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Π-SEMIGROUP FOR INVARIANT UNDER TRANSLATIONS TIME SCALES AND ABSTRACT WEIGHTED PSEUDO ALMOST PERIODIC FUNCTIONS WITH APPLICATIONS

CHAO WANG, RAVI P. AGARWAL, AND DONAL O'REGAN

Department of Mathematics, Yunnan University, Kunming, Yunnan 650500,
People's Republic of China
chaowang@ynu.edu.cn
Department of Mathematics, Texas A&M University-Kingsville, TX 78363-8202,
Kingsville, TX, USA
Agarwal@tamuk.edu
School of Mathematics, Statistics and Applied Mathematics, National University
of Ireland, Galway, Ireland
donal.oregan@nuigalway.ie

ABSTRACT. In this paper, we introduce and discuss the concept of a Π-semigroup for invariant under translations time scales and the concept of abstract weighted pseudo almost periodic functions in Banach spaces. As an application, we obtain conditions for the existence of weighted pseudo almost periodic solutions for a class of neutral functional differential equations on time scales.

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1. INTRODUCTION

The basic calculus on time scales (see [1, 2]) was introduced by Hilger in [3] to unify continuous and discrete analysis, and to study dynamic equations on time scales [4, 5, 6, 7, 8]. Almost periodic, asymptotically almost periodic, and pseudo-almost periodic solutions for differential, and difference equations arise naturally in biology, economics and physics [9, 10, 11, 12, 13]. The concept of almost periodic functions on time scales in $\mathbb{R}^n$ was proposed and investigated in [15, 16]. In this paper, we introduce weighted pseudo almost periodic functions and define a new concept, namely, a Π-semigroup for invariant under translations time scales. This concept provides a new method to investigate abstract differential equations on time scales, and we provide sufficient conditions for the existence of weighted pseudo almost periodic mild solutions for a class of neutral functional differential equations on time scales.

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The organization of this paper is as follows. In Section 2, we introduce some preliminary results needed in the later sections. In Section 3, we introduce a Π-semigroup for invariant under translations time scales, and in Section 4, we introduce the concept of abstract weighted pseudo almost periodic functions. Finally in Section 5, we study the existence of weighted pseudo almost periodic solutions for abstract neutral functional differential equations on time scales.

2. PRELIMINARIES

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \varrho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are, respectively, defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \varrho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\varrho(t) = t$, left-scattered if $\varrho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

Throughout this paper, $X = (X, \|\,.\,\|)$ denotes a Banach space. We collect the following basic concepts and results from Ref. [1].

Definition 2.1. Let $D \subset \mathbb{T}$ be an open set, $f : D \to X$ and $t \in \mathbb{T}^\kappa$. If there exists a $B : \mathbb{T} \to X$ with the property that given any $\varepsilon > 0$, there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap D$ for some $\delta > 0$) such that

$$\|[f(\sigma(t)) - f(s)] - [\sigma(t) - s]B\| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U,$$

then we say that f is Δ-differentiable at t, and B is called the Δ-derivative of f at t. In what follows, we shall denote the Δ-derivative of f at t by $f^\Delta(t)$.

Lemma 2.2. *Assume that X is a Banach space, $f : \mathbb{T} \to X$ and $t \in \mathbb{T}^\kappa$. Then we have the following*

- (i)**:** *If f is differentiable at t, then f is continuous at t.*
- (ii)**:** *If f is continuous at t and t is right-scattered, then f is differentiable at t with*
$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$
- (iii)**:** *If t is right-dense, then f is differentiable at t if and only if the limit*
$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$
exists as a finite number. In this case
$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv): *If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Definition 2.3. A function $f : \mathbb{T} \to X$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.4. A function $f : \mathbb{T} \to X$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense point in $\mathbb{T}$.

Lemma 2.5. *Assume that $f : \mathbb{T} \to X$.*

- *(i)*: *If f is continuous, then f is rd-continuous.*
- *(ii)*: *If f is rd-continuous, then f is regulated.*
- *(iii)*: *The jump operator σ is rd-continuous.*
- *(iv)*: *If f is regulated or rd-continuous, then so is f^{σ}.*
- *(v)*: *Assume that f is continuous. If $g : \mathbb{T} \to X$ is regulated or rd-continuous, then $f \circ g$ also has the same property.*

A function $p : \mathbb{T} \to X$ is called regressive provided $I + \mu(t)p(t)$ is invertible for all $t \in \mathbb{T}^k$, where I is the identity operator. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, X)$.

Remark 2.6. An $n \times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called regressive provided

$$I + \mu(t)A(t) \text{ is invertible for all } t \in \mathbb{T},$$

and the class of all such regressive and rd-continuous functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

Lemma 2.7. *Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^{\Delta}(t) = f(t) \quad \textit{holds for all } t \in D.$$

Definition 2.8. Assume that $f : \mathbb{T} \to X$ is a regulated function. Any function F as in Lemma 2.7 is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta t = F(t) + C,$$

where $C \in X$ is an arbitrary element independent of t and F is a pre-antiderivative of f. We define the definite integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to X$ is called an antiderivative of $f : \mathbb{T} \to X$ provided

$$F^{\Delta}(t) = f(t) \quad \text{holds for all} \quad t \in \mathbb{T}^{\kappa}.$$

Definition 2.9. Let $t_0 \in \mathbb{T}$ and assume that $A : X \to X$ is regressive and $Y : \mathbb{T} \to X$. The unique solution of the IVP

$$Y^{\Delta}(t) = AY(t), \quad Y(t_0) = Y_0,$$

is called the operator exponential function at t_0, and it is denoted by $e_A(\cdot, t_0)Y_0$.

3. Π-SEMIGROUP FOR INVARIANT UNDER TRANSLATIONS TIME SCALES AND MOVING-OPERATORS

To introduce the concept of a Π-semigroup for invariant under translations time scales, we need the following basic definitions.

Definition 3.1 ([15, 17])**.** A time scale $\mathbb{T}$ is called a invariant under translations time scale (i.e., almost periodic time scale) if

$$\Pi := \big\{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\big\} \neq \{0\}.$$

In what follows, we denote by $\mathbb{T}^{\tau} = \{t + \tau : t \in \mathbb{T}\}$. It follows that if $\tau \in \Pi$ is a nonzero real number, then $\mathbb{T} = \mathbb{T}^{\tau}$ if and only if $\mathbb{T}$ is invariant under translations, i.e., $\mathbb{T}$ coincides exactly with $\mathbb{T}^{\tau}$ if $\mathbb{T}$ is invariant under translations. In fact, Definition 3.1 has the following equivalent form:

Definition 3.2. A time scale $\mathbb{T}$ is called invariant under translations time scale if

$$\Pi := \{\tau \in \mathbb{R} : \mathbb{T} \cap \mathbb{T}^{\pm\tau} = \mathbb{T}\} \neq \{0\}.$$

Theorem 3.3. *If $\mathbb{T}$ is an invariant under translations time scale, then the graininess function $\mu : \mathbb{T} \to \mathbb{R}^{+}$ is a periodic function.*

Proof. Assume that $\mathbb{T}$ is invariant under translation, then by Definition 3.2, we have $\mathbb{T} = \mathbb{T}^{\tau}$.

If t is a right dense point in $\mathbb{T}$, then $t+\tau$ is also a right dense point in $\mathbb{T}^{\tau}$, so $t+\tau$ is a right dense point in $\mathbb{T}$. Hence, $\mu(t+\tau)-\mu(t) = \sigma(t+\tau)-\sigma(t)-\tau = t+\tau-t-\tau = 0$, i.e., $\mu(t+\tau) = \mu(t)$.

If t is a right scattered point in $\mathbb{T}$, then $t+\tau$ is also a right scattered point in $\mathbb{T}^{\tau}$, so $t+\tau$ is a right scattered point in $\mathbb{T}$. Without loss of generality, we assume that $\tau \in \Pi$ and $\tau > 0$. It follows from $\sigma(t) > t$ that $\sigma(t) + \tau > t + \tau$, so we have

$$\sigma(t) + \tau \geq \sigma(t + \tau) > t + \tau. \tag{3.1}$$

From (3.1), we find

$$\sigma(t) - t = \mu(t) \geq \sigma(t + \tau) - (t + \tau) = \mu(t + \tau) > 0. \tag{3.2}$$

If in (3.2), $\mu(t) > \mu(t+\tau)$, then clearly μ is decreasing at all right scattered points in $\mathbb{T}$, which leads to a contradiction because $\mathbb{T}$ is an invariant under translations time scale. Hence, $\mu(t+\tau) = \mu(t)$. Therefore, μ is periodic. This completes the proof. □

Definition 3.4. The set Π in Definition 3.1 is called an invariant translations set for $\mathbb{T}$.

Theorem 3.5. *Let $\mathbb{T}$ be an invariant under translations time scale and Π be an invariant translations set. Then*

- (a)**:** Π *is a time scale.*
- (b)**:** *For all $\tau_1, \tau_2 \in \Pi$, we have $\tau_1 + \tau_2 \in \Pi$.*
- (c)**:** *For all $\tau_1, \tau_2, \tau_3 \in \Pi$, we have $(\tau_1 + \tau_2) + \tau_3 = \tau_1 + (\tau_2 + \tau_3)$.*
- (d)**:** *There exists an element $0 \in \Pi$, such that for all elements $\tau \in \Pi$, the equation $0 + \tau = \tau + 0 = \tau$ holds.*
- (e)**:** *For each $\tau \in \Pi$, there exists an element $-\tau \in \Pi$ such that $\tau + (-\tau) = 0$, where 0 is the identity element.*
- (f)**:** *For all $\tau_1, \tau_2 \in \Pi$, we have $\tau_1 + \tau_2 = \tau_2 + \tau_1$.*

Proof. All the parts follow from the definition, so we omit the details. □

Remark 3.6. Because Π is a time scale, we denote its graininess function as $\mu_\Pi : \Pi \to \mathbb{R}^+$, and its forward jump operator as $\sigma_\Pi(\tau_1) = \inf\{\tau_2 \in \Pi : \tau_2 > \tau_1\}$.

From Theorem 3.5, the following result follows immediately.

Theorem 3.7. *The pair $(\Pi, +)$ is an Abelian group.*

From the proof of Proposition 4.4 in Ref. [18], the following lemma is also immediate.

Lemma 3.8. *For the time scale Π, Π^κ has constant graininess, that is $\mu_\Pi(\tau) = h$, $\tau \in \Pi^\kappa$, for some $h \in \mathbb{R} \cup \{\infty\}$, $h \geq 0$.*

Remark 3.9. Since Π is an Abelian group, $\sup \Pi = +\infty,\ \inf \Pi = -\infty$.

