x=l} = 0, \quad t > 0. \tag{2.5}$$

It should be noted that the homogeneous form of (1.4) associated with the Green's function problem, namely $D_t^{1-\alpha}\partial G_\alpha/\partial x = 0$, is implied by (2.3).

Utilizing the Green's function, it is possible to express $T(x,t)$ for the local problem as

$$T(x,t) = \int_0^l G_\alpha(x,t|x_0,0)T_0(x_0)dx_0 + \int_0^t G_\alpha(x,t-s|0,0)F[1-T(0,s)]ds. \tag{2.6}$$

By setting $x = 0$, a Volterra equation for $T(0,t) \equiv u(t)$ follows as

$$u(t) = h(t) + \int_0^t k(t-s)F[1-u(s)]ds, \quad t \ge 0, \tag{2.7}$$

where

$$h(t) = \int_0^l G_\alpha(0,t|x_0,0)T_0(x_0)dx_0,$$

and

$$k(t) = G_\alpha(0,t|0,0).$$

The analysis of (2.7) depends upon the properties of $h(t)$ and $k(t)$, which both depend upon the behavior of $G_\alpha(x,t|x_0,0)$. It was demonstrated in [11] that $G_\alpha(x,t|x_0,0)$ is related to the Green's function for the classical diffusion problem by

$$G_\alpha(x,t|x_0,0) = \int_0^\infty f_\alpha(z) G_1(x, t^\alpha z|x_0,0)dz, \tag{2.8}$$

where $G_1(x,t|x_0,0)$ satisfies (2.1) with $\alpha = 1$. The function $f_\alpha(z)$ is defined by the inverse Mellin transform

$$f_\alpha(z) = M_z^{-1}\Big[\frac{\Gamma(r)}{\Gamma(1-\alpha+\alpha r)}\Big] = \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{j!\Gamma(1-\alpha-\alpha j)}, \quad z \geq 0,$$

where $f_\alpha(z) \geq 0$, for $z \geq 0$ and $f_\alpha(z) \to 0$ exponentially as $z \to \infty$. In [4], the properties of $G_\alpha(x,t|x_0,0)$ are examined in order to determine the behavior of $k(t)$ and $h(t)$. Since $G_1(x,t|x_0,0) \geq 0$ for both the Neumann and Dirichlet cases, it follows that

$$G_\alpha(x,t|x_0,0) \geq 0, \quad 0 \leq x \leq l, \quad t \geq 0. \tag{2.9}$$

For the Neumann case, an expression for $k(t)$ was determined in [4] as

$$k(t) = k_N(t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\int_0^\infty f_\alpha(z)\exp\left\{-\frac{n^2\pi^2}{l^2}t^\alpha z\right\}dz,$$

which has the asymptotic behavior

$$k_N(t) = \frac{1}{l} + \frac{2}{l}\frac{1}{t^\alpha}\sum_{n=1}^{\infty}\int_0^\infty f_\alpha(z/t^\alpha)\exp\left\{-\frac{n^2\pi^2}{l^2}z\right\}dz \sim \frac{1}{l} \quad \text{as} \quad t \to \infty. \tag{2.10}$$

For the Dirichlet case, $k(t)$ was determined in [4] as

$$k(t) = k_D(t) = \frac{2}{l}\sum_{n=1}^{\infty}\int_0^\infty f_\alpha(z)\exp\left\{-\left(\frac{2n-1}{2}\right)^2\frac{\pi^2}{l^2}t^\alpha z\right\}dz,$$

which has the asymptotic behavior

$$k_D(t) \sim \frac{2}{l}\int_0^\infty f_\alpha(z)\exp\left\{-\frac{\pi^2}{4l^2}t^\alpha z\right\}dz \sim \frac{8l}{\pi^2 t^\alpha} \quad \text{as} \quad t \to \infty. \tag{2.11}$$

