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GLOBAL EXISTENCE RESULTS FOR FUNCTIONAL EVOLUTION EQUATIONS WITH DELAY AND RANDOM EFFECTS

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ABSTRACT. We study the existence of mild solutions of a functional evolution equation with infinite delay and random effects. We use a random fixed point theorem with stochastic domain.
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1. INTRODUCTION

Functional evolution equations play a very important role in describing many phenomena of physics, mechanics, biology, etc. For more details on this theory and on its applications we refer to the monographs of Hale and Verduyn Lunel [18], Kolmanovskii and Myshkis [21], and Wu [31], and the references therein. Recently, many authors have studied the existence of various models of semilinear evolution equations with finite and infinite delay in Fréchet space; for instance, we refer to the book by Abbas and Benchohra [1] and to the papers by Baghli and Benchohra [4, 5, 6]. On the other hand, different fields of engineering problems which are of current interest in unbounded domains have also received the attention of researchers; see [2, 26, 27].

The nature of a dynamic system in engineering or natural sciences depends on the accuracy of the information obtained concerning the parameters that describe that system. If the knowledge about a dynamic system is precise, then a deterministic dynamical system arises. Yet, in most cases, the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without

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uncertainties. When knowledge about the parameters of a dynamic system are of a statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians. We refer the reader to the monographs [7, 30], the papers [10, 9, 11, 29] and the references therein. We also refer the reader to recent results in [23, 24, 25]. There are real world phenomena with anomalous dynamics such as signals transmissions through strong magnetic fields, atmospheric diffusion of pollution, network traffic, the effect of speculations on the profitability of stocks in financial markets, and so on, where the classical models are not sufficiently good to describe these features.

In this work we prove the existence of mild solutions of the following functional differential equation with delay and random effects (random parameters) of the form:

$$y'(t,w) = A(t)y(t,w) + f(t, y_t(\cdot,w), w), \quad \text{a.e. } t \in J := [0,\infty), \tag{1.1}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0],\ w \in \Omega, \tag{1.2}$$

where (Ω, F, P) is a complete probability space, $f : J\times\mathcal{B}\times\Omega \to E$, $\phi \in \mathcal{B}\times\Omega$ are given random functions which represent random nonlinearity of the system, $\{A(t)\}_{0\le t<+\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $0 \le s \le t < +\infty$, $\mathcal{B}$ is the phase space to be specified later, and $(E, |\cdot|)$ is a real Banach space. For any function y defined on $(-\infty,+\infty)\times\Omega$ and any $t \in J$ we denote by $y_t(\cdot,w)$ the element of $\mathcal{B}\times\Omega$ defined by $y_t(\theta,w) = y(t+\theta,w), \theta \in (-\infty,0]$. Here $y_t(\cdot,w)$ represents the history of the state from time $-\infty$, up to the present time t. We assume that the histories $y_t(\cdot,w)$ belong to some abstract phases $\mathcal{B}$, to be specified later.

To our knowledge, the literature on the global existence of random evolution equations with delay is very limited, so the present paper can be considered as a contribution to such a class of equations.

2. **PRELIMINARIES**

In this section we present briefly some notations, definitions, and theorems which are used throughout this work.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [17] and follow the terminology used in [19]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty,0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, T) \to E, T > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:

(i) $y_t \in \mathcal{B}$;

(ii) There exists a positive constant H such that $|y(t)| \le H\|y_t\|_{\mathcal{B}}$;

(iii) There exist two functions $L(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with L continuous and bounded, and M locally bounded such that:

$$\|y_t\|_{\mathcal{B}} \le L(t) \sup\{\ |y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Denote

$$K_T = \sup\{L(t) : t \in J\},$$

and

$$M_T = \sup\{M(t) : t \in J\}.$$

Remark 2.1. 1. (ii) is equivalent to $|\phi(0)| \le H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.

2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can satisfy $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \le 0$.
3. From the equivalence in part 1 of this remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$, we necessarily have that $\phi(0) = \psi(0)$.

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ into E. Finally, by $BC := BC([0, +\infty)$ we denote the Banach space of bounded and continuous functions from $[0, \infty)$ into E, equipped with the standard norm

$$\|y\|_{BC} = \sup_{t \in [0,\infty)} |y(t)|.$$

Definition 2.2. A map $f : J \times \mathcal{B} \times \Omega \to E$ is said to be Carathéodory if

(i) $t \to f(t, y, w)$ is measurable for all $y \in \mathcal{B}$ and for all $w \in \Omega$.