For convenience, we let $\Pi^+ = [0, +\infty) \cap \Pi$. Let X be a Banach space, and $T_\tau : X \to X$ be a transformation. Obviously, $\{T_\tau : \tau \in \Pi\}$ is a set containing only one parameter. We define the multiplication as

$$T_{\tau_1} T_{\tau_2} = T_{\tau_1 + \tau_2}. \tag{3.3}$$

It follows that

$$T_{\tau_1}\big(T_{\tau_2} T_{\tau_3}\big) = \big(T_{\tau_1} T_{\tau_2}\big) T_{\tau_3} = T_{\tau_1 + \tau_2 + \tau_3},$$

$I = T_0$ is the identity, and $T_{-\tau}$ is the inverse element of T_τ. From these definitions, the following theorem is clear.

Theorem 3.10. *$\{T_\tau : \tau \in \Pi\}$ is an operator group with respect to the multiplication defined by (3.3). It is an Abelian group.*

In view of Theorem 3.10, we are now in a position to introduce some basic concepts which will be needed to define a Π-semigroup for an invariant under translations time scales.

Definition 3.11. Let $\mathbb{T}$ be an invariant under translations time scales, and $\{T_\tau\}$ be a family of bounded linear operators on Banach space X. If for all $\tau_1, \tau_2 \in \Pi^+$ the following holds:

$$T_{\tau_1+\tau_2} = T_{\tau_1} T_{\tau_2}, \tag{3.4}$$

then $\{T_\tau : \tau \in \Pi^+\}$ is called a one-parameter operator semigroup; if (3.4) holds for all $\tau \in \Pi$, we call $\{T_\tau : \tau \in \Pi\}$ a one-parameter operator group.

Definition 3.12. Let $\mathbb{T}$ be an invariant under translations time scales, and $\{T_\tau : \tau \in \Pi^+\}$ be an operator group on a Banach space X, i.e.,

$$T_{\tau_1} T_{\tau_2} = T_{\tau_1+\tau_2}, \quad \tau_1, \tau_2 \in \Pi^+, \quad T_0 = I.$$

If for every $\tau_0 \geq 0$ and any $\varepsilon > 0$, there is a neighborhood U of τ_0 (i.e., $U = (\tau_0 - \delta, \tau_0 + \delta) \cap \Pi^+$ for some $\delta > 0$) such that

$$\|T_\tau x - T_{\tau_0} x\| < \varepsilon \quad \text{for all } \tau \in U,$$

then we call $\{T_\tau : \tau \in \Pi^+\}$ the strong-continuous operator semigroup or the Π-semigroup.

Theorem 3.13. *Let $\mathbb{T}$ be an invariant under translations time scale, and $\{T_\tau : \tau \in \Pi^+\}$ be an operator semigroup on the Banach space X, and for any $x \in X$ and any $\varepsilon > 0$ there exists a neighborhood $U = (\tau_1 - \delta, \tau_1 + \delta) \cap \Pi^+$ for some $\delta > 0$, such that*

$$\|T_{|\sigma_\Pi(\tau_1)-\tau_2|} x - x\| \leq \varepsilon \quad \textit{for all } \tau_2 \in U, \tag{3.5}$$

then $\{T_\tau : \tau \in \Pi^+\}$ is a Π-semigroup.

Proof. For any $L \in \Pi$, $L > 0$, we claim that

$$\sup\{\|T_\tau\| : \tau \in [0, L]_\Pi\} < +\infty. \tag{3.6}$$

For any $x \in X$, we can take, $h \in \Pi$, $h > 0$, $c > 0$ such that

$$\sup\{\|T_\tau x\| : \tau \in [0, h]_\Pi\} \leq c.$$

Now for $\tau \in [0, L]_\Pi$, let $\tau = kh + r$, $r \in \Pi$, where $k \leq \frac{L}{h}$, $0 \leq r < h$. Then, it follows that

$$\|T_\tau x\| = \|T_{kh} T_r x\| \leq \|T_{kh}\| c.$$

Hence (3.6) holds. In what follows we let $M := \sup\{\|T_\tau\| : \tau \in [0, L]_\Pi\}$.

For any $\varepsilon > 0$, there is δ, such that for $\tau_2 \in (\tau_1 - \delta, \tau_1 + \delta)_{\Pi^+}$, we have

(i): If $\tau_2 > \tau_1$, then $\sigma_\Pi(\tau_1) = \tau_1$, and we have

$$\|T_{\tau_2}x - T_{\tau_1}x\| \leq \|T_{\sigma_\Pi(\tau_1)}(T_{\tau_2-\sigma_\Pi(\tau_1)} - I)x + T_{\tau_1}(T_{\sigma_\Pi(\tau_1)-\tau_1} - I)x\| \leq 2M\varepsilon.$$

In the above $\sigma_\Pi(\tau_1) = \tau_1$. In fact, if $\sigma_\Pi(\tau_1) > \tau_1$, then τ_1 is a right scattered point, which implies that $\tau_2 = \tau_1$, and this contradicts $\tau_2 > \tau_1$.

(ii): If $\tau_2 \leq \tau_1$, then $\tau_2 \leq \tau_1 \leq \sigma_\Pi(\tau_1)$, which yields $0 \leq \sigma_\Pi(\tau_1) - \tau_1 \leq \sigma_\Pi(\tau_1) - \tau_2$. Hence, we find

$$\|T_{\tau_2}x - T_{\tau_1}x\| \leq \|T_{\tau_2}(I - T_{\sigma_\Pi(\tau_1)-\tau_2})x + T_{\tau_1}(T_{\sigma_\Pi(\tau_1)-\tau_1} - I)x\| \leq 2M\varepsilon.$$

Therefore, for $\tau_2 \in (\tau_1 - \delta, \tau_1 + \delta)_{\Pi^+}$, the following holds:

$$\|T_{\tau_2}x - T_{\tau_1}x\| \leq 2M\varepsilon.$$

Hence, $\{T_\tau : \tau \in \Pi^+\}$ is a Π-semigroup and (3.5) holds. This completes the proof. □

In the following, we introduce the definition of infinitesimal generator of a Π-semigroup.

Definition 3.14. Let $\mathbb{T}$ be an invariant under translations time scale and $\{T_\tau : \tau \in \Pi^+\}$ be a Π-semigroup on a Banach space X. Let $\mathscr{D}$ denote a subset of X, which has the property that for each $x \in \mathscr{D}$ there exists a $y \in X$ such that for any $\varepsilon > 0$, there is a neighborhood $U = (\tau_1 - \delta, \tau_1 + \delta)_{\Pi^+}$ for some $\delta > 0$, which satisfy

$$\|(T_{|\sigma_\Pi(\tau_1)-\tau_2|} - I)x - y|\sigma_\Pi(\tau_1) - \tau_2|\| < \varepsilon|\sigma_\Pi(\tau_1) - \tau_2|, \quad \tau_2 \in U. \tag{3.7}$$

We define $A : \mathscr{D} \to X$ satisfying $Ax = y$, where y is fixed by (3.7). In what follows we call this A the infinitesimal generator of this Π-semigroup.

Remark 3.15. From (3.7), it follows that

$$\left\| \frac{T_{|\sigma_\Pi(\tau_1)-\tau_2|} - I}{\sigma_\Pi(\tau_1) - \tau_2} x - Ax \right\| < \varepsilon.$$

By Lemma 3.8, $\mu_\Pi(\tau) \equiv h > 0$ is a constant, thus A is independent of the variable τ.

(i): If $\mathbb{T} = \mathbb{R}$, then $\Pi = \mathbb{R}$. Thus from (3.7), we have

$$\lim_{t\to 0} \left\| \frac{1}{t}(T_t - I)x - y \right\| = 0,$$

and hence

$$A = \lim_{t\to 0} \frac{1}{t}(T_t - I).$$

(ii): If $\mathbb{T} = h\mathbb{Z}$, then $\Pi = h\mathbb{Z}$. Thus from (3.7), we find

$$\left\| \frac{1}{h}(T_h - I) - y \right\| = 0,$$

and hence

$$A = \frac{1}{h}(T_h - I).$$

Definition 3.16 ([20])**.** A linear operator T from one topological vector space X to another one Y is said to be densely defined if the domain of T is a dense subset of X and the range of T is contained within Y.

Theorem 3.17. *Let $\mathbb{T}$ be an invariant under translations time scale, $\{T_\tau : \tau \in \Pi^+\}$ be a Π-semigroup on Banach space X satisfying* (3.5)*, and A be the infinitesimal generator of the Π-semigroup. Then A is a closed densely defined operator and for every $x \in \mathscr{D}(A)$, the following holds:*

$$(T_\tau x)^{\Delta_\Pi} = A(T_\tau x) = T_\tau Ax, \tag{3.8}$$

that is

$$(T_\tau x) - x = \int_0^\tau AT_s x\Delta_\Pi s = \int_0^\tau T_s Ax\Delta_\Pi s, \tag{3.9}$$

where $\mathscr{D}(A)$ denotes the domain of the operator A and Δ_Π is the differential operator over the time scale Π.

Proof. First we show that A is a densely defined operator. Note that for any $x \in X$, we have

$$\begin{aligned}\left\|\int_0^{|\sigma_\Pi(\tau_1)-\tau_2|} T_\theta x\Delta_\Pi\theta - |\sigma_\Pi(\tau_1)-\tau_2|x\right\| &= \left\|\int_0^{|\sigma_\Pi(\tau_1)-\tau_2|}(T_\theta x - x)\Delta_\Pi\theta\right\| \\ &\leq |\sigma_\Pi(\tau_1)-\tau_2| \sup_{0\leq\theta\leq|\sigma_\Pi(\tau_1)-\tau_2|}\|T_\theta x - x\| \\ &< |\sigma_\Pi(\tau_1)-\tau_2|\varepsilon. \end{aligned} \tag{3.10}$$

Let $y = \int_0^\tau T_\theta x\Delta_\Pi\theta$, then

$$\begin{aligned} T_{|\sigma_\Pi(\tau_1)-\tau_2|}y - y &= \int_0^\tau (T_{\theta+|\sigma_\Pi(\tau_1)-\tau_2|} - T_\theta x)\Delta_\Pi\theta \\ &= \int_{|\sigma_\Pi(\tau_1)-\tau_2|}^{\tau+|\sigma_\Pi(\tau_1)-\tau_2|} T_\theta x\Delta_\Pi\theta - \int_0^\tau T_\theta x\Delta_\Pi\theta \\ &= \int_\tau^{\tau+|\sigma_\Pi(\tau_1)-\tau_2|} T_\theta x\Delta_\Pi\theta - \int_0^{|\sigma_\Pi(\tau_1)-\tau_2|} T_\theta x\Delta_\Pi\theta \\ &= \int_0^{|\sigma_\Pi(\tau_1)-\tau_2|} T_\theta(T_\tau x)\Delta_\Pi\theta - \int_0^{|\sigma_\Pi(\tau_1)-\tau_2|} T_\theta x\Delta_\Pi\theta. \end{aligned}$$

Since (3.10) holds for any $x \in X$, then it follows that

$$\begin{aligned} &\|(T_{|\sigma_\Pi(\tau_1)-\tau_2|}y - y) - |\sigma_\Pi(\tau_1)-\tau_2|(T_\tau x - x)\| \\ &= \left\|\int_0^{|\sigma_\Pi(\tau_1)-\tau_2|} T_\theta(T_\tau x - x)\Delta_\Pi\theta - |\sigma_\Pi(\tau_1)-\tau_2|(T_\tau x - x)\right\| \\ &\leq |\sigma_\Pi(\tau_1)-\tau_2|\varepsilon. \end{aligned}$$

Therefore, $y \in \mathscr{D}(A)$, so $\overline{\mathscr{D}(A)} = X$.