In each of these cases, it is seen that

$$k(t) > 0, \quad k'(t) < 0, \quad 0 \leq t < \infty.$$

To obtain bounds on $h(t)$, it is necessary to examine the properties of

$$\tilde{T}(x,t) \equiv \int_0^l G_\alpha(x,t|x_0,0)T_0(x_0)dx_0.$$

For the Neumann case, $\tilde{T}(x,t)$ satisfies (1.1) with the initial condition $\tilde{T}_0(x) \equiv 1$, together with the boundary conditions $\frac{\partial \tilde{T}}{\partial x}|_{x=0} = 0$ and $\frac{\partial \tilde{T}}{\partial x}|_{x=l} = 0$. This problem has the unique solution $\tilde{T}(x,t) \equiv 1$, and hence for the Neumann case,

$$\int_0^l G_\alpha(x,t|x_0,0)dx_0 = 1, \quad 0 \le x \le l, \quad t \ge 0.$$

For the Dirichlet case, $\tilde{T}(x,t)$ satisfies (1.1) with the initial condition $\tilde{T}_0(x) \equiv 1$, together with the boundary conditions $\frac{\partial \tilde{T}}{\partial x}|_{x=0} = 0$ and $\tilde{T}|_{x=l} = 0$. In this problem, it is first noted that

$$0 \le \int_0^l G_1(x,t^\alpha z|x_0,0)dx_0 \le 1 \quad \text{for} \quad 0 \le x \le l, \quad t \ge 0,$$

which follows from the maximum principle for classical diffusion. Then from (2.8), we have

$$\int_0^l G_\alpha(x,t|x_0,0)dx_0 \le 1$$

for the Dirichlet case.

Thus follows for both the Neumann and Dirichlet cases,

$$0 \le h_0 \equiv \min h(t) \le h(t) \le 1 - \delta.$$

Having established appropriate bounds on $k(t)$ and $h(t)$, we turn to an analysis of (2.7). We will first establish the existence of a unique solution that is continuous with $0 \le u(t) < 1$ for $0 \le t < t^* < \infty$. This will be shown by demonstrating the contraction properties of the operator A defined by

$$(Au)(t) \equiv h(t) + \int_0^t k(t-s)F[1-u(s)]ds, \quad t > 0,$$

where $u(t)$ belongs to the space of continuous functions that satisfy

$$0 \le u(t) \le M < 1, \quad 0 \le t < t^*.$$

Clearly the operator A maps continuous functions into continuous functions. We also need a bound on the mapping such that

$$(Au)(t) \le 1 - \delta + F(1-M)I(t) \le M, \quad M < 1, \quad 0 \le t < t^*, \tag{2.12}$$

where

$$I(t) \equiv \int_0^t k(s)ds.$$

For the contraction, consider

$$\sup_{0 \le t \le t^*} |Au_1 - Au_2| \le I(t)\frac{\partial F}{\partial u}(1-M) \sup_{0 \le t \le t^*} |u_1 - u_2|. \tag{2.13}$$

Thus, the contraction requires that

$$I(t)\frac{\partial F}{\partial u}(1-M) < 1, \quad 0 \le t < t^*. \tag{2.14}$$

To satisfy both (2.12) and (2.14), the optimal M is such that

$$\frac{M-(1-\delta)}{F(1-M)} = \frac{1}{F_u(1-M)}.$$

Hence the largest t^* is obtained from

$$I(t^*) = \Lambda \equiv \max_{0\leq M<1}\left[\frac{M-(1-\delta)}{F(1-M)}\right].$$

This establishes the existence of a unique solution of (2.7) which remains below the quenching level for $t < t^*$.

Next we will demonstrate the existence of a $t^{**} < \infty$ such that the solution of (2.7) must have achieved the quenching level. That is, (2.7) implies that $u(t) \geq 1$ for $t > t^{**}$. To establish this result, we assume the contrary, namely that $0 \leq u(t) < 1$ for $0 \leq t \leq t^{**}$. Then follows

$$u(t) = Au(t) \geq h(t) + J(t) \geq h_0 + J(t),$$

where

$$J(t) \equiv \int_0^t k(t^{**}-s)F[1-u(s)]ds.$$

Then

$$\begin{aligned} J'(t) &= k(t^{**}-t)F[1-u(t)] \\ &\geq k(t^{**}-t)F[1-h_0-J(t)]. \end{aligned}$$

Integration yields

$$\int_0^{J(t^{**})} \frac{dJ}{F(1-h_0-J)} \geq \int_0^{t^{**}} k(t^{**}-t)dt = \int_0^{t^{**}} k(s)ds = I(t^{**}),$$

or

$$\int_{h_0}^{J(t^{**})+h_0} \frac{dr}{F(1-r)} \geq I(t^{**}).$$

If $J(t^{**}) + h_0 \geq 1$, then there is a contradiction of $u(t) < 1$. Thus if t^{**} is such that

$$\int_{h_0}^{1} \frac{dr}{F(1-r)} \equiv \kappa = I(t^{**}), \tag{2.15}$$

then t^{**} becomes an upper bound on the existence of a solution $u(t)$, such that $u(t) < 1$.

