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SINGULARLY PERTURBED MULTI-SCALE SWITCHING DIFFUSIONS

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ABSTRACT. This work is concerned with singularly perturbed multi-scale switching diffusions. The switching process is a two-time-scale Markov chain with slow and fast components subject to weak and strong interactions. In the model, there are two small parameters ε and δ. The first one highlights the fast changing part of the switching process, and the other delineates the slow diffusion. We treat the case that ε and δ are related in that $\varepsilon = \delta^\gamma$. Under certain conditions, asymptotic expansions of the probability densities for the underlying processes are developed. The approach is constructive and the asymptotic series are rigorously justified with error bounds.

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1. INTRODUCTION

This work studies the analysis of probability densities of multi-scale switching diffusions. The motivation comes from many problems in optimization and control in which continuous dynamics and discrete events coexist and the systems display multi-scales; see, for example, [9, 10] for a spectrum of manufacturing models, [15] for applications to ecology systems, [11] for hierarchical decomposition and aggregation in economic systems, and [8] for problems arising in queueing theory and applications; see also [18] for a comprehensive treatment of switching diffusion processes and a collection of applications in [16].

This paper is concerned with multi-scale switching diffusion. To be more specific, consider the switching diffusion process $(x^{\varepsilon,\delta}(\cdot), \alpha^\varepsilon(\cdot))$ given by

$$dx^{\varepsilon,\delta}(t) = b(x^{\varepsilon,\delta}(t), \alpha^\varepsilon(t))dt + \sqrt{\delta}\sigma(x^{\varepsilon,\delta}(t), \alpha^\varepsilon(t))dw(t),\ \ x^{\varepsilon,\delta}(0) = x, \alpha^\varepsilon(0) = i, \tag{1.1}$$

for appropriate functions $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$, where ε and δ are two small positive parameters, $\alpha^\varepsilon(t)$ is a two-time-scale Markov chain with slow and fast components subject to weak and strong interactions. The weak and strong interactions of the system are

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modeled by assuming the generator of the underlying Markov chain $Q^\varepsilon(t) = (q^\varepsilon_{ij}(t))$ to be of the form

$$Q^\varepsilon(t) = \widetilde{Q}(t)/\varepsilon + \widehat{Q}(t), \tag{1.2}$$

with $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ being generators of continuous-time Markov chains. Note that $\widetilde{Q}(t)$ governs the rapidly changing part and $\widehat{Q}(t)$ describes the slow component. The state space of $\alpha^\varepsilon(t)$ is given by a finite set $\mathcal{M} = \{1, \dots, m\}$. Intuitively, when ε and δ are sufficiently small, $\alpha^\varepsilon(t)$ converges rapidly to its stationary distribution and the intensity of the diffusion is negligible. Therefore, the dynamic system is close to a deterministic one. However, when we look at the joint probability distribution of $x^{\varepsilon,\delta}(\cdot)$ and $\alpha^\varepsilon(\cdot)$, the randomness cannot be ignored completely due to its various impact on the dynamics of the underlying system. Much research work has been devoted to such perturbed dynamic systems; see [1, 12, 13] for studies on diffusion processes with small diffusion. For studies on asymptotic properties of (1.1), we refer the reader to [4]. In [14], we constructed an asymptotic expansion of the expectation of $g(x^{\varepsilon,\delta}(t), \alpha^\varepsilon(t))$ for some appropriate function $g(\cdot,\cdot)$. Corresponding to (1.1), we can write down the associated operator

$$\begin{aligned} Lf(x,i,t) = {} & \frac{\partial}{\partial t} f(x,i,t) + \frac{1}{2}\sigma^2(x,i)\frac{d^2}{dx^2} f(x,i,t) \\ & + b(x,i)\frac{d}{dx} f(x,i,t) + \sum_{j=1}^m q^\varepsilon_{ij}(t) f(x,j,t),\ i \in \mathcal{M}, \end{aligned}$$

for a suitably smooth function $f(\cdot, i, \cdot)$ for each $i \in \mathcal{M}$. Our current effort in this paper is to further explore the limiting behavior of the process $x^{\varepsilon,\delta}(\cdot)$ by constructing asymptotic expansions of its probability densities, which are associated with the adjoint operator L^*. In the next section, we define the operator, but suppress the superscript $*$ for notation simplification. We consider the case in which the small parameters δ and ε are related by $\varepsilon = \delta^\gamma$ for some $\gamma > 0$. Depending on γ, $\lim_{\varepsilon\to 0} \dfrac{\delta}{\varepsilon}$ is a nonzero constant ($\gamma = 1$), or 0 ($\gamma < 1$), or ∞ ($\gamma > 1$). These three different cases could yield different outcomes.