Next, we will show that (3.8) and (3.9) hold. Since

$$\frac{(T_{|\sigma_\Pi(\tau_1)-\tau_2|}-I)T_{\tau_1}x}{|\sigma_\Pi(\tau_1)-\tau_2|}=\frac{T_{\tau_1}(T_{|\sigma_\Pi(\tau_1)-\tau_2|}-I)x}{|\sigma_\Pi(\tau_1)-\tau_2|}=Ax,$$

we have

$$\begin{aligned}&\|T_{\tau_1}(T_{|\sigma_\Pi(\tau_1)-\tau_2|}-I)x-|\sigma_\Pi(\tau_1)-\tau_2|T_{\tau_1}Ax\|\\ &\le\|T_{\tau_1}\|\|(T_{|\sigma_\Pi(\tau_1)-\tau_2|}-I)x-|\sigma_\Pi(\tau_1)-\tau_2|Ax\|\le\|T_{\tau_1}\|\varepsilon|\sigma_\Pi(\tau_1)-\tau_2|,\end{aligned}\tag{3.11}$$

and so, $T_\tau x\in\mathscr{D}(A)$. From (3.11), we also have

$$\|(T_{|\sigma_\Pi(\tau_1)-\tau_2|}-I)x-|\sigma_\Pi(\tau_1)-\tau_2|Ax\|\le\varepsilon|\sigma_\Pi(\tau_1)-\tau_2|.\tag{3.12}$$

(i)**:** If $\tau_2>\tau_1$, then from (3.12) and Theorem 3.13 it follows that

$$\begin{aligned}&\|(T_{\sigma_\Pi(\tau_1)}-T_{\tau_2})x-(\sigma_\Pi(\tau_1)-\tau_2)T_{\tau_1}Ax\|\\ &\quad\le\|T_{\sigma_\Pi(\tau_1)}(I-T_{\tau_2-\sigma_\Pi(\tau_1)})x-(\sigma_\Pi(\tau_1)-\tau_2)T_{\sigma_\Pi(\tau_1)}Ax\\ &\qquad+(\sigma_\Pi(\tau_1)-\tau_2)T_{\sigma_\Pi(\tau_1)}Ax-(\sigma_\Pi(\tau_1)-\tau_2)T_{\tau_1}Ax\|\\ &\quad\le\|T_{\sigma_\Pi(\tau_1)}\|\|(\tau_2-\sigma_\Pi(\tau_1))Ax-(I-T_{\tau_2-\sigma_\Pi(\tau_1)})x\|\\ &\qquad+\|T_{\tau_1}\|\|(I-T_{\sigma_\Pi(\tau_1)-\tau_1})Ax\|(\tau_2-\sigma_\Pi(\tau_1))\\ &\quad\le M\varepsilon(\tau_2-\sigma_\Pi(\tau_1)),\end{aligned}$$

where $M:=\sup\{\|T_\tau\|:\tau\in[0,L]_\Pi\}$, and $L\in\Pi$ is any fixed positive constant. In the above it is necessary that $\sigma_\Pi(\tau_1)=\tau_1$, since if $\sigma_\Pi(\tau_1)>\tau_1$, then τ_1 is right scattered point, which implies that $\tau_2=\tau_1$, and this contradictions our assumption that $\tau_2>\tau_1$.

(ii)**:** If $\tau_2\le\tau_1$, then it follows from $\tau_2\le\tau_1\le\sigma_\Pi(\tau_1)$ that $0\le\tau_1-\tau_2\le\sigma_\Pi(\tau_1)-\tau_2$. Hence, from (3.12) and Theorem 3.13, we obtain

$$\begin{aligned}&\|(T_{\sigma_\Pi(\tau_1)}-T_{\tau_2})x-(\sigma_\Pi(\tau_1)-\tau_2)T_{\tau_1}Ax\|\\ &\quad\le\|T_{\tau_2}(T_{\sigma_\Pi(\tau_1)-\tau_2}-I)x-(\sigma_\Pi(\tau_1)-\tau_2)T_{\tau_2}Ax\\ &\qquad+(\sigma_\Pi(\tau_1)-\tau_2)T_{\tau_2}Ax-(\sigma_\Pi(\tau_1)-\tau_2)T_{\tau_1}Ax\|\\ &\quad\le\|T_{\tau_2}\|\|(T_{\sigma_\Pi(\tau_1)-\tau_2}-I)x-(\sigma_\Pi(\tau_1)-\tau_2)Ax)\|\\ &\qquad+\|T_{\tau_2}\|\|(I-T_{\tau_1-\tau_2})Ax\|(\sigma_\Pi(\tau_1)-\tau_2)\\ &\quad\le M\varepsilon(\sigma_\Pi(\tau_1)-\tau_2),\end{aligned}$$

where $M:=\sup\{\|T_\tau\|:\tau\in[0,L]_\Pi\}$, and $L\in\Pi$ is any fixed positive constant.

Therefore, $(T_\tau x)^{\Delta_\Pi}=T_\tau Ax=AT_\tau x$. Since (3.9) is the integral form of (3.8), we can conclude that (3.9) holds.

Finally, we show that A is a closed operator. Let $x_n \in \mathscr{D}(A)$, $x_n \to x$, $Ax_n \to y$, then by (3.12), we have

$$\begin{aligned}\|(T_{|\sigma_\Pi(\tau_1)-\tau_2|} - I)x - |\sigma_\Pi(\tau_1) - \tau_2|y\| &= \lim_{n\to\infty} \|(T_{|\sigma_\Pi(\tau_1)-\tau_2|} - I)x_n - |\sigma_\Pi(\tau_1) - \tau_2|Ax_n\| \\ &\leq \varepsilon|\sigma_\Pi(\tau_1) - \tau_2|.\end{aligned}$$

Hence, $x \in \mathscr{D}(A)$ and $Ax = y$, that is, A is a closed operator. This completes the proof. □

Theorem 3.18. *Let $\mathbb{T}$ be an invariant under translations time scale and X be a Banach space. Assume that $\{T_\tau : \tau \in \Pi^+\}$ is a Π-semigroup, A is the infinitesimal generator of the Π-semigroup and $\mathscr{D}(A) = X$, $e_A(\tau_1 + \tau_2, 0) = e_A(\tau_1, 0)e_A(\tau_2, 0)$ for all $\tau_1, \tau_2 \in \Pi^+$. Then,*

$$T_\tau = e_A(\tau, 0),\ \tau \in \Pi^+,$$

where $\mathscr{D}(A)$ denotes the domain of A.

Proof. From Theorem 3.17, we have

$$\big(e_A(\tau, 0)x\big)^{\Delta_\Pi} = Ae_A(\tau, 0)x = e_A(\tau, 0)Ax.$$

Further, since $e_A(\tau, 0)$ is Δ-differentiable on Π, from Definition 2.1, for any $\varepsilon > 0$, there is δ, such that for $\tau_2 \in (\tau_1 - \delta, \tau_1 + \delta)_{\Pi^+}$, it follows that

$$\|(e_A(\sigma_\Pi(\tau_1), 0) - e_A(\tau_2, 0))x - (\sigma_\Pi(\tau_1) - \tau_2)Ae_A(\tau_1, 0)x\| \leq \varepsilon|\sigma_\Pi(\tau_1) - \tau_2|, \tag{3.13}$$

and hence:

(i)**:** If $\tau_2 > \tau_1$, then it follows from (3.13) that

$$\begin{aligned}&\|e_A(\sigma_\Pi(\tau_1), 0)[I - e_A(\tau_2 - \sigma_\Pi(\tau_1), 0)x - (\sigma_\Pi(\tau_1) - \tau_2)e_A(\tau_1, \sigma_\Pi(\tau_1))Ax]\| \\ &\quad\leq \|e_A(\sigma_\Pi(\tau_1), 0)\|\|[I - e_A(\tau_2 - \sigma_\Pi(\tau_1), 0)x - (\sigma_\Pi(\tau_1) - \tau_2)e_A(\tau_1, \sigma_\Pi(\tau_1))Ax]\| \\ &\quad\leq M\varepsilon|\sigma_\Pi(\tau_1) - \tau_2|.\end{aligned}$$

In the above $\sigma_\Pi(\tau_1) = \tau_1$. Indeed, if $\sigma_\Pi(\tau_1) > \tau_1$, then τ_1 is a right scattered point, and then $\tau_2 = \tau_1$, which is a contradiction since $\tau_2 > \tau_1$.

(ii)**:** If $\tau_2 \leq \tau_1$, then it follows from $\tau_2 \leq \tau_1 \leq \sigma_\Pi(\tau_1)$, that $0 \leq \tau_1 - \tau_2 \leq \sigma_\Pi(\tau_1) - \tau_2$. Hence, from (3.13) we find

$$\begin{aligned}&\|e_A(\tau_2, 0)[(e_A(\sigma_\Pi(\tau_1) - \tau_2, 0) - I)x - (\sigma_\Pi(\tau_1) - \tau_2)Ax \\ &\qquad + (\sigma_\Pi(\tau_1) - \tau_2)(I - e_A(\tau_1, \tau_2))Ax]\| \\ &\quad\leq \|e_A(\tau_2, 0)\|\|[(e_A(\sigma_\Pi(\tau_1) - \tau_2, 0) - I)x - (\sigma_\Pi(\tau_1) - \tau_2)Ax]\| \\ &\qquad + M\varepsilon|\sigma_\Pi(\tau_1) - \tau_2| \\ &\quad\leq 2M\varepsilon|\sigma_\Pi(\tau_1) - \tau_2|,\end{aligned}$$

where $M := \sup\{\|e_A(\tau, 0)\| : \tau :\in [0, L]_\Pi\}$, and $L \in \Pi$ is any fixed positive constant.