From the asymptotic behavior of $k(t)$ as $t \to \infty$ in the Neumann case (2.10) and in the Dirichlet case (2.11), it is clear that $I(t) \to \infty$ as $t \to \infty$ in both cases. Thus, there always exists a $t^{**} < \infty$ such that (2.15) is satisfied. Hence the first part of the quenching criteria (1.11) is fulfilled, namely $u(t) \to 1$ as $t \to \hat{t}$, $t^* \leq \hat{t} \leq t^{**} < \infty$.

To complete the investigation of quenching, we consider $u'(t)$ by differentiation of (2.7) to obtain

$$u'(t) = h'(t) + \frac{d}{dt}\int_0^t k(t-s)F[1-u(s)]ds.$$

We note that

$$\begin{aligned}\int_0^t k(t-s)F[1-u(s)]ds = &\ -I(t-s)F[1-u(s)]|_{s=0}^{s=t}\\ &+\int_0^t I(t-s)\frac{\partial F}{\partial u}[1-u(s)]u'(s)ds\\ = &\ I(t)F[1-u(0)]-I(0)F[1-u(t)]\\ &+\int_0^t I(t-s)\frac{\partial F}{\partial u}[1-u(s)]u'(s)ds.\end{aligned}$$

Then follows

$$u'(t)=h'(t)+k(t)F[1-u(0)]+\int_0^t k(t-s)\frac{\partial F}{\partial u}[1-u(s)]u'(s)ds. \tag{2.16}$$

Since $u(t)$ has been shown to exist as a unique continuous function for $0\le t<t^*$ with $0\le u(t)\le M<1$, then (2.16) can be expressed as

$$u'(t)=\tilde{h}(t)+\int_0^t \tilde{k}(t,s)u'(s)ds,$$

which represents a linear integral equation for $u'(t)$ where

$$\tilde{h}(t)\equiv h'(t)+u(t)F[1-u(0)]\quad\text{and}\quad \tilde{k}(t,s)\equiv k(t-s)\frac{\partial F}{\partial u}[1-u(s)].$$

Standard theory of linear Volterra equations can then be applied to establish the existence of a continuous $u'(t)$ as long as $u(t)<1$ exists.

We note that $\tilde{k}(t,s)\ge 0$. We know that $u(t)\to 1^-$ as $t\to\hat{t}$, $t^*\le\hat{t}\le t^{**}$. Thus if $h'(t)+k(t)F[1-u(0)]>0$ for $t<\hat{t}$, then $u'(t)>0$ for $0<t<\hat{t}$. This follows by noting that all terms in the Neumann series solution for $u'(t)$ will be positive.

To see that $u'(t)\to\infty$ as $t\to\hat{t}$, note that $k(t-s)\ge k(t)$ and hence

$$\begin{aligned}u'(t)\ &\ge h'(t)+k(t)F[1-u(0)]+k(t)\int_0^t \frac{\partial F}{\partial u}[1-u(s)]u'(s)ds\\ &=h'(t)+k(t)F[1-u(0)]+k(t)\left[F[1-u(s)]|_{s=0}^{s=t}\right]\\ &=h'(t)+k(t)F[1-u(t)].\end{aligned} \tag{2.17}$$

Since $u(t)\to 1$ as $t\to\hat{t}$, it follows that $F[1-u(t)]\to\infty$ as $t\to\hat{t}$, and (2.17) implies that $u'(t)\to\infty$ as $t\to\hat{t}$.

Thus, for both the Neumann and Dirichlet cases, we have established the existence of a $\hat{t}<\infty$ such that

$$u(t)\to 1,\quad u'(t)\to\infty,\quad\text{as}\quad t\to\hat{t},\quad t^*\le\hat{t}\le t^{**},$$

which is the definition of quenching.