(ii) $y \to f(t, y, w)$ is continuous for almost each $t \in J$ and for all $w \in \Omega$.

(iii) $w \to f(t, y, w)$ is measurable for all $y \in \mathcal{B}$, and almost each $t \in J$.

In what follows, we assume that $\{A(t),\ t \ge 0\}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 2.3. A family of bounded linear operators

$$\{U(t, s)\}_{(t,s)\in\Delta} : U(t, s) : E \to E,\ (t, s) \in \Delta := \{(t, s) \in J \times J : 0 \le s \le t < +\infty\}$$

is called an evolution system if the following properties are satisfied:

1. $U(t, t) = I$ where I is the identity operator in E,

2. $U(t,s)\,U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t < +\infty$,
3. $U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(s,t) \in \Delta$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)\,y$ is continuous.

More details on evolution systems and their properties could be found on the books of Ahmed [3], Engel and Nagel [13] and Pazy [28].

Lemma 2.4 (Corduneanu [8]). *Let $C \subset BC(J,E)$ be a set satisfying the following conditions:*

(i): *C is bounded in $BC(J,E)$;*
(ii): *the functions belonging to C are equicontinuous on any compact interval of J;*
(iii): *the set $C(t) := \{y(t) : y \in C\}$ is relatively compact on any compact interval of J;*
(iv): *the functions from C are equiconvergent, i.e., given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|y(t) - y(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $y \in C$.*

Then C is relatively compact in $BC(J,E)$.

Theorem 2.5 (Schauder fixed point [16]). *Let B be a closed, convex and nonempty subset of a Banach space E. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of E. Then N has at least one fixed point in B.*

Let Y be a separable Banach space with the Borel σ-algebra B_Y. A mapping $y : \Omega \longrightarrow Y$ is said to be a random variable with values in Y if for each $B \in B_Y, y^{-1}(B) \in F$. A mapping $T : \Omega \times Y \longrightarrow Y$ is called a random operator if $T(\cdot, y)$ is measurable for each $y \in Y$ and is generally expressed as $T(w,y) = T(w)y$; we will use these two expressions alternatively.

Let y be a mapping of $J \times \Omega$ into X. y is said to be a stochastic process if for each $t \in J$ the function $y(t,\cdot)$ is measurable.

Next, we will give a very useful random fixed point theorem with stochastic domain.

Definition 2.6 ([12]). Let C be a mapping from Ω into 2^Y. A mapping $T : \{(w,y) : w \in \Omega \wedge y \in C(w)\} \longrightarrow Y$ is called a 'random operator with stochastic domain C' if and only if C is measurable (i.e., for all closed $A \subseteq Y, \{w \in \Omega : C(w) \cap A \neq \emptyset\} \in F$) and for all open $D \subseteq Y$ and all $y \in Y, \{w \in \Omega : y \in C(w) \wedge T(w,y) \in D\} \in F$. The mapping T will be called 'continuous' if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \longrightarrow Y$ is called a 'random (stochastic) fixed point of T' if and only if for p-almost all $w \in \Omega, y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subseteq Y, \{w \in \Omega : y(w) \in D\} \in F$ ('y is measurable').

Remark 2.7. If $C(w) \equiv Y$, then the definition of random operator with stochastic domain coincides with the definition of random operator.

Lemma 2.8 ([12]). *Let $C : \Omega \longrightarrow 2^Y$ be measurable with $C(w)$ closed, convex and solid (i.e., $\operatorname{int} C(w) \neq \emptyset$) for all $w \in \Omega$. We assume that there exists a measurable function $y_0 : \Omega \longrightarrow Y$ with $y_0 \in \operatorname{int} C(w)$ for all $w \in \Omega$. Let T be a continuous random operator with stochastic domain C such that for every $w \in \Omega, \{y \in C(w) : T(w)y = y\} \neq \emptyset$. Then T has a stochastic fixed point.*

3. EXISTENCE OF MILD SOLUTIONS

Now we give our main existence result for problem (1.1)–(1.2). Before stating and proving this result, we give the definition of a mild random solution.