From (i), (ii), we obtain

$$\|(e_A(|\sigma_\Pi(\tau_1)-\tau_2|,0)-I)x-|\sigma_\Pi(\tau_1)-\tau_2|Ax\|\leq 2M\varepsilon|\sigma_\Pi(\tau_1)-\tau_2|.$$

Therefore, A is the infinitesimal generator of $\{T_\tau : \tau\in\Pi^+\}$. This completes the proof. □

Remark 3.19. If (i) $\mathbb{T}=\mathbb{R}$, then $\Pi=\mathbb{R}$, $T_\tau=e_A(\tau,0)=e^{A\tau}$. Clearly, it satisfies $T_{\tau_1+\tau_2}=T_{\tau_1}T_{\tau_2}$. If (ii) $\mathbb{T}=\mathbb{Z}$, then $\Pi=\mathbb{Z}$, $T_\tau=e_A(\tau,0)=(I+A)^\tau$, which also satisfies $T_{\tau_1+\tau_2}=T_{\tau_1}T_{\tau_2}$.

Now we will introduce a new concept that will be needed later.

Definition 3.20. Let A be the infinitesimal generator of the Π-semigroup. We call $\tilde{e}_A(t,t_0), t_0\in\mathbb{T}$ the exponential function generated by A on the time scale $\mathbb{T}$. We also let $\mathscr{T}_t=\tilde{e}_A(t,t_0)$ and call $\mathscr{T}_t$ the moving-operator on $\mathbb{T}$.

Remark 3.21. In Figure 1 we give a relationship between $\mathbb{T}$, Π, A, and $\mathscr{T}_t$. Note that if $\mathbb{T}=\Pi$, then the Π-semigroup will strictly include the continuous ($\mathbb{T}=\mathbb{R}$) and the discrete ($\mathbb{T}=\mathbb{Z}$) case of a $\mathscr{C}_0$-semigroup.

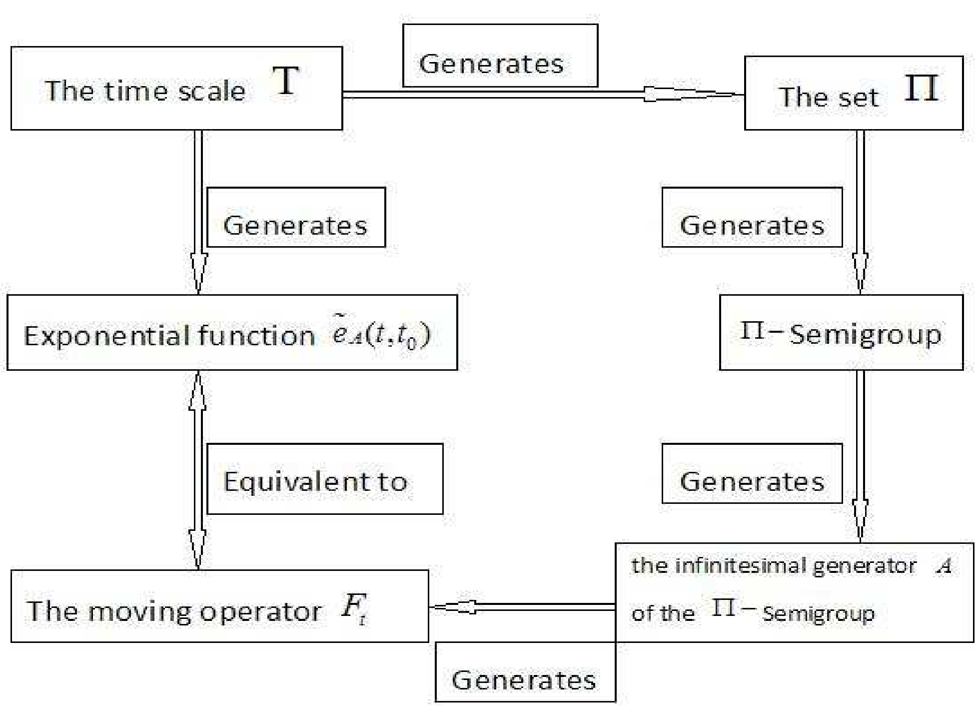

Figure 1. The generation relationship of $\mathbb{T}$, Π-semigroup, A, $\mathscr{T}_t$.

Let X be a Banach space, and consider the following system:

$$x^\Delta=Ax(t),\quad x(t_0)=x_0,\quad t_0\in\mathbb{T}, \tag{3.14}$$

where A is the infinitesimal generator of a Π-semigroup satisfying all the conditions in Theorem 3.18, and $x : \mathbb{T} \to X$.

Theorem 3.22. *The fundamental solution of the system* (3.14) *can be expressed as*

$$x(t) = \mathscr{T}_t x_0,$$

Proof. From Definition 3.20, $\mathscr{T}_t = \tilde{e}_A(t, t_0)$, and hence

$$x^\Delta = (\mathscr{T}_t x_0)^\Delta = A\mathscr{T}_t x(t_0) = Ax(t).$$

Therefore, $\mathscr{T}_t x_0$ is the fundamental solution of (3.14). This completes the proof. □

From Theorem 3.22, the following result follows immediately.

Theorem 3.23. *Let A be the infinitesimal generator of the* Π*-semigroup, and let $\mathscr{T}_t$ be the moving-operator on $\mathbb{T}$. Then*

$$(\mathscr{T}_t x)^\Delta = A(\mathscr{T}_t x) = \mathscr{T}_t A x,$$

that is

$$(\mathscr{T}_t x) - x = \int_{t_0}^{t} A\mathscr{T}_s x \Delta s = \int_{t_0}^{t} \mathscr{T}_s A x \Delta s.$$

Remark 3.24. Note the Π-semigroups studied in this paper are more general than the $\mathscr{C}_0$-semigroups introduced in [21]. If we let $\mathbb{T} = \Pi$, we obtain that the Π-semigroups in this paper turn into the $\mathscr{C}_0$-semigroups in [21]. If $\Pi \neq \mathbb{T}$, for example, $\Pi \cap \mathbb{T} = \emptyset$ (see Example 1.2 from Ref. [16]), the results in [21] cannot be applied to study abstract dynamic equations on time scales.

4. ABSTRACT WEIGHTED PSEUDO ALMOST PERIODIC FUNCTIONS

In this section, we shall assume that $\mathbb{T}$ is an invariant under translations time scale. We introduce abstract weighted pseudo almost periodic functions on time scales. For this, we need the following definitions.

Definition 4.1 ([16])**.** Let $\mathbb{T}$ be an invariant under translations time scale and let X be a Banach space. A function $f : \mathbb{T} \times X \to X$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in X$ if the ε-translation set of f

$$E\{\varepsilon, f, D\} = \{\tau \in \Pi : \|f(t+\tau, x) - f(t, x)\| < \varepsilon, \quad \text{for all } (t, x) \in \mathbb{T} \times D\}$$

is a relatively dense set in Π for all $\varepsilon > 0$ and for each compact subset D of X; that is, for any given $\varepsilon > 0$ and each compact subset D of X, there exists a constant $l(\varepsilon, D) > 0$ such that each interval of length $l(\varepsilon, D)$ contains a $\tau(\varepsilon, D) \in E\{\varepsilon, f, D\}$ such that

$$\|f(t+\tau, x) - f(t, x)\| < \varepsilon, \quad \text{for all } t \in \mathbb{T} \times D.$$

Here, τ is called the ε-translation number of f and $l(\varepsilon, D)$ is called the inclusion length of $E\{\varepsilon, f, D\}$.

Now we state several results which can be proved by following the same lines as in Ref. [14] and Refs. [15, 16]. We will use them in the applications later. Below X is a Banach space

Theorem 4.2. *Let $f : \mathbb{T} \times X \to X$ be almost periodic in t uniformly for $x \in X$. Then it is uniformly continuous and bounded on $\mathbb{T} \times D$; here D is any compact subset of X.*

Corollary 4.1. *Let $f : \mathbb{T} \to X$ be almost periodic. Then it is uniformly continuous and bounded on $\mathbb{T}$.*

Theorem 4.3. *If $F : \mathbb{R} \times X \to X$ is almost periodic in t uniformly for $x \in X$, then $F(t, x)$ is also continuous on $\mathbb{T} \times X$ and almost periodic in t uniformly for $x \in X$.*

Corollary 4.2. *If $F : \mathbb{R} \to X$ is almost periodic, then $F(t)$ is almost periodic on $\mathbb{T}$.*

Theorem 4.4. *If $f_n : \mathbb{T} \times X \to X, n = 1, 2, \ldots$ are almost periodic in t for $x \in X$, and the sequence $\{f_n(t, x)\}$ uniformly converges to $f(t, x)$ on $\mathbb{T} \times D$, then $f(t, x)$ is almost periodic in t uniformly for $x \in X$; here D is any compact subset of X.*

Corollary 4.3. *If $f_n : \mathbb{T} \to X, n = 1, 2, \ldots$ are almost periodic on $\mathbb{T}$, and the sequence $\{f_n(t)\}$ uniformly converges to $f(t)$ on $\mathbb{T}$, then $f(t)$ is almost periodic on $\mathbb{T}$.*

Theorem 4.5. *Let $f : \mathbb{T} \times X \to X$ be almost periodic in t uniformly for $x \in X$, and let*

$$F(t, x) = \int_0^t f(s, x)\Delta s.$$

Then $F : \mathbb{T} \times X \to X$ is almost periodic in t uniformly for $x \in X$ if and only if $F(t, x)$ is bounded on $\mathbb{T} \times D$; here D is any compact subset of X.

Next, we will prove a theorem that will be needed later.

Theorem 4.6. *If $u(t) : \mathbb{T} \to X$ and $\hat{g}(t) : \mathbb{T} \to \Pi$ are almost periodic, and $E\{\varepsilon, u\} \cap E\{\varepsilon, \hat{g}\} \neq \emptyset$, then $u\big(t - \hat{g}(t)\big)$ is almost periodic.*

Proof. Since $u : \mathbb{T} \to X$ is almost periodic, for any $\varepsilon > 0$, there exists a τ such that

$$\big\| u\big(t + \tau - \hat{g}(t)\big) - u\big(t - \hat{g}(t)\big) \big\| < \frac{\varepsilon}{2}.$$

Now from Theorem 4.2, we find

$$\big\| u\big(t + \tau - \hat{g}(t + \tau)\big) - u\big(t + \tau - \hat{g}(t)\big) \big\| < \frac{\varepsilon}{2}.$$

Hence, it follows that

$$\begin{aligned}\big\|u\big(t+\tau-\hat{g}(t+\tau)\big)-u\big(t-\hat{g}(t)\big)\big\| &= \big\|u\big(t+\tau-\hat{g}(t+\tau)\big)-u\big(t+\tau-\hat{g}(t)\big)\big\| \\ &\quad +\big\|u\big(t+\tau-\hat{g}(t)\big)-u\big(t-\hat{g}(t)\big)\big\| \\ &\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

This completes the proof. □

Definition 4.7 ([22, 23])**.** Assume that X is a Banach space, $f:\mathbb{T}\times X\to X$ and $T\in\Pi$. Then $m(f)$ is called mean-value of $f(t,x)$ if $m(f)=\lim\limits_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T}f(t,x)\Delta t\in X$, where $t_0\in\mathbb{T}$.