It is worthwhile to compare the quenching result established here for a subdiffusive medium with that of a classical diffusive medium investigated in [9]. For the Neumann case, quenching occurs in both types of media. For the Dirichlet case associated with the classical diffusive medium, it is always possible to adjust the physical parameters so that quenching does not occur. This is in distinct contrast with the subdiffusive medium where quenching can not be prevented.

3. Nonlocal Problem

We now consider the problem in which the nonlinear flux condition at $x = 0$ given by (1.4) depends upon the average temperature of the bar $\bar{u}(t)$, as defined by (1.6). Here the integral representation of $T(x,t)$ in terms of the Green's function $G_\alpha(x,t|x_0,0)$ has the form

$$T(x,t) = \int_0^l G_\alpha(x,t|x_0,0)T_0(x_0)dx_0 + \int_0^t G_\alpha(x,t-s|0,0)F[1-\bar{u}(s)]ds. \tag{3.1}$$

From the properties of the various quantities in (3.1), it is clear that $T(x,t) \geq 0$. Hence the definition of $\bar{u}(t)$ in (1.6) can be replaced by

$$\bar{u}(t) = \frac{1}{l}\int_0^l T(x,t)dx.$$

For the Neumann case in which we impose $\frac{\partial T}{\partial x}|_{x=l} = 0$, it is not necessary to convert (3.1) to an integral equation for $\bar{u}(t)$ as was done for the local problem. There is a fortunate simplification that occurs by applying $\frac{1}{l}\int_0^l(\cdot)dx$ to the governing partial differential equation (1.1) so that

$$\frac{d}{dt}\left[\frac{1}{l}\int_0^l T(x,t)dx\right] = \frac{1}{l}\left[D_t^{1-\alpha}\frac{\partial T}{\partial x}\bigg|_{x=l} - D_t^{1-\alpha}\frac{\partial T}{\partial x}\bigg|_{x=0}\right].$$

Thus follows

$$\bar{u}'(t) = \frac{1}{l}F[1-\bar{u}(t)], \quad t > 0, \tag{3.2}$$

together with the initial condition

$$\bar{u}(0) = \frac{1}{l}\int_0^l T_0(x)dx \leq 1-\delta.$$

Integration of (3.2) yields

$$\int_{\bar{u}(0)}^{\bar{u}(t)} \frac{dy}{F(1-y)} = \frac{t}{l}, \tag{3.3}$$

which represents an implicit form of the exact solution of the Neumann case. From (3.3) it is clear that $\bar{u}(t) \to 1$ as $t \to \hat{t}$, where

$$\hat{t} = l\int_{\bar{u}(0)}^1 \frac{dy}{F(1-y)}.$$

Then follows from (3.2) that $\bar{u}'(t) \to \infty$ as $t \to \hat{t}$. This implies that quenching always occurs for the Neumann case of the nonlocal problem.

For the Dirichlet case in which we impose $T|_{x=l} = 0$, there is no simplification as found in the Neumann case. An analysis similar to the local problem is employed.

To derive an integral equation for $\bar{u}(t)$, apply $\frac{1}{l}\int_0^l(\cdot)dx$ to (3.1) to obtain

$$\bar{u}(t) = \bar{h}(t) + \int_0^t \bar{k}(t-s)F[1-\bar{u}(s)]ds, \quad t \geq 0, \tag{3.4}$$

where

$$\bar{h}(t) = \frac{1}{l}\int_0^l \int_0^l G_\alpha(x,t|x_0,0)T_0(x_0)dx_0dx,$$

and

$$\bar{k}(t) = \frac{1}{l}\int_0^l G_\alpha(x,t|0,0)dx. \tag{3.5}$$

Both $\bar{h}(t)$ and $\bar{k}(t)$ involve $\frac{1}{l}\int_0^l G_\alpha(x,t|x_0,0)dx$. An expression for this quantity follows from (2.8) as

$$\frac{1}{l}\int_0^l G_\alpha(x,t|x_0,0)dx = \int_0^\infty f_\alpha(z)\frac{1}{l}\int_0^l G_1(x,t^\alpha z|x_0,0)dxdz, \tag{3.6}$$

where

$$G_1(x,z|x_0,0) = \frac{2}{l}\sum_{n=1}^\infty \cos\left(\frac{(2n-1)\pi x_0}{2l}\right)\cos\left(\frac{(2n-1)\pi x}{2l}\right)\exp\left[-\frac{(2n-1)^2\pi^2}{4l^2}z\right]. \tag{3.7}$$