Definition 3.1. A stochastic process $y : J \times \Omega \to E$ is said to be a random mild solution of problem (1.1)–(1.2) if $y(t,w) = \phi(t,w)$, $t \in (-\infty, 0]$ and the restriction of $y(\cdot, w)$ to the interval $[0, \infty)$ is continuous and satisfies the following integral equation:

$$y(t,w) = U(t,0)\phi(0,w) + \int_0^t U(t,s) f(s, y_s(\cdot, w), w) ds, \quad t \in J. \tag{3.1}$$

We will need to introduce the following hypotheses which are be assumed hereafter

(H_1) There exist a constant $M \geq 1$ and $\alpha > 0$ such that

$$\|U(t,s)\|_{B(E)} \leq M e^{-\alpha(t-s)} \quad \text{for every } (s,t) \in \Delta.$$

(H_2) The function $f : \mathbb{R}^+ \times \mathcal{B} \times \Omega \to E$ is Carathéodory.

(H_3) There exist functions $\psi : J \times \Omega \to \mathbb{R}^+$ and $p : J \times \Omega \to \mathbb{R}^+$ such that for each $w \in \Omega$, $\psi(\cdot, w)$ is a continuous nondecreasing function and $p(\cdot, w)$ is integrable with:

$$|f(t,u,w)| \leq p(t,w)\ \psi(\|u\|_{\mathcal{B}}, w) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

(H_4) For each $w \in \Omega, \phi(\cdot, w)$ is continuous and for each $t, \phi(t, \cdot)$ is measurable.

(H_5) For each $(t,s) \in \Delta$ we have

$$\lim_{t \to +\infty} \int_0^t e^{-\alpha(t-s)} p(s,w) ds = 0.$$

Theorem 3.2. *Suppose that hypotheses (H_1)–(H_5) are valid, then the problem* (1.1)–(1.2) *has at least one mild random solution on* $(-\infty, \infty)$.

Proof. Let Y be the space defined by

$$Y = \{y : \mathbb{R} \to E \text{ such that } y|_J \in BC(J,E) \text{ and } y_0 \in \mathcal{B}\},$$

(where we denote by $y|_J$ the restriction of y to J), endowed with the uniform convergence topology, and $N : \Omega \times Y \to Y$ be the random operator defined by

$$(N(w)y)(t) = U(t,0)\ \phi(0,w) + \int_0^t U(t,s)\ f(s,y_s,w)\ ds, \quad t \in J. \tag{3.2}$$

Then we show that the mapping defined by (4) is a random operator. To do this, we need to prove that for any $y \in Y$, $N(\cdot)(y) : \Omega \longrightarrow Y$ is a random variable. First, we prove that $N(\cdot)(y) : \Omega \longrightarrow Y$ is measurable since the mapping $f(t,y,\cdot)$, $t \in J$, $y \in Y$, is measurable by assumption (H_2) and (H_4).

Let $R(w)$ be any measurable positive function and consider the set-valued map $D : \Omega \longrightarrow 2^Y$ defined by

$$D(w) = \{y \in Y : \|y\| \le R(w)\}.$$

$D(w)$ is bounded, closed, convex and solid for all $w \in \Omega$. Then D is measurable by Lemma 17 (see [20]).

Next, let $w \in \Omega$ be fixed. Then for any $y \in D(w)$, and by assumption (A1), we get

$$\begin{aligned}\|y_s\|_{\mathcal{B}} &\le L(s)|y(s)| + M(s)\|y_0\|_{\mathcal{B}}\\ &\le K_T|y(s)| + M_T\|\phi\|_{\mathcal{B}},\end{aligned}$$

and by (H_3), we have

$$\begin{aligned}|(N(w)y)(t)| &\le \|U(t,0)\|_{B(E)}|\phi(0,w)| + \int_0^t \|U(t,s)\|_{B(E)}|f(s,y_s,w)|ds\\ &\le Me^{-\alpha t}\|\phi\|_{\mathcal{B}} + M\ \int_0^t e^{-\alpha(t-s)}p(s,w)\ \psi\left(\|y_s\|_{\mathcal{B}},w\right) ds.\end{aligned}$$

Then, we have

$$|(N(w)y)(t)| \le M\|\phi\|_{\mathcal{B}} + M\|p\|_{L^1}\psi(K_T R(w) + M_T\|\phi\|_{\mathcal{B}}, w).$$