From Theorem 3.2 in Ref. [23], the following result follows immediately:

Theorem 4.8. *Let X be a Banach space, and let $f:\mathbb{T}\times X\to X$ be almost periodic in t uniformly for $x\in X$. Then $m(f)$ exists uniformly for $x\in X$.*

Remark 4.9. If $f(t,x)$ is almost periodic in t uniformly for $x\in X$, then $\|f(t,x)\|$ is almost periodic in t uniformly for $x\in X$. This follows from the fact that for any $\varepsilon>0$, there is a $\tau\in E\{\varepsilon,f,D\}$ such that

$$\big|\|f(t+\tau,x)\|-\|f(t,x)\|\big|\le\|f(t+\tau,x)-f(t,x)\|<\varepsilon.$$

Hence, by Theorem 4.8, we have $m(\|f\|)=\lim\limits_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T}\|f(t,x)\|\Delta t$ exists uniformly for $x\in X$, where $t_0\in\mathbb{T}$.

Let X,Y be two Banach spaces endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. We denote by $B(X,Y)$ the Banach space of all bounded linear operators from X to Y. This is denoted as $B(X)$ when $X=Y$. Now $BC(\mathbb{T},X)$ is the space of bounded continuous function from $\mathbb{T}$ to X equipped with the supremum norm defined by

$$\|u\|_\infty=\sup_{t\in\mathbb{T}}\|u(t)\|_X.$$

Let U be the collection of all functions (weights) $\rho:\mathbb{T}\to(0,+\infty)$ which are locally integrable over $\mathbb{T}$ and are such that $\rho(t)>0$ for almost each $t\in\mathbb{T}$. For each $r>0$ and $\rho\in U$, we set

$$m(r,\rho,t_0)=\int_{t_0-r}^{t_0+r}\rho(t)\Delta t,\ \text{where}\, t_0\in\mathbb{T},\ r\in\Pi.$$

Let

$$U_\infty=\big\{\rho\in U:\lim_{r\to\infty}m(r,\rho)=\infty\big\};\quad U_B=\Big\{\rho\in U_\infty:\rho\,\text{is bounded and}\inf_{t\in\mathbb{T}}\rho(t)>0\Big\}.$$

Before introducing the concept of weighted pseudo almost periodic functions, we need to define the "weighted ergodic" spaces $PAP_0(\mathbb{T},X,\rho)$ and $PAP_0(\mathbb{T}\times X,X,\rho)$.

Let $\rho \in U_\infty$. We define

$$\begin{aligned} PAP_0(\mathbb{T}, X, \rho) &= \Big\{ f \in BC(\mathbb{T}, X) : \lim_{r\to\infty} \frac{1}{m(r, \rho, t_0)} \int_{t_0-r}^{t_0+r} \|f(t)\|_X \rho(t)\Delta t = 0, \\ &\qquad \text{where } r \in \Pi,\ t_0 \in \mathbb{T} \Big\} \end{aligned}$$

and

$$\begin{aligned} PAP_0(\mathbb{T}\times X, X, \rho) &= \Big\{ f \in BC(\mathbb{T}\times X, X) : \\ &\qquad \lim_{r\to\infty} \frac{1}{m(r, \rho, t_0)} \int_{t_0-r}^{t_0+r} \|f(t,x)\|_X \rho(t)\Delta t = 0 \\ &\qquad \text{uniformly for } x \in X,\ \text{ where } r \in \Pi,\ t_0 \in \mathbb{T} \Big\}. \end{aligned}$$

Remark 4.10. If $\rho(t) \equiv 1$, $t_0 = 0 \in \mathbb{T}$, then $PAP_0(\mathbb{T}, X, \rho)$ and $PAP_0(\mathbb{T}\times X, X, \rho)$ are reduced to ergodic spaces $PAP_0(\mathbb{T}, X)$ and $PAP_0(\mathbb{T}\times X, X)$ respectively, which are defined as

$$PAP_0(\mathbb{T}, X) = \Big\{ f \in BC(\mathbb{T}, X) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(t)\|\Delta t = 0 \Big\}$$

and

$$\begin{aligned} PAP_0(\mathbb{T}\times X, X) &= \Big\{ f \in BC(\mathbb{T}\times X, X) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(t,x)\|\Delta t = 0, \\ &\qquad \text{uniformly for } x \in X \Big\}. \end{aligned}$$

Definition 4.11. A function $f : \mathbb{T} \to X$ is called pseudo almost periodic, if $f = g + \phi$ where $g \in AP(\mathbb{T}, X)$ and $\phi \in PAP_0(\mathbb{T}, X)$.

We denote the set of all such functions by $PAP(\mathbb{T}, X)$.

Definition 4.12. A function $f : \mathbb{T}\times X \to X$ is called pseudo almost periodic, if $f = g + \phi$ where $g \in AP(\mathbb{T}\times X, X)$ and $\phi \in PAP_0(\mathbb{T}\times X, X)$.

We denote the set of all such functions by $PAP(\mathbb{T}\times X, X)$.

Definition 4.13. Let $\rho \in U_\infty$. A function $f \in BC(\mathbb{T}, X)$ is called weighted pseudo almost periodic, if $f = g + \phi$ where $g \in AP(\mathbb{T}, X)$ and $\phi \in PAP_0(\mathbb{T}, X, \rho)$.

We denote the set all such functions by $WPAP(\mathbb{T}\times X, X, \rho)$.

Definition 4.14. Let $\rho \in U_\infty$. A function $f \in BC(\mathbb{T}\times X, X)$ is called weighted pseudo almost periodic, if $f = g + \phi$ where $g \in AP(\mathbb{T}\times X, X)$ and $\phi \in PAP_0(\mathbb{T}\times X, X, \rho)$.

We denote the set all such functions by $WPAP(\mathbb{T}\times X, X, \rho)$. Since the space $WPAP(\mathbb{T}\times X, X)$ is a particular case of the space $WPAA(\mathbb{T}\times X, X)$ (i.e., the weighted pseudo almost periodic functions on time scales are a particular case of the weighted pseudo almost automorphic functions on time scales) and some of the results in U_B are also true for U_∞, Theorems 4.15–4.18 can be deduced directly from Ref. [24].

Theorem 4.15. *Let $\rho\in U_\infty$ be fixed. Then the decomposition of a weighted pseudo almost periodic function $f=g+\phi$ where $g\in AP(\mathbb{T}, X)$ and $\phi\in PAP_0(\mathbb{T}, X, \rho)$ is unique.*

Theorem 4.16. *If $\rho\in U_\infty$, then $(WPAP(\mathbb{T}, X, \rho), \|\cdot\|_\infty)$ is a Banach space.*

Theorem 4.17. *If $f\in WPAP(\mathbb{T}, X, \rho)$, then $f(t)$ is bounded on $\mathbb{T}$.*

Theorem 4.18. *If $f\in WPAP(\mathbb{T}\times X, X, \rho)$, then $f(t,x)$ is bounded on $\mathbb{T}\times D$; here D is any compact subset of X.*

Following the definition of Δ-measurability in Ref. [25], we introduce the following concept:

Definition 4.19. A closed subset C of $\mathbb{T}$ is said to be a weighted ergodic zero set in $\mathbb{T}$ if

$$\frac{\mu_\Delta(C\cap([t_0-r, t_0+r]\cap\mathbb{T}))}{m(r,\rho,t_0)}\to 0 \quad \text{as } r\to\infty, \text{ where } t_0\in\mathbb{T}.$$

Using this concept the following Theorems 4.20–4.24 directly follow from Ref. [24].

Theorem 4.20. *A function $\phi\in PAP_0(\mathbb{T}\times X, X, \rho)$ if and only if for $\varepsilon>0$, the set $C_\varepsilon=\{t\in\mathbb{T}:\|\phi(t,x)\|\geq\varepsilon\}$ is a weighted ergodic zero subset in $\mathbb{T}$.*

Theorem 4.21. *If $\rho\in U_\infty$, then the following hold:*

- *(i)*: *A function $\phi\in PAP_0(\mathbb{T}\times D, X, \rho)$ if and only if $\|\phi(t,x)\|^2\in PAP_0(\mathbb{T}\times D, \mathbb{R}, \rho)$; here D is any compact subset of X.*
- *(ii)*: *$\phi\in PAP_0(\mathbb{T}\times X, X, \rho)$ if and only if the norm function $\|\phi(\cdot, x)\|\in PAP_0(\mathbb{T}\times X, \mathbb{R}, \rho)$.*

Theorem 4.22. *If $f\in WPAP(\mathbb{T}\times X, X, \rho)$ and g is its almost periodic component, then*

$$g(\mathbb{T}\times D)\subset\overline{f(\mathbb{T}\times D)}$$

and

$$\|f\|_\infty\geq\|g\|_\infty\geq\inf_{(t,x)\in\mathbb{T}\times D}\|g(t,x)\|_X\geq\inf_{(t,x)\in\mathbb{T}\times D}\|f(t,x)\|_X,$$

where $f(\mathbb{T}\times D)$ denotes the value field of f on $\mathbb{T}\times D$, $\overline{f(\mathbb{T}\times D)}$ denotes the closure of $f(\mathbb{T}\times D)$ and is the same as $g(\mathbb{T}\times D)$; here D is an arbitrary compact subset of X.

Theorem 4.23. *If $f \in WPAP(\mathbb{T} \times X, X, \rho)$ satisfy the Lipschitz condition*

$$\|f(t,x) - f(t,y)\|_X \leq L_f \|x - y\|_X, \text{ for all } x, y \in X,\ t \in \mathbb{T},$$

then $\phi_0 \in WPAP(\mathbb{T}, X, \rho)$ implies $f(\cdot, \phi_0(\cdot)) \in WPAP(\mathbb{T}, X, \rho)$.

Theorem 4.24. *Assume that $f, g \in WPAP(\mathbb{T}, X, \rho)$, then $f \pm g$, $f \cdot g \in WPAP(\mathbb{T}, X, \rho)$.*

In what follows, we will consider linear abstract differential equations on time scales which are based on the Π-semigroup introduced in Section 3.