In [9], it is shown that

$$\tilde{k}_M(\tau) \equiv \frac{1}{l}\int_0^l G_1(x,\tau|0,0)dx$$

has the properties

$$\tilde{k}_M(\tau) \geq 0, \quad \tilde{k}'_M(\tau) \leq 0, \quad \tau \geq 0.$$

This implies that

$$\bar{k}(t) \geq 0, \quad \bar{k}'(t) \leq 0, \quad t \geq 0.$$

To obtain bounds on $\bar{h}(t)$, it is necessary to examine the properties of

$$\bar{T}(t) \equiv \frac{1}{l}\int_0^l \int_0^l G_\alpha(x,t|x_0,0)dx_0dx.$$

From (2.9) and (3.6) it is found that

$$\begin{aligned} 0 \leq \bar{T}(t) &= \int_0^\infty f_\alpha(z)\left\{\frac{8}{\pi^2}\sum_{n=1}^\infty \frac{1}{(2n-1)^2}\exp\left[-\frac{(2n-1)^2\pi^2}{4l^2}t^\alpha z\right]\right\}dz \\ &\leq \int_0^\infty f_\alpha(z)\left\{\frac{8}{\pi^2}\sum_{n=1}^\infty \frac{1}{(2n-1)^2}\right\}dz = 1. \end{aligned}$$

Then follows the bounds on $\bar{h}(t)$ as

$$0 \leq \bar{h}_0 \equiv \min \bar{h}(t) \leq \bar{h}(t) \leq 1-\delta.$$

To obtain the asymptotic behavior of $\bar{k}(t)$ as $t \to \infty$ in this Dirichlet case, it follows from (3.5), (3.6), and (3.7) that

$$\bar{k}(t) = \bar{k}_D(t) = \frac{4}{\pi l} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \int_0^{\infty} f_\alpha(z) \exp\left[-\frac{(2n-1)^2\pi^2}{4l^2} t^\alpha z \right] dz.$$

For $z = \xi t^{-\alpha}$,

$$\bar{k}_D(t) = \frac{4}{\pi l} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \int_0^{\infty} f_\alpha(\xi t^{-\alpha}) \exp\left[-\frac{(2n-1)^2\pi^2}{4l^2} \xi \right] d\xi \right\} t^{-\alpha}.$$

Hence

$$\bar{k}_D(t) \sim \frac{4}{\pi l} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \int_0^{\infty} \exp\left[-\frac{(2n-1)^2\pi^2}{4l^2} \xi \right] d\xi \right\} t^{-\alpha}, \quad \text{as} \quad t \to \infty,$$

$$= \left\{ \frac{16l}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} \right\} t^{-\alpha}, \quad \text{as} \quad t \to \infty.$$

This behavior allows us to conclude that

$$\bar{I}(t) \equiv \int_0^t \bar{k}(s) ds$$

is such that $\bar{I}(t) \to \infty$ as $t \to \infty$.

It is now possible to apply the same analysis to (3.4) as that for the local problem (2.7) in Section 1 to establish the existence of a $\hat{t} < \infty$ such that

$$\bar{u}(t) \to 1, \quad \bar{u}'(t) \to \infty \quad \text{as} \quad t \to \hat{t}, \quad t^* \leq \hat{t} \leq t^{**},$$

where t^* and t^{**} are defined by

$$\bar{I}(t^*) = \Lambda, \quad \bar{I}(t^{**}) = \kappa.$$

Since $\bar{I}(t) \to \infty$ as $t \to \infty$, there always exists a $t^{**} < \infty$ such that $\bar{I}(t^{**}) = \kappa$ thereby assuring that quenching will occur. This is in contrast to the classical diffusion case where quenching can be avoided by an appropriate selection of the material properties.