Switching diffusion models can represent naturally system dynamics in a wide variety of applications, and hence become increasingly important in system optimization and control. To solve related problems, we need to have a thorough understanding on the structure of the underlying probability density. To illustrate the importance and applications of the asymptotic expansion approach, consider the following motivational example whose detailed discussions can be found in [17, 18]. Suppose that the process $\big(x^{\varepsilon,\delta}(t), \alpha^\varepsilon(t)\big)$ satisfies (1.1) and one wants to minimize an objective function

$$J^{\varepsilon,\delta}(\theta) = E\int_0^T G(x^{\varepsilon,\delta}(t), \alpha^\varepsilon(t), t, \theta)dt = \int_0^T \int \sum_{i=1}^m G(x,i,t,\theta) p_i^{\varepsilon,\delta}(x,t)dxdt,$$

over $[0,T]$ for a given $T \in (0,\infty)$, where θ is a parameter, $p_i^{\varepsilon,\delta}(x,t)$ is the probability density of $\big(x^{\varepsilon,\delta}(t),\alpha^\varepsilon(t)\big)$ with a given initial probability density, and $G(\cdot)$ is a suitable function. The presence of the multi-scale structure together with the weak and strong interactions among states of $\alpha^\varepsilon(t)$ makes analysis and solutions very complicated. One cannot obtain closed form solutions except in some special cases. It is thus vital to reduce the complexity of the underlying problem. We will show in this paper that $p_i^{\varepsilon,\delta}(x,t)$ has a uniform asymptotic expansion. This expansion allows us to also obtain an asymptotic expansion for $J^{\varepsilon,\delta}(\theta)$. Therefore, we can approximate a complex problem by an asymptotic problem which is much simpler than the original one and is easier to analyze.

Although the motivation of our study comes from treating switching diffusion processes, the approach that we are using and the solution method are mainly analytic. The original asymptotic expansion method can be found in [5]. For some related works on asymptotic expansions of systems of the Kolmogorov-Fokker-Planck equations associated with switching diffusions, we refer the reader to [6, 7]. One of the distinct features of this work is: We treat multi-scale systems of rapid switchings and slow diffusion with two small parameters ε and δ. The ε and δ are related through $\varepsilon = \delta^\gamma$ so that different values of γ lead to different behaviors of the underlying systems. It should be noted that different from previous works on asymptotic expansions (of systems of forward equations arising from singularly perturbed switching diffusions; see [6, 7]), in lieu of non-degenerate second order partial differential equations, we have to deal with certain first order linear equations.

The rest of the paper is arranged as follows. Section 2 begins with the problem formulation and presents certain preliminary results. In Section 3, we work with the case that $\varepsilon = \delta^2$. In particular, we construct the matched asymptotic expansions so that the outer expansions are smooth and the initial layer corrections decay exponentially fast. Then, the asymptotic expansions are validated and uniform error bounds are obtained. Section 4 treats the case $\delta = \varepsilon^2$ with its unique features. Finally, we give some further remarks in Section 5.

2. FORMULATION

We work with a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P)$, where $\{\mathcal{F}_t\}$ is a family of σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \le t$. We assume that $\mathcal{F}_0$ is complete, i.e., it contains all null sets. Let $w(\cdot)$ be a one-dimensional standard Brownian motion defined in the filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P)$. Throughout the paper, we work with a finite horizon $t \in [0,T]$ for some $T > 0$. We use v' to denote the transpose of $v \in \mathbb{R}^{m_1\times m_2}$ with $m_1, m_2 \ge 1$. We also use the notation $\mathbb{1} = (1,\dots,1)' \in \mathbb{R}^m$. Suppose $\varepsilon > 0$ and $\delta > 0$ are small parameters. Since we only work with the case that $\varepsilon = \delta^\gamma$, we only use superscripts ε to indicate that the objects we are working with depend on parameters ε and δ. For notational simplicity, K and κ_0 are generic positive constants. Their values may change for different appearances.