Suppose that $X(t)$ is the fundamental solution of the linear system:

$$x^{\Delta} = Ax, \tag{4.1}$$

where A is the infinitesimal generator of a Π-semigroup that satisfies all the conditions in Theorem 3.18 and $x : \mathbb{T} \to X$.

Now following Ref. [22], we introduce the following definition:

Definition 4.25. Eq. (4.1) is said to admit exponential dichotomy if there is a projection P and positive numbers $\alpha > 0$ and $\beta \geq 1$ such that

$$\|X(t)PX^{-1}(s)\|_{B(X)} \leq \beta e_{\ominus\alpha}(t,s),\ \ t \geq s, \tag{4.2}$$

$$\|X(t)(I-P)X^{-1}(s)\|_{B(X)} \leq \beta e_{\ominus\alpha}(s,t),\ \ s \geq t. \tag{4.3}$$

Let $F : \mathbb{T} \to X$ and consider the system

$$x^{\Delta} = Ax + F(t). \tag{4.4}$$

In view of Definition 4.1 and Theorem 4.19 in Ref. [15], we state the following theorem:

Theorem 4.26. *Let A be the infinitesimal generator of a Π-semigroup and all the conditions in Theorem 3.18 are satisfied, and $F(t)$ is almost periodic. If (4.1) admits an exponential dichotomy, then (4.4) has a unique almost periodic solution*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}\big(\sigma(s)\big)F(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}\big(\sigma(s)\big)F(s)\Delta s,$$

where $X(t)$ is the fundamental solution of (4.1).

From Theorem 4.26, the following corollary follows immediately.

Corollary 4.4. *Suppose (4.1) admits exponential dichotomy, that is, there exist constants $\alpha > 0$, $\beta \geq 1$ such that (4.2) and (4.3) hold. Then for each almost periodic function $F(t)$, both $\int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))F(s)\Delta s$ and $\int_{t}^{\infty} X(t)(I-P)X^{-1}(\sigma(s))F(s)\Delta s$ are almost periodic.*

5. APPLICATIONS TO NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS ON TIME SCALES

Qualitative analysis such as periodicity, almost periodicity, and stability of functional differential equations was studied by many researchers (see [26, 27, 28, 29, 30, 31] and the references cited therein). In [28], Islam and Raffoul examined the periodic solutions of a nonlinear neutral equations of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)x(t) + \frac{\mathrm{d}}{\mathrm{d}t}F\big(t, x(t-g(t))\big) + G\big(t, x(t), x(t-g(t))\big), \tag{5.1}$$

where $A(t)$ is a nonsingular $n \times n$ matrix with continuous real-valued functions as its elements, and the functions $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous in their respective arguments. In [30], Abbas and Bahuguna discussed the almost periodic solutions of (5.1) when $A(t)$ is almost periodic and $F(t, u)$, $G(t, u, v)$ are almost periodic. They assumed that

- $(A1)$: The family $\{A(t), t \in \mathbb{R}\}$ of operators in X generates an exponential stable evolution system $\{U(t, s), t \geq s\}$.
- $(A2)$: $\{U(t, s), t \geq s\}$, satisfies the condition that, for each $\varepsilon > 0$, there exists a number $l_\varepsilon > 0$ such that each interval of length l_ε contains a number τ with the property that

$$\|U(t+\tau, s+\tau) - U(t, s)\|_{B(X)} < Me^{-\frac{\delta}{2}(t-s)}\varepsilon.$$

Motivated by these works in this section we provide sufficient conditions which ensure the existence and uniqueness of weighted pseudo almost periodic solutions of the following system of neutral differential equations:

$$x^{\Delta}(t) = Ax(t) + F^{\Delta}\big(t, x(t-g(t))\big) + G\big(t, x(t), x(t-g(t))\big), \ t \in \mathbb{T}, \tag{5.2}$$

where $\mathbb{T}$ is an invariant under translations time scale and A is the infinitesimal generator of a Π-semigroup that satisfies all the conditions in Theorem 3.18, $F : \mathbb{T} \times X \to X$ is almost periodic in t uniformly for $x \in X$, $G : \mathbb{T} \times X \times X \to X$ is almost periodic in t uniformly for $(x, y) \in X \times X$, $g : \mathbb{T} \to \Pi$. Using the properties of weighted pseudo almost periodic functions in the previous sections and the exponential dichotomy of a linear differential equations together with Krasnoselskii's fixed point theorem, we obtain conditions that guarantee the existence of weighted pseudo almost periodic solutions of (5.2).

Lemma 5.1 ([20]). *Let $\mathbb{M}$ be a closed convex nonempty subset of X. Suppose that B and C map $\mathbb{M}$ into X such that*

- (i): *x, $y \in \mathbb{M}$ implies $Bx + Cy \in \mathbb{M}$,*
- (ii): *C is continuous and $C\mathbb{M}$ is contained in a compact set,*
- (iii): *B is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ with $z = Bz + Cz$.

With respect to (5.2), we shall assume the following conditions:

(H_1): There exist positive numbers L_F, L_G such that for each x_i, $y_i \in X$, $i = 1, 2$, and all $t \in \mathbb{T}$,

$$\|F(t, x_1) - F(t, x_2)\|_X \leq L_F \|x_1 - x_2\|_X \tag{5.3}$$

and

$$\|G(t, x_1, y_1) - G(t, x_2, y_2)\|_X \leq L_G(\|x_1 - x_2\|_X + \|y_1 - y_2\|_X); \tag{5.4}$$

(H_2): A is the infinitesimal generator of the Π-semigroup and all the conditions in Theorem 3.18 are satisfied, and $F \in WPAP(\mathbb{T} \times X, X, \rho)$ and $G \in WPAP(\mathbb{T} \times X \times X, X, \rho)$;

(H_3): Eq. (4.1) admits exponential dichotomy, that is, there exists constants $\alpha > 0$, $\beta \geq 1$, such that (4.2) and (4.3) hold.

(H_4): The weight $\rho : \mathbb{T} \to (0, \infty)$ is continuous and

$$\limsup_{s \to \infty} \left[\frac{\rho(s + \tau)}{\rho(s)}\right] < \infty, \ \limsup_{r \to \infty} \left[\frac{m(r + \tau, \rho, t_0)}{m(r, \rho, t_0)}\right] < \infty$$

for every $\tau \in \Pi$, $t_0 \in \mathbb{T}$.

Lemma 5.2. *Under the condition (H_4), the space $PAP_0(\mathbb{T}, X, \rho)$ is translation invariant, that is, for each $\phi \in PAP_0(\mathbb{T}, X, \rho)$ and $u \in \Pi$, we have $t \to \phi(t - u) \in PAP_0(\mathbb{T}, X, \rho)$.*

Proof. Let $\phi \in PAP_0(\mathbb{T}, X, \rho)$. Then for each $u \in \Pi$, $u > 0$, we have

$$\begin{aligned}
0 &\leq \frac{1}{m(r, \rho, t_0)} \int_{t_0 - r}^{t_0 + r} \|\phi(t - u)\|_X \rho(t) \Delta t \\
&= \frac{1}{m(r, \rho, t_0)} \int_{t_0 - r - u}^{t_0 + r - u} \|\phi(t)\|_X \rho(t + u) \Delta t \\
&= \frac{1}{m(r, \rho, t_0)} \bigg(\int_{t_0 - r - u}^{t_0 + r + u} \|\phi(t)\|_X \rho(t + u) \Delta t \\
&\quad - \int_{t_0 - r - u}^{t_0 + r + u} \|\phi(t)\|_X \rho(t + u) \Delta t \bigg) \\
&\leq \frac{1}{m(r, \rho, t_0)} \int_{t_0 - r - u}^{t_0 + r + u} \|\phi(t)\|_X \rho(t + u) \Delta t
\end{aligned}$$

and for each $u\in\Pi$, $u<0$, we have

$$\begin{aligned}
0 &\le \frac{1}{m(r,\rho,t_0)}\int_{t_0-r}^{t_0+r}\|\phi(t-u)\|_X\rho(t)\Delta t\\
&= \frac{1}{m(r,\rho,t_0)}\int_{t_0-r-u}^{t_0+r-u}\|\phi(t)\|_X\rho(t+u)\Delta t\\
&= \frac{1}{m(r,\rho,t_0)}\bigg(\int_{t_0-r+u}^{t_0+r-u}\|\phi(t)\|_X\rho(t+u)\Delta t\\
&\quad -\int_{t_0-r+u}^{t_0-r-u}\|\phi(t)\|_X\rho(t+u)\Delta t\bigg)\\
&\le \frac{1}{m(r,\rho,t_0)}\int_{t_0-r+u}^{t_0+r-u}\|\phi(t)\|_X\rho(t+u)\Delta t.
\end{aligned}$$

Thus it follows that

$$0\le \frac{1}{m(r,\rho,t_0)}\int_{t_0-r}^{t_0+r}\|\phi(t-u)\|_X\rho(t)\Delta t\le \frac{1}{m(r,\rho,t_0)}\int_{t_0-r-|u|}^{t_0+r+|u|}\|\phi(t)\|_X\rho(t+u)\Delta t.$$

Now from condition (H_4) and the fact that $\phi\in PAP_0(\mathbb{T},X,\rho)$, we find

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{m(r,\rho,t_0)}\int_{t_0-r-|u|}^{t_0+r+|u|}\|\phi(t)\|_X\rho(t+u)\Delta t\\
&\quad\le \lim_{r\to\infty}\frac{m(r+|u|,\rho,t_0)}{m(r,\rho,t_0)}\frac{\rho(\xi_u+u)}{\rho(\xi_u)}\frac{1}{m(r+|u|,\rho,t_0)}\int_{t_0-r-|u|}^{t_0+r+|u|}\|\phi(t)\|_X\rho(t)\Delta t=0,
\end{aligned}$$

where $\xi_u\in(t_0-r-|u|,t_0+r+|u|)_{\mathbb{T}}$.

Hence, we have

$$\lim_{r\to\infty}\frac{1}{m(r,\rho,t_0)}\int_{t_0-r}^{t_0+r}\|\phi(t-u)\|_X\rho(t)\Delta t=0,$$

that is, $t\to\phi(t-u)\in PAP_0(\mathbb{T},X,\rho)$. Therefore, the space $PAP_0(\mathbb{T},X,\rho)$ is translation invariant. This completes the proof. $\square$

To prove our results, we define a mapping Φ as follows

$$\begin{aligned}
(\Phi u)(t) \;=\;& F\big(t,u(t-g(t))\big)+\int_{-\infty}^{t}X(t)PX^{-1}\big(\sigma(s)\big)G\big(s,u(s),u(s-g(s))\big)\Delta s\\
&-\int_{t}^{\infty}X(t)(I-P)X^{-1}\big(\sigma(s)\big)G\big(s,u(s),u(s-g(s))\big)\Delta s,
\end{aligned}$$

where $X(t)$ is the fundamental solution of (4.1).