4. **Local Problem with Advection**

We now reconsider the local problem examined in Section 2 while allowing for the effect of advection. It is suggested in [6] and [7] that advection in a subdiffusive medium can be modeled by adding another fractional derivative term to the basic subdiffusion equation. Thus, (1.1) is replaced by

$$\frac{\partial \hat{T}}{\partial t} = v D_t^{1-\alpha} \frac{\partial \hat{T}}{\partial x} + D_t^{1-\alpha} \frac{\partial^2 \hat{T}}{\partial x^2}, \quad 0 < x < l, \quad t > 0, \tag{4.1}$$

where $v > 0$ is the advection speed. The initial condition is

$$\hat{T}(x, 0) = \hat{T}_0(x), \quad 0 \leq x \leq l; \quad 0 \leq \hat{T}_0 \leq 1 - \delta, \quad 0 < \delta < 1.$$

The nonlinear flux condition at $x = 0$ is

$$D_t^{1-\alpha}\frac{\partial \hat{T}}{\partial x}\bigg|_{x=0} = -F[1-\hat{T}(0,t)], \quad t > 0,$$

where $F(1-\hat{T})$ has the same properties as specified in (1.7) and (1.8). At $x = l$, we will only consider the Neumann case as given by

$$\frac{\partial \hat{T}}{\partial x}(l,t) = 0, \quad t > 0.$$

To obtain an integral representation of $\hat{T}(x,t)$ analogous to (2.6), we use the Green's function $\hat{G}_\alpha(x,t|x_0,0)$ that satisfies

$$D_t^{1-\alpha}\frac{\partial^2 \hat{G}_\alpha}{\partial x^2} + vD_t^{1-\alpha}\frac{\partial \hat{G}_\alpha}{\partial x} - \frac{\partial \hat{G}_\alpha}{\partial t} = -\delta(x-x_0)\delta(t), \quad 0 < x, x_0 < l, \quad t > 0^-,$$

$$\hat{G}_\alpha|_{t=0^-} = 0, \quad 0 \le x \le l,$$

$$D_t^{1-\alpha}\frac{\partial \hat{G}_\alpha}{\partial x}\bigg|_{x=0} = 0, \quad \frac{\partial \hat{G}_\alpha}{\partial x}\bigg|_{x=l} = 0, \quad t > 0.$$

Then $\hat{T}(x,t)$ can be expressed as

$$\hat{T}(x,t) = \int_0^l \hat{G}_\alpha(x,t|x_0,0)\hat{T}_0(x_0)dx_0 + \int_0^t \hat{G}_\alpha(x,t-s|0,0)F[1-\hat{T}(0,s)]ds.$$

By setting $x = 0$, a Volterra equation for $\hat{T}(0,t) \equiv \hat{u}(t)$ follows as

$$\hat{u}(t) = \hat{h}(t) + \int_0^t \hat{k}(t-s)F[1-\hat{u}(s)]ds, \quad t \ge 0, \tag{4.2}$$

where

$$\hat{h}(t) = \int_0^l \hat{G}_\alpha(0,t|x_0,0)\hat{T}_0(x_0)dx_0,$$

and

$$\hat{k}(t) = \hat{G}_\alpha(0,t|0,0).$$

The relationship between the Green's function $\hat{G}_\alpha(x,t|x_0,0)$ for subdiffusion with advection and that for classical diffusion with advection $\hat{G}_1(x,t|x_0,0)$ is the same as (2.8), namely

$$\hat{G}_\alpha(x,t|x_0,0) = \int_0^\infty f_\alpha(z)\hat{G}_1(x,t^\alpha z|x_0,0)dz. \tag{4.3}$$

The Green's function problem of classical diffusion with advection takes the form

$$\frac{\partial^2 \hat{G}_1}{\partial x^2} + v\frac{\partial \hat{G}_1}{\partial x} - \frac{\partial \hat{G}_1}{\partial t} = -\delta(x-x_0)\delta(t), \quad 0 < x, x_0 < l, \quad t > 0^-, \tag{4.4}$$

$$\hat{G}_1|_{t=0^-} = 0, \quad 0 \le x \le l; \quad \frac{\partial \hat{G}_1}{\partial x}\bigg|_{x=0} = 0, \quad \frac{\partial \hat{G}_1}{\partial x}\bigg|_{x=l} = 0, \quad t > 0.$$

Since $\hat{G}_1(x,t|x_0,0) \ge 0$, (2.8) provides that $\hat{G}_\alpha(x,t|x_0,0) \ge 0$.