Suppose that $\alpha^\varepsilon(t)$ is a continuous-time Markov chain, independent of the Brownian motion $w(t)$, with generator $Q^\varepsilon(t) = \widetilde{Q}(t)/\varepsilon + \widehat{Q}(t)$ and a finite state space $\mathcal{M} = \{1, \dots, m\}$, where $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ are the generators of continuous-time Markov chains. Recall that a generator $\widetilde{Q}(t) \in \mathbb{R}^{m\times m}$ is said to be weakly irreducible if the system of equations

$$\widetilde{\nu}(t)\widetilde{Q}(t) = 0,\ \sum_{i=1}^{m} \widetilde{\nu}_i(t) = 1,$$

has a unique solution $\widetilde{\nu}(t) = (\widetilde{\nu}_1(t), \dots, \widetilde{\nu}_m(t)) \in \mathbb{R}^{1\times m}$ satisfying $\widetilde{\nu}_i(t) \geq 0$ for each $i \in \mathcal{M}$. Such a nonnegative solution is termed a quasi-stationary distribution; see [17, p. 23]. The state space of the process $(x^\varepsilon(t), \alpha^\varepsilon(t))$ is $\mathbb{S}\times\mathcal{M}$, where $\mathbb{S}$ is the unit circle. By identifying the endpoints 0 and 1, let $x \in [0, 1]$ be the coordinates in $\mathbb{S}$. The existence and uniqueness of such switching diffusion processes can be found in [3, Section 2.2].

Suppose that $b(\cdot,\cdot,\cdot) : \mathbb{R}\times\mathcal{M}\times[0, T] \mapsto \mathbb{R}$ and that $\sigma(\cdot,\cdot,\cdot) : \mathbb{R}\times\mathcal{M}\times[0, T] \mapsto \mathbb{R}$. Consider the switching diffusion $(x^\varepsilon(\cdot), \alpha^\varepsilon(\cdot))$ given by (1.1). It is known that the probability density of the process satisfying

$$\int_\Gamma p_i^\varepsilon(x, t)dx = P\left(x^\varepsilon(t) \in \Gamma, \alpha^\varepsilon(t) = i\right),\ \Gamma \subset [0, 1],$$

is the solution to the Kolmogorov-Fokker-Planck equation

$$\begin{cases} \dfrac{\partial p_i^\varepsilon}{\partial t} = \delta\mathcal{D}_i p_i^\varepsilon + \mathcal{L}_i p_i^\varepsilon + \displaystyle\sum_{j=1}^{m} p_j^\varepsilon q_{ji}^\varepsilon, \\ p_i^\varepsilon(x, 0) = g_i(x),\ i \in \mathcal{M}, \end{cases} \tag{2.1}$$

where $g_i(x) \geq 0$,

$$\mathcal{D}_i\cdot = \frac{\partial^2\big(\sigma^2(x, i)\cdot\big)}{2\,\partial x^2}, \quad \mathcal{L}_i\cdot = -\frac{\partial\big(b(x, i)\cdot\big)}{\partial x},$$

and $g(x) = (g_1(x), \dots, g_m(x))$ is the initial probability density of $(x^\varepsilon(t), \alpha^\varepsilon(t))$. Denote $p^\varepsilon(x, t) = \big(p_1^\varepsilon(x, t, \dots, p_m^\varepsilon(x, t)\big)$. In view of the periodicity given above,

$$\int_{[0,1]} g(x)\mathbb{1}dx = 1.$$

Our main interest is to derive asymptotic properties of the solution of (2.1) when $\varepsilon \to 0$. For convenience, we define

$$\begin{aligned} \mathcal{L}p^\varepsilon(x, t) &:= \Big(\mathcal{L}_1 p_1^\varepsilon(x, t), \dots, \mathcal{L}_m p_m^\varepsilon(x, t)\Big) \in \mathbb{R}^{1\times m}, \\ \mathcal{D}p^\varepsilon(x, t) &:= \Big(\mathcal{D}_1 p_1^\varepsilon(x, t), \dots, \mathcal{D}_m p_m^\varepsilon(x, t)\Big) \in \mathbb{R}^{1\times m}, \end{aligned}$$

and sometimes we suppress the variable (x, t). Then (2.1) can be rewritten as

$$\frac{\partial p^{\varepsilon}(x,t)}{\partial t} = \delta \mathcal{D} p^{\varepsilon}(x,t) + \mathcal{L} p^{\varepsilon}(x,t) + p^{\varepsilon}(x,t)\frac{\widetilde{Q}(t)}{\varepsilon} + p^{\varepsilon}(x,t)\widehat{Q}(t). \tag{2.2}$$

To carry out the desired asymptotic analysis, we need the following assumptions:

(A1) There is a $T > 0$ such that for all $t \in [0, T]$, the generator $\widetilde{Q}(t)$ is weakly irreducible. There is an $n \geq 1$ such that $\widetilde{Q}(\cdot) \in C^{n+3}([0, T])$ and $\widehat{Q}(\cdot) \in C^{n+3}([0, T])$.