Lemma 5.3. *The operator Φu is weighted pseudo almost periodic if u is weighted pseudo almost periodic.*

Proof. Let $u(t)$ be weighted pseudo almost periodic. Now from (H_1), (H_4), Theorem 4.23 and Lemma 5.2, it is clear that $F\big(t,u(t-g(t))\big)$ and $G\big(t,u(t),u(t-g(t))\big)$ are also weighted pseudo almost periodic.

Now we will show that $H_1(t) = \int_{-\infty}^{t} X(t)PX^{-1}\big(\sigma(s)\big)G\big(s, u(s), u(s-g(s))\big)\Delta s$ is weighted pseudo almost periodic. Let

$$G\big(t, u(t), u(t-g(t))\big) = G_1(t) + \phi(t),$$

where $G_1 \in AP(\mathbb{T}, X)$ and $\phi \in PAP_0(\mathbb{T}, X, \rho)$. Then

$$H_1(t) = \int_{-\infty}^{t} X(t)PX^{-1}\big(\sigma(s)\big)G_1(s)\Delta s + \int_{-\infty}^{t} X(t)PX^{-1}\big(\sigma(s)\big)\phi(s)\Delta s.$$

Since $G_1(t)$ is almost periodic, it follows from Corollary 4.4 that $\int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))\times G_1(s)\Delta s$ is almost periodic.

Let $h(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))\phi(s)\Delta s$. In order to show $H_1 \in WPAP(\mathbb{T}, X, \rho)$, we only need to show that $h \in PAP_0(\mathbb{T}, X, \rho)$, that is

$$\lim_{r\to\infty} \frac{1}{m(r,\rho,t_0)} \int_{t_0-r}^{t_0+r} \|h(t)\|_X \rho(t)\Delta t = 0.$$

It follows from (H_3) and $e_{\ominus\alpha}(t, \sigma(s)) \le 1$ for $t \ge s$ that

$$\begin{aligned}
&\frac{1}{m(r,\rho,t_0)} \int_{t_0-r}^{t_0+r} \|h(t)\|_X \rho(t)\Delta t \\
&\quad\le \frac{1}{m(r,\rho,t_0)} \int_{t_0-r}^{t_0+r} \rho(t)\Delta t \int_{-\infty}^{t} \beta e_{\ominus\alpha}(t,\sigma(s))\|\phi(s)\|_X \Delta s \\
&\quad\le \frac{\beta}{m(r,\rho,t_0)} \int_{t_0-r}^{t_0+r} \rho(t)\Delta t \int_{-\infty}^{t} \|\phi(s)\|_X \Delta s \\
&\quad= \frac{\beta}{m(r,\rho,t_0)} \int_{0}^{\infty} \Delta u \int_{t_0-r}^{t_0+r} \rho(t)\|\phi(t-u)\|_X \Delta t.
\end{aligned}$$

Now from (H_4) and Lemma 5.2, $PAP_0(\mathbb{T}, X, \rho)$ translation invariant, we find that $t \to \phi(t-u) \in PAP_0(\mathbb{T}, X, \rho)$ for each $u \in \Pi$. Thus we have

$$\lim_{r\to\infty} \frac{1}{m(r,\rho,t_0)} \int_{t_0-r}^{t_0+r} \rho(t)\|\phi(t-u)\|_X \Delta t = 0,$$

for each $u \in \Pi$. This implies that

$$\lim_{r\to\infty} \frac{1}{m(r,\rho,t_0)} \int_{t_0-r}^{t_0+r} \|h(t)\|_X \rho(t)\Delta t = 0,$$

that is, $h \in PAP_0(\mathbb{T}, X, \rho)$, and hence $H_1 \in WPAP(\mathbb{T}, X, \rho)$.

Finally, let

$$H_2(t) = \int_{t}^{\infty} X(t)(I-P)X^{-1}\big(\sigma(s)\big)G\big(s, u(s), u(s-g(s))\big)\Delta s.$$

In a similar way we see that $H_2 \in WPAP(\mathbb{T}, X, \rho)$. Thus from Theorem 4.24, we find that $\Phi u \in WPAP(\mathbb{T}, X, \rho)$ for $u \in WPAP(\mathbb{T}, X, \rho)$. This completes the proof. □

Next we will apply Krasnoselskii's fixed point theorem. Let

$$(\Phi u)(t) = (Bu)(t) + (Cu)(t)$$

where $B, C : WPAP(\mathbb{T}, X, \rho) \to WPAP(\mathbb{T}, X, \rho)$ are given by

$$(Bu)(t) = F\big(t, u(t - g(t))\big) \tag{5.5}$$

and

$$\begin{aligned}(Cu)(t) &= \int_{-\infty}^{t} X(t)PX^{-1}\big(\sigma(s)\big)G\big(s, u(s), u(s - g(s))\big)\Delta s \\ &\quad - \int_{t}^{\infty} X(t)(I - P)X^{-1}\big(\sigma(s)\big)G\big(s, u(s), u(s - g(s))\big)\Delta s.\end{aligned} \tag{5.6}$$

Lemma 5.4. *The operator B is contraction provided $L_F < 1$.*

Proof. From (5.3), we have

$$\begin{aligned}\big\|B\big(\phi(t)\big) - B\big(\psi(t)\big)\big\|_X &= \big\|F\big(t, \phi(t - g(t))\big) - F\big(t, \psi(t - g(t))\big)\big\|_X \\ &\le L_F\big\|\phi\big(t - g(t)\big) - \psi\big(t - g(t)\big)\big\|_X \le L_F\|\phi - \psi\|_\infty.\end{aligned}$$

Since $L_F < 1$, B is contraction. This completes the proof. $\square$

Lemma 5.5. *The operator C is continuous and the image $C\mathbb{D}$ is contained in a compact set, where $\mathbb{D} = \{u \in WPAP(\mathbb{T}, X, \rho) : \|u\|_\infty \le k\}$, and k is a fixed constant.*

Proof. Clearly, we have

$$\begin{aligned}\|(Cu)(t)\|_X &\le \int_{-\infty}^{t} \big\|X(t)PX^{-1}\big(\sigma(s)\big)\big\|_{B(X)}\big\|G\big(s, u(s), u(s - g(s))\big)\big\|_X\Delta s \\ &\quad + \int_{t}^{\infty} \big\|X(t)(I - P)X^{-1}\big(\sigma(s)\big)\big\|_{B(X)}\big\|G\big(s, u(s), u(s - g(s))\big)\big\|_X\Delta s.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|(Cu)(\cdot)\|_\infty &\le \big\|G\big(\cdot, u(\cdot), u(\cdot - g(\cdot))\big)\big\|_\infty\bigg(\int_{-\infty}^{t} \big\|X(t)PX^{-1}\big(\sigma(s)\big)\big\|_{B(X)}\Delta s \\ &\quad + \int_{t}^{\infty} \big\|X(t)(I - P)X^{-1}\big(\sigma(s)\big)\big\|_{B(X)}\Delta s\bigg) \\ &\le \big\|G\big(\cdot, u(\cdot), u(\cdot - g(\cdot))\big)\big\|_\infty\bigg(\int_{-\infty}^{t} \beta e_{\ominus\alpha}\big(t, \sigma(s)\big)\Delta s \\ &\quad + \int_{t}^{\infty} \beta e_{\ominus\alpha}\big(\sigma(s), t\big)\Delta s\bigg) \\ &= \beta\bigg(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}\bigg)\big\|G\big(\cdot, u(\cdot), u(\cdot - g(\cdot))\big)\big\|_\infty.\end{aligned} \tag{5.7}$$

To see that C is continuous, we let $u, v \in AP(\mathbb{T}, X)$. Given $\varepsilon > 0$, take $\delta = \dfrac{\varepsilon}{2L_G\beta(\frac{1}{\alpha} - \frac{1}{\ominus\alpha})}$. Then we have

$$\begin{aligned}
&\|(Cu)(t) - (Cv)(t)\|_X \\
&\quad \le \int_{-\infty}^{t} \big\|X(t)PX^{-1}\big(\sigma(s)\big)\big\|_{B(X)} \big\|G\big(s, u(s), u(s-g(s))\big) \\
&\quad\quad - G\big(s, v(s), v(s-g(s))\big)\big\|_X \Delta s \\
&\quad\quad + \int_{t}^{\infty} \big\|X(t)(I-P)X^{-1}\big(\sigma(s)\big)\big\|_{B(X)} \big\|G\big(s, u(s), u(s-g(s))\big) \\
&\quad\quad - G\big(s, v(s), v(s-g(s))\big)\big\|_X \Delta s \\
&\quad \le \int_{-\infty}^{t} \beta e_{\ominus\alpha}\big(t, \sigma(s)\big)\big(L_G\|u(s) - v(s)\|_X \\
&\quad\quad + L_G\big\|u\big(s-g(s)\big) - v\big(s-g(s)\big)\big\|_X\big)\Delta s \\
&\quad\quad + \int_{t}^{\infty} \beta e_{\ominus\alpha}\big(\sigma(s), t\big)\big(L_G\|u(s) - v(s)\|_X \\
&\quad\quad + L_G\big\|u\big(s-g(s)\big) - v\big(s-g(s)\big)\big\|_X\big)\Delta s \\
&\quad \le 2L_G\|u - v\|_\infty \left(\int_{-\infty}^{t} \beta e_{\ominus\alpha}\big(t, \sigma(s)\big)\Delta s + \int_{t}^{\infty} \beta e_{\ominus\alpha}\big(\sigma(s), t\big)\Delta s \right) \\
&\quad = 2L_G\beta\left(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}\right)\|u - v\|_\infty < \varepsilon,
\end{aligned}$$

whenever $\|u - v\|_\infty < \delta$. This proves that C is continuous. For $\mathbb{D} = \{u \in WPAP(\mathbb{T}, X) : \|u\|_\infty \le k\}$, to show that the image $C\mathbb{D}$ is contained in a compact set, let $\{u_n\}$ be a sequence in $\mathbb{D}$. In view of (5.4), we have

$$\begin{aligned}
\big\|G\big(t, u(t), v(t)\big)\big\|_X &\le \big\|G\big(t, u(t), v(t)\big) - G(t, 0, 0)\big\|_X + \|G(t, 0, 0)\|_X \\
&\le L_G(\|u(t)\|_X + \|v(t)\|_X) + a \le L_G(\|u\|_\infty + \|v\|_\infty) + a \\
&\le L_G(2k + a), \qquad (5.8)
\end{aligned}$$

where $a = \|G(t, 0, 0)\|_X$. From (5.7) and (5.8), we find

$$\|Cu_n\|_\infty \le L_G(2k + a)\beta\left(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}\right) := L. \qquad (5.9)$$