To determine the properties of $\hat{k}(t)$, it is useful to solve for $\hat{G}_1(x,t|x_0,0)$ in terms of the eigenfunctions $\{\Phi_n(x)\}$ that satisfy

$$\frac{\partial^2\Phi_n}{\partial x^2}+v\frac{\partial\Phi_n}{\partial x}+\lambda_n\Phi_n=0, 0<x<l, \tag{4.5}$$

$$\Phi_n'(0)=0, \quad \Phi_n'(l)=0.$$

The solution of (4.5) is given by

$$\Phi_n(x)=e^{-\frac{vx}{2}}\phi_n(x), \quad \phi_n(x)=\frac{2n\pi}{l}\cos\left(\frac{n\pi x}{l}\right)+v\sin\left(\frac{n\pi x}{l}\right),$$

$$\lambda_n=\frac{n^2\pi^2}{l^2}+\frac{v^2}{4}, \quad n=1,2,\cdots$$

Using an eigenfunction expansion to solve (4.4) yields

$$\hat{G}_1(x,t|x_0,0)=2le^{-\frac{v^2}{4}t}e^{-\frac{v}{2}(x-x_0)}\sum_{n=1}^{\infty}\frac{\phi_n(x_0)\phi_n(x)}{4n^2\pi^2+v^2l^2}e^{-\frac{n^2\pi^2}{l^2}t}.$$

Then follows from (4.3),

$$\hat{G}_\alpha(x,t|x_0,0)=2le^{-\frac{v}{2}(x-x_0)}\sum_{n=1}^{\infty}\frac{\phi_n(x_0)\phi_n(x)}{4n^2\pi^2+v^2l^2}\int_0^\infty f_\alpha(z)e^{-(\frac{v^2}{4}+\frac{n^2\pi^2}{l^2})zt^\alpha}dz.$$

Thus we obtain an expression for the kernel $\hat{k}(t)$ as

$$\hat{k}(t)=\hat{G}_\alpha(0,t|0,0)=\frac{8\pi^2}{l}\sum_{n=1}^{\infty}\frac{n^2}{4n^2\pi^2+v^2l^2}\int_0^\infty f_\alpha(z)e^{-(\frac{v^2}{4}+\frac{n^2\pi^2}{l^2})zt^\alpha}dz. \tag{4.6}$$

From (4.6), it is easily seen that

$$\hat{k}(t)>0, \quad \hat{k}'(t)<0, \quad 0\le t<\infty.$$

The asymptotic behavior of $\hat{k}(t)$ as $t\to\infty$ is obtained from (4.6) as

$$\begin{aligned}\hat{k}(t)&=\frac{8\pi^2}{l}\frac{1}{t^\alpha}\sum_{n=1}^{\infty}\frac{n^2}{4n^2\pi^2+v^2l^2}\int_0^\infty f_\alpha(st^{-\alpha})e^{-(\frac{v^2}{4}+\frac{n^2\pi^2}{l^2})s}ds\\ &\sim\frac{8\pi^2}{l}\frac{1}{t^\alpha}\sum_{n=1}^{\infty}\frac{n^2}{4n^2\pi^2+v^2l^2}\int_0^\infty e^{-(\frac{v^2}{4}+\frac{n^2\pi^2}{l^2})s}ds\\ &\sim\frac{8\pi^2}{l}\frac{1}{t^\alpha}\sum_{n=1}^{\infty}\frac{4l^2n^2}{(4n^2\pi^2+v^2l^2)^2}, \quad \text{as} \quad t\to\infty.\end{aligned}$$

This behavior allows us to conclude that

$$\hat{I}(t)\equiv\int_0^t\hat{k}(s)ds$$

is such that $\hat{I}(t)\to\infty$ as $t\to\infty$.