(A2) For each $i \in \mathcal{M}$, $b(\cdot, i)$, $\sigma(\cdot, i)$, and $g(\cdot, i)$ are periodic with period 1. Moreover, $b(\cdot, i)$, $\sigma(\cdot, i)$, and $g(\cdot, i)$ are $2(n + 3)$-times continuously differentiable.

(A3) For given real-valued functions $h_1(\cdot, \cdot) \in C^{n_1,n_2}([0, 1]\times[0, T])$ and $h_2(\cdot) \in C^{n_1}([0, 1])$, with n_1, n_2 being integers, $1 \leq n_1 \leq 2(n + 3)$, and $1 \leq n_2 \leq n + 3$, the system

$$\begin{cases} \dfrac{\partial h(x,t)}{\partial t} - \mathcal{L}\big(h(x,t)\widetilde{\nu}(t)\big) = h_1(x,t), \\ h(x,0) = h_2(x), \end{cases} \tag{2.3}$$

has a unique solution $h(\cdot, \cdot) \in C^{n_1,n_2}([0, 1] \times [0, T])$.

Remark 2.1. Condition (A1) indicates that the weak irreducibility of $\widetilde{Q}(t)$ implies the existence of the unique quasi-stationary distribution $\widetilde{\nu}(t) = (\widetilde{\nu}_1(t), \dots, \widetilde{\nu}_m(t)) \in \mathbb{R}^{1\times m}$. Condition (A2) ensures certain smoothness of coefficients in operator $\mathcal{D}$, $\mathcal{L}$, and the initial probability density $g(\cdot)$. Our analysis in the next sections shows that in constructing an asymptotic expansion of order n, we only need (A3) holds for certain functions $h_1(\cdot, \cdot)$ and $h_2(\cdot)$.

Now we present several lemmas needed in the subsequent development, which are essential for constructing asymptotic expansions.

Lemma 2.2. *Suppose that a constant matrix $Q \in \mathbb{R}^{m\times m}$ is a generator of a continuous-time Markov chain and that Q is weakly irreducible with $\nu = (\nu_1, \dots, \nu_m)$ being the quasi-stationary distribution associated with Q.*

(a) *Then for any $z \in \mathbb{R}^{1\times m}$, the equation*

$$yQ = z, \tag{2.4}$$

has a solution if and only if $z\mathbb{1} = 0$. Moreover, suppose that y_1 and y_2 are two solutions of (2.4). Then $y_1 - y_2 = c_0\nu$ for some $c_0 \in \mathbb{R}$.

(b) *Define $Q_c = \begin{pmatrix} \mathbb{1} & Q \end{pmatrix} \in \mathbb{R}^{m\times(m+1)}$. Then*

$$\operatorname{rank}(Q_c) = m, \ \operatorname{rank}(Q_c Q_c') = m.$$

(c) *Define $\widehat{z} = \begin{pmatrix} 0 & z \end{pmatrix}$. Any solution of (2.4) can be written as $\eta = c_0\nu + \xi$, where $c_0 \in \mathbb{R}$ is an arbitrary constant and $\xi = \widehat{z}Q_c'(Q_cQ_c')^{-1}$ is the unique solution of (2.4) satisfying $\xi\mathbb{1} = 0$.*

(d) *There are constants $K > 0$ and $\kappa_0 > 0$ such that*

$$|\exp(Qs) - \overline{P}| \le K \exp(-\kappa_0 s),\ \ s > 0,$$

where $\overline{P} = \mathbb{1}(\nu_1, \dots, \nu_m) \in \mathbb{R}^{m \times m}$.

Proof. The first part of (a) follows from the Fredholm alternative (see [17, Corollary A.38]). If y_1 and y_2 are two solutions of (2.4), we have $Q'(y_1 - y_2)' = 0$. Because Q is weakly irreducible, the rank of $Q' = m - 1$. So the null space of Q' is one dimensional. Consequently, the null space is spanned by $\nu' = (\nu_1, \dots, \nu_m)' \in \mathbb{R}^{m\times 1}$. This yields that $y_1 - y_2 = c_0 \nu$ for some $c_0 \in \mathbb{R}$. Part (b) follows from the weak irreducibility of Q. We proceed to prove part (c). Consider the system

$$\xi Q = z, \quad \xi \mathbb{1} = 0,$$

that can be rewritten as $\xi Q_c = \widehat{z}$. Thus its solution can be represented by $\xi = \widehat{z} Q_c'(Q_c Q_c')^{-1}$. The proof of part (d) can be found in [17, Lemma 4.4]. □