Set

$$C_1 = M\|\phi\|_{\mathcal{B}}, \quad C_2 = M\|p\|_{L^1}, \quad C_3 = M_T\|\phi\|_{\mathcal{B}}, \quad C_4 = K_T$$

and define the set-valued map

$$\mathcal{G}(\omega) = \{r \ge 0 : C_1 + C_2\psi(C_3 + C_4 r, w) \le r\}.$$

Under a suitable choice of the constantes C_2 and C_4 we can easily show that the inequality

$$C_1 + C_2\psi(C_3 + C_4 r, w) \le r,$$

has at least one solution, and hence the set-valued map $\mathcal{G}$ is nonempty valued. The continuity of ψ implies that $\mathcal{G}$ has closed values. Notice that

$$\mathcal{G}(\omega) = D(w) \circ h(r,w),$$

where h is the function defined by

$$h(r, w) = C_1 + C_2\psi(C_3 + C_4 r, w).$$

Since D and h are measurable, the set-valued map $\mathcal{G}$ is measurable. The celebrated Kuratowski-Ryll-Nardzewski selection theorem ([15], Theorem 19.7) implies that the set-valued map $\mathcal{G}$ has a measurable selection. Thus

$$\|(N(w)y)\| \leq M\|\phi\|_{\mathcal{B}} + M\psi(D_T, w)\|p\|_{L^1} \leq R(w).$$

This implies that N is a random operator with stochastic domain D and $F(w) : D(w) \longrightarrow D(w)$ for each $w \in \Omega$.

Step 1: N is continuous.

Let y^n be a sequence such that $y^n \longrightarrow y$ in Y. Then

$$\begin{aligned}
|(N(w)y^n)(t) - (N(w)y)(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} \, |f(s, y^n_s, w) - f(s, y_s, w)| \, ds. \\
&\leq M \int_0^t e^{-\alpha(t-s)} \, |f(s, y^n_s, w) - f(s, y_s, w)| \, ds.
\end{aligned}$$

Since $f(s, \cdot, w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$\|f(\cdot, y^n_\cdot, w) - f(\cdot, y_\cdot, w)\|_{L^1} \to 0 \text{ as } n \to +\infty.$$

Thus N is continuous.

Step 2: We shall prove that for every $w \in \Omega$, $\{y \in D(w) : N(w)y = y\} \neq \emptyset$. For this, we use Schauder's theorem. First, we will show that $N(D(w))$ is relatively compact using Corduneanu's lemma.

(a) First, it is clear that the assumption (i) holds. Then we will demonstrate that $N(D(w))$ is an equicontinuous set for each closed bounded interval $[0, T]$ in J. Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_2 > \tau_1$, $D(w)$ be a bounded set as in Step 2, and $y \in D(w)$. Then

$$\begin{aligned}
|(N(w)y)(\tau_2) - (N(w)y)(\tau_1)| &\leq \|U(\tau_2, 0) - U(\tau_1, 0)\|_{B(E)}\|\phi\|_{\mathcal{B}} \\
&\quad + \Big| \int_0^{\tau_1} [U(\tau_2, s) - U(\tau_1, s)] f(s, y_s, w) \, ds \Big| \\
&\quad + \Big| \int_{\tau_1}^{\tau_2} U(\tau_2, s) f(s, y_s, w) \, ds \Big| \\
&\leq \|U(\tau_2, 0) - U(\tau_1, 0)\|_{B(E)}\|\phi\|_{\mathcal{B}} \\
&\quad + \int_0^{\tau_1} |U(\tau_2, s) - U(\tau_1, s)||f(s, y_s, w)| \, ds \\
&\quad + \int_{\tau_1}^{\tau_2} |U(\tau_2, s)||f(s, y_s, w)| \, ds \\
&\leq \|U(\tau_2, 0) - U(\tau_1, 0)\|_{B(E)}\|\phi\|_{\mathcal{B}}
\end{aligned}$$

$$+\psi(D_T,w)\int_0^{\tau_1}\|U(\tau_2,s)$$
$$-U(\tau_1,s)\|_{B(E)}p(s,w)ds$$
$$+M\psi(D_T,w)e^{-\alpha(\tau_2-s)}\int_{\tau_1}^{\tau_2}p(s,w)ds.$$

The right-hand of the above inequality tends to zero as $\tau_2-\tau_1\to 0$, as N is bounded and equicontinuous.

(b) Now we will prove that $Z(t,w)=\{(N(w)y)(t): y\in D(w)\}$ is precompact in E. Let $t\in[0,T]$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t$. For $y\in D(w)$ we define

$$(N_\epsilon(w)y)(t)=U(t,0)\phi(0,w)+U(t,t-\epsilon)\int_0^{t-\epsilon}U(t-\epsilon,s)f(s,y_s,w)\,ds.$$

Since $U(t,s)$ is a compact operator and the set $Z_\epsilon(t,w)=\{(N_\epsilon(w)y)(t): y\in D(w)\}$ is the image of a bounded subset of E, then $Z_\epsilon(t,w)$ is precompact in E for every ϵ, $0<\epsilon<t$. Moreover

$$\begin{aligned}|(N(w)y)(t)-(N_\epsilon(w)y)(t)| &\le \int_{t-\epsilon}^{t}\|U(t,s)\|_{B(E)}|f(s,y_s,w)|ds\\ &\le M\psi(D_T,w)e^{-\alpha(t-s)}\int_{t-\epsilon}^{t}p(s,w)ds.\end{aligned}$$

Therefore the set $Z(t,w)=\{(N(w)y)(t): y\in D(w)\}$ is precompact in E.