To obtain bounds on $\hat{h}(t)$, it is necessary to examine the properties of

$$\tilde{\hat{T}}(x,t)\equiv\int_0^l\hat{G}_\alpha(x,t|x_0,0)dx_0,$$

which satisfies (4.1) with the initial condition $\tilde{\hat{T}}_0 \equiv 1$ together with the boundary condition $\frac{\partial \tilde{\hat{T}}}{\partial x}|_{x=0} = 0$ and $\frac{\partial \tilde{\hat{T}}}{\partial x}|_{x=l} = 0$. This problem has the unique solution $\tilde{\hat{T}}(x,t) \equiv 1$, and hence

$$\int_0^l \hat{G}_\alpha(x,t|x_0,0)dx_0 = 1, \quad 0 \le x \le 1, \quad t \ge 0.$$

Thus follows

$$0 \le \hat{h}_0 \equiv \min \hat{h}(t) \le \hat{h}(t) \le 1 - \delta.$$

It is now possible to apply the same analysis to (4.2) as that for the local problem (2.7) in Section 2 to establish the existence of a $\hat{t} < \infty$ such that

$$\hat{u}(t) \to 1, \quad \hat{u}'(t) \to \infty \quad \text{as} \quad t \to \hat{t}, \quad t^* \le \hat{t} \le t^{**},$$

where t^* and t^{**} are defined by

$$\hat{I}(t^*) = \Lambda, \quad \hat{I}(t^{**}) = \kappa.$$

Since $\hat{I}(t) \to \infty$ as $t \to \infty$, there always exists a $t^{**} < \infty$ such that $\hat{I}(t^{**}) = \kappa$ thereby assuring that quenching will occur. Thus the addition of advection does not allow the suppression of quenching in a subdiffusive medium.

5. **Conclusion**

We have analyzed the behavior of the solution to a mathematical model for boundary quenching in a subdiffusive medium. The nonlinear flux condition that simulates quenching is examined for both local and nonlocal effects. The nonquenching boundary is constrained by a homogeneous condition of either Neumann or Dirichlet type. In all situations, it was determined that quenching occurs. This is in distinct contrast with the classical diffusion problem in which the Dirichlet case demonstrates that quenching can be avoided by sufficient enhancement of the material properties.

An extension of the model to include advection reveals that quenching always occurs regardless of the magnitude of the advection speed. This result is also in contrast to the classical diffusion problem where quenching is prevented by a sufficiently large advection speed.

REFERENCES

[1] C. Y. Chan and P. Tragoonsirisak, Critical source and critical width for a parabolic quenching problem in an infinite strip, *Neural Parallel Sci. Comput.*, 22:539–546, 2014.

[2] C. Y. Chan, A quenching criterion for a multi-dimensional parabolic problem due to a concentrated nonlinear source, *J. Comput. Appl. Math.*, 235:3724–3727, 2011.

[3] C. Y. Chan and H. T. Liu, Does quenching for degenerate parabolic equations occur at the boundaries?, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 8:121–128, 2001.

[4] C. M. Kirk and W. E. Olmstead, Thermal blow-up in a subdiffusive medium due to a nonlinear boundary flux, *Fract. Calc. Appl. Anal.*, 17:191–205, 2014.

[5] C. M. Kirk and C. A. Roberts, A Review of Quenching Results in the Context of Nonlinear Volterra Integral Equations, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 10:343–356, 2003.

[6] R. Metzler and J. Klafter, The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A*, 37:161–208, 2004.

[7] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach., *Phys. Rep.*, 339:1–77, 2000.

[8] C. A. Roberts, Recent results on blow-up and quenching for nonlinear Volterra equations, *Journal of Computational and Applied Mathematics*, 205:736–743, 2007.

[9] C. A. Roberts and W. E. Olmstead, Local and non-local boundary quenching, *Math. Meth. Appl. Sci.*, 22:1465–1484, 1999.

[10] J. Trujillo, Fractional models: Sub and super-diffusives, and undifferentiable solutions, *Innovation in Engineering Computational Technology* (eds. V.H.V. Topping, G. Montero, and R. Montenegro), Sax-Coburg Publ., UK, 371–402, 2006.

[11] M. M. Wyss and W. Wyss, Evolution, its fractional extension and generalization, *Fract. Calc. Appl. Anal.,* 4:273–284, 2001.