Next, we calculate $(Cu_n)^\Delta(t)$ and show that it is uniformly bounded. Clearly,

$$(Cu_n)^\Delta(t) = A(Cu_n)(t) + G\big(t, u_n(t), u_n(t - g(t))\big).$$

Since A is a bounded operator, there exists a positive constant K such that $\|A\| \le K$. This, when combined with (5.8) and (5.9) implies

$$\|(Cu_n)^\Delta\|_\infty \le KL + L_G(2k + a).$$

Thus the sequence $\{(Cu_n)(t)\}$ is uniformly bounded and equi-continuous. Hence, by the Ascoli-Arzelà Theorem, $C\mathbb{D}$ is compact. The completes the proof. $\square$

Theorem 5.6. *Suppose that* (H_1)–(H_4) *hold,* $b = \|F(t,0)\|_\infty$ *and* $a = \|G(t,0,0)\|_\infty$*. If there exists a constant* k*, such that*

$$L_F k + b + \beta\left(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}\right)L_G(2k+a) \leq k, \tag{5.10}$$

where α, β *are constants given in* (4.2) *and* (4.3)*, then* (5.2) *has a weighted pseudo almost periodic solution in* $\mathbb{M} = \{u \in WPAP(\mathbb{T}, X, \rho) : \|u\| \leq k\}$*.*

Proof. Note that condition (5.10) implies that $L_F < 1$. Thus in view of Lemma 5.4 the mapping B defined by (5.5) is contraction. The mapping C defined by (5.6) is continuous by Lemma 5.5 and $C\mathbb{M}$ is contained in a compact set. Now, for u, $v \in \mathbb{M}$, we have

$$\begin{aligned}
&\|(Bu)(t) + (Cv)(t)\|_X \\
&\quad \leq \big\|F\big(t, u(t-g(t))\big) - F(t,0)\big\|_X + \|F(t,0)\|_X \\
&\qquad + \int_{-\infty}^{t} \big\|X(t)PX^{-1}\big(\sigma(s)\big)\big\|_{B(X)} \big\|G\big(s, v(s), v(s-g(s))\big)\big\|_X \Delta s \\
&\qquad + \int_{t}^{\infty} \big\|X(t)(I-P)X^{-1}\big(\sigma(s)\big)\big\|_{B(X)} \big\|G\big(s, v(s), v(s-g(s))\big)\big\|_X \Delta s \\
&\quad \leq L_F\|u\|_\infty + b + \beta\left(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}\right)L_G(2k+a) \leq k.
\end{aligned}$$

Thus $Bu + Cv \in \mathbb{M}$. Therefore all the conditions of Krasnoselskii's theorem are satisfied, and as a consequence there exists a fixed point $z \in \mathbb{M}$ such that $z = Bz + Cz$, i.e., (5.2) has a weighted pseudo almost periodic solution in $\mathbb{M}$. This completes the proof. □

Theorem 5.7. *Suppose that* (H_1)–(H_4) *hold. Further, suppose that*

$$L_F + 2L_G\beta\left(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}\right) < 1. \tag{5.11}$$

Then (5.2) *has a unique weighted pseudo almost periodic solution.*

Proof. It follows from Lemma 5.3 that Φu maps $WPAP(\mathbb{T}, X, \rho)$ to $WPAP(\mathbb{T}, X, \rho)$. Thus for u, $v \in WPAP(\mathbb{T}, X, \rho)$, we have

$$\begin{aligned}
\|\Phi u(t) - \Phi v(t)\|_X &\leq L_F\|u-v\|_\infty + \int_{-\infty}^{t} \big\|X(t)PX^{-1}\big(\sigma(s)\big)\big\|_{B(X)} 2L_G\|u-v\|_\infty \Delta s \\
&\quad - \int_{t}^{\infty} \big\|X(t)(I-P)X^{-1}\big(\sigma(s)\big)\big\|_{B(X)} 2L_G\|u-v\|_\infty \Delta s \\
&\leq \left[L_F + 2L_G\beta\left(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}\right)\right]\|u-v\|_\infty.
\end{aligned}$$

Since $L_F + 2L_G\beta(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}) < 1$, Φ is a contractive mapping. Therefore, Φ has a unique fixed point $u^* \in WPAP(\mathbb{T}, X, \rho)$. We conclude that $u^*(t)$ is the unique weighted pseudo almost periodic solution of (5.2). This completes the proof. □

An example. Let $\rho(t)=1+t^2+t\sigma(t)+\sigma^2(t)$, and let $\mathbb{T}$ be an invariant under translations time scale. For this $\rho(t)$, we have

$$\begin{aligned} m(r,\rho) &= \int_{-r}^{r}\big[1+t^2+t\sigma(t)+\sigma^2(t)\big]\Delta t \\ &= \int_{-r}^{r}\big[1+t^2\cdot 1+(t+\sigma(t))\sigma(t)\big]\Delta t \\ &= \int_{-r}^{r}\big[1+t^2\cdot 1+(t^2)^{\Delta}\sigma(t)\big]\Delta t=\int_{-r}^{r}\big[1+(t^3)^{\Delta}\big]\Delta t=2(r+r^3). \end{aligned}$$

Thus condition (H_4) holds. Now consider the following perturbed differential equations for small ε_1 and ε_2 on $\mathbb{T}$:

$$x^{\Delta}=Ax+F^{\Delta}\big(t,x(t-g(t))\big)+G\big(t,x(t),x(t-g(t))\big),\ t\in\mathbb{T}^+, \tag{5.12}$$

where

$$x=\begin{pmatrix}x_1\\x_2\end{pmatrix},\quad A=\begin{pmatrix}-5&0\\0&-5\end{pmatrix},\quad \text{and}\quad \mu(t)\neq\frac{1}{5},$$

$$F\big(t,x(t-g(t))\big)=\begin{pmatrix}0\\ \varepsilon_1(\sin t+\sin\sqrt{2}t+e_{\ominus 3}(t,0))x_1^2(t-g(t))\end{pmatrix}$$

and

$$G\big(t,x(t),x(t-g(t))\big)=\begin{pmatrix}0\\ \varepsilon_2(\cos t+\cos\sqrt{2}t+e_{\ominus 3}(t,0))-\varepsilon_2x_1^2(t)x_2(t)\end{pmatrix}.$$

It is clear that $\sin t+\sin\sqrt{2}t$ and $\cos t+\cos\sqrt{2}t$ are almost periodic, and by Theorem 1.120 in Ref. [1][L'Hôpital's Rule], we find that $\lim_{t\to+\infty}\frac{\rho(t)}{e_3(t,0)}=0$. Thus, $e_{\ominus 3}(t,0)\rho(t)$ is bounded, and hence we have

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}e_{\ominus 3}(t,0)\rho(t)\Delta t=\lim_{r\to\infty}\frac{1}{2(r+r^3)}\int_{-r}^{r}e_{-3}(t,0)\rho(t)\Delta t=0.$$

Thus $F\in WPAP(\mathbb{T}\times\mathbb{R}^2,\mathbb{R}^2,\rho)$ and $G\in(\mathbb{T}\times\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R}^2,\rho)$.

Obviously, $I+\mu(t)A$ is invertible for all $\mathbb{T}$, so $A\in\mathcal{R}$. We claim that $x^{\Delta}=Ax$ admits an exponential dichotomy. In fact, the eigenvalues of the coefficient matrix A are $\lambda_1=\lambda_2=-5$, and thus in Theorem 5.35 (Putzer Algorithm) in Ref. [1], the P matrices are given by

$$P_0=I=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad\text{and}\quad P_1=(A-\lambda_1 I)P_0=A+5I=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

We choose

$$r_1^{\Delta}=-5r_1,\ r_1(t_0)=1\quad\text{and}\quad r_2^{\Delta}=r_1-5r_2,\ r_2(t_0)=0.$$

Solving the first IVP for r_1 we get $r_1=e_{-5}(t,t_0)$. Solving the second IVP, i.e.,

$$r_2^{\Delta}=-5r_2+e_{-5}(t,t_0),\ r_2(t_0)=0,$$

we obtain

$$r_2 = e_{-5}(t,t_0)\int_{t_0}^{t}\frac{\Delta s}{1-5\mu(s)}.$$

Now using Theorem 5.35 (Putzer Algorithm) in Ref. [1], we get

$$e_A(t,t_0) = r_1(t)P_0 + r_2(t)P_1 = e_{-5}(t,t_0)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Thus,

$$\|X(t)PX^{-1}(s)\| = \left\|e_{-5}(t,t_0)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}e_{\ominus -5}(s,t_0)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\| \le \sqrt{2}e_{\ominus\frac{5}{2}}(t,s).$$

Thus we can take $\beta = \sqrt{2}$, $\alpha = \frac{5}{2}$ so that $x^{\Delta} = Ax$ admits an exponential dichotomy. Define $\mathbb{M} = \{u \in WPAP(\mathbb{T},\mathbb{R}^2,\rho) : \|u\| \le k\}$, where k is a fixed constant. For $x(t) = (x_1(t),x_2(t))$, $y(t) = (y_1(t),y_2(t)) \in \mathbb{M}$, we have

$$\big\|F\big(t,x_1(t-g(t))\big) - F\big(t,x_2(t-g(t))\big)\big\| \le 6\varepsilon_1 k\|x_1-x_2\|,$$

$$\big\|G\big(t,x_1(t),y_1(t-g(t))\big) - G\big(t,x_2(t),y_2(t-g(t))\big)\big\| \le \varepsilon_2 k^2\|x_1-x_2\|.$$

Thus if we take $L_F = 6\varepsilon_1 k$, $L_G = \varepsilon_2 k^2$, $\beta = \sqrt{2}$, $\alpha = \dfrac{5}{2}$ then (5.10) holds. Therefore, with these choices the conditions of Theorem 5.6 are satisfied. In conclusion, (5.12) has a weighted pseudo almost periodic solution in $\mathbb{M}$.

Moreover, for each positive number k, if $\varepsilon_1,\varepsilon_2$ are small enough such that

$$L_F + 2L_G\beta\left(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}\right) = 6\varepsilon_1 k + 2\varepsilon_2 k^2\sqrt{2}\frac{4+5\mu(t)}{5} < 1,$$

then (5.12) has a unique weighted pseudo almost periodic solution in $\mathbb{M}$.

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