(c) Finally, it remains to show that N is equiconvergent.

Let $y\in D(w)$. Then from (H_1) and (H_3), we have

$$|(N(w)y)(t)|\le Me^{-\alpha t}\|\phi\|_{\mathcal{B}}+M\int_0^t e^{-\alpha(t-s)}p(s,w)\ \psi\left(D_T,w\right)ds.$$

It follows immediately by (H_5) that $|(N(w)y)(t)|\longrightarrow 0$ as $t\to+\infty$. Then

$$\lim_{t\to+\infty}|(N(w)y)(t)-(N(w)y)(+\infty)|=0,$$

which implies that N is equiconvergent.

As a consequence of Steps 1–2 and (a), (b), (c), we can conclude that $N(w): D(w)\to D(w)$ is continuous and compact. From Schauder's theorem, we deduce that $N(w)$ has a fixed point $y(w)$ in $D(w)$. Since $\bigcap_{w\in\Omega}D(w)\neq\emptyset$, the hypothesis that a measurable selection of int D exists holds. By Lemma 2.8, the random operator N has a stochastic fixed point $y^*(w)$, which is a random mild solution of the random problem (1.1)–(1.2).

4. **AN EXAMPLE**

Consider the following functional partial differential equation:

$$\frac{\partial}{\partial t}z(t,x,w)=a(t,x)\frac{\partial^2}{\partial x^2}z(t,x,w)+C_0(w)K(w)e^{-t}\int_{-\infty}^{0}\frac{\exp(z(t+s,x,w))}{1+s^2}ds, \tag{4.1}$$

$x \in [0, \pi], \ t \in [0, +\infty)$,

$$z(t, 0, w) = z(t, \pi, w) = 0, \quad t \in [0, +\infty), \quad w \in \Omega, \tag{4.2}$$

$$z(s, x, w) = z_0(s, x, w), \quad s \in (-\infty, 0], \quad x \in [0, \pi], \quad w \in \Omega, \tag{4.3}$$

where $a(t, \xi)$ is a continuous function which is uniformly Hölder continuous in t, K and C_0 are a real-valued random variable.

Let $E = L^2[0, \pi]$ and (Ω, F, P) be a complete probability space, and define $A(t)$ by

$$A(t)v = a(t, \xi)v''$$

with domain

$$D(A) = \{v \in E, v, v' \text{ are absolutely continuous }, v'' \in E, \ v(0) = v(\pi) = 0\}.$$

Then $A(t)$ generates an evolution system $U(t, s)$ satisfying assumption (H_1) (see [14, 22]).

Let $\mathcal{B} = BUC(\mathbb{R}^-; E)$ be the space of bounded uniformly continuous functions endowed with the norm,

$$\|\phi\| = \sup_{s \leq 0} |\phi(s)| \quad \text{ for } \phi \in \mathcal{B}.$$

If we put $\phi \in BCU(\mathbb{R}^-; E)$, $x \in [0, \pi]$ and $w \in \Omega$,

$$y(t, x, w) = z(t, x, w), \quad t \in [0, T],$$

$$\phi(s, x, w) = z_0(s, x, w), \quad s \in (-\infty, 0], \quad x \in [0, \pi], \quad w \in \Omega.$$

Set

$$f(t, \varphi(x), w) = \int_{-\infty}^{0} e^{-t} \varphi(s, x, w) ds,$$

with

$$\varphi(s, x, w) = \exp(z(t + s, x, w)).$$

The function $f(t, \varphi(x), w)$ is Carathéodory, and satisfies (H_2) with

$$p(t, w) = K(w) \frac{\pi}{2} e^{-t} \quad \text{ and } \quad \psi(x, w) = |C_0(w)| e^x.$$

Then the problem (1.1)–(1.2) is an abstract formulation of the problem (4.1)–(4.3), and conditions (H_1)–(H_5) are satisfied. Theorem 3.2 implies that the random problem (4.1)–(4.3) has at least one random mild solution.

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