3. ASYMPTOTIC EXPANSION: $\varepsilon = \delta^2$

This section focuses on the case $\varepsilon = \delta^\gamma$ for a positive integer $\gamma \ge 2$, i.e., ε goes to zero much faster than δ. To be more specific, let $\varepsilon = \delta^2$. A similar argument can be used for $\gamma \ge 3$. To proceed, we seek asymptotic expansions of $p^\varepsilon(x,t)$ of the form

$$\begin{aligned} \Phi_n^\varepsilon(x,t) + \Psi_n^\varepsilon(x,t) &= \sum_{k=0}^n (\sqrt{\varepsilon})^k \varphi_k(x,t) + \sum_{k=0}^n (\sqrt{\varepsilon})^k \psi_k(x,\tau) \\ &= \sum_{k=0}^n \delta^k \varphi_k(x,t) + \sum_{k=0}^n \delta^k \psi_k(x,\tau), \end{aligned} \tag{3.1}$$

where $\tau = t/\varepsilon$ is the stretched-time variables,

$$\begin{aligned} \varphi_k(x,t) &= \big(\varphi_k(x,1,t), \dots, \varphi_k(x,m,t)\big) \in \mathbb{R}^{1\times m}, \\ \psi_k(x,\tau) &= \big(\psi_k(x,1,\tau), \dots, \psi_k(x,m,\tau)\big) \in \mathbb{R}^{1\times m}. \end{aligned}$$

The $\varphi_k(x,t)$ are called regular terms or outer expansion terms, and the $\psi_k(x,\tau)$ are the initial layer corrections. We aim to obtain asymptotic expansions of order n, and derive the uniform error bounds. For the purposes of error bound estimates, we need to calculate six extra terms $\varphi_k(x,t)$, $\psi_k(x,\tau)$ with $k = n+1, n+2, n+3$. This will become clear when we carry out the error analysis.

First let us look at the regular part of the asymptotic expansions $\Phi_{n+3}^\varepsilon(x,t) = \sum_{k=0}^{n+3} \delta^k \varphi_k(x,t)$. Substituting it into (2.2), we obtain

$$\sum_{k=1}^{n+4} \delta^k \mathcal{D}\varphi_{k-1} + \sum_{k=0}^{n+3} \delta^k \mathcal{L}\varphi_k = \sum_{k=0}^{n+3} \delta^k \frac{\partial \varphi_k}{\partial t} - \sum_{k=-2}^{n+1} \delta^k \varphi_{k+2} \widetilde{Q}(t) - \sum_{k=0}^{n+3} \delta^k \varphi_k \widehat{Q}(t).$$

Comparing coefficients of like powers of δ^k for $k=-2,\dots,n+1$ leads to

(3.2)
$$
\begin{aligned}
&\varphi_0(x,t)\widetilde{Q}(t)=0,\\
&\varphi_1(x,t)\widetilde{Q}(t)=0,\\
&\varphi_2(x,t)\widetilde{Q}(t)=\frac{\partial\varphi_0(x,t)}{\partial t}-\mathcal{L}\varphi_0(x,t)-\varphi_0(x,t)\widehat{Q}(t),\\
&\varphi_{k+2}(x,t)\widetilde{Q}(t)=\frac{\partial\varphi_k(x,t)}{\partial t}-\mathcal{L}\varphi_k(x,t)-\varphi_k(x,t)\widehat{Q}(t)-\mathcal{D}\varphi_{k-1},\ k=1,\dots,n+1.
\end{aligned}
$$

Define the stretched variable $\tau=t/\delta^2$ as in the usual practice of singular perturbation method. Substituting the initial layer corrections

$$\Psi^{\varepsilon}_{n+3}(x,\tau)=\sum_{k=0}^{n+3}\delta^k\psi_k(x,\tau)$$

into (2.1), we obtain

$$\sum_{k=0}^{n+3}\delta^{k+1}\mathcal{D}\psi_k+\sum_{k=0}^{n+3}\delta^k\mathcal{L}\psi_k+\sum_{k=0}^{n+3}\delta^k\psi_k\widehat{Q}(t)=\sum_{k=0}^{n+3}\delta^{k-2}\Big(\frac{\partial\psi_k}{\partial\tau}-\psi_k\widetilde{Q}(t)\Big), \tag{3.3}$$

Owing to the smoothness of $\widetilde{Q}(\cdot)$, a truncated Taylor expansion about $t=0$ up to order k leads to

$$\widetilde{Q}(t)=\widetilde{Q}(\delta^2\tau)=\sum_{j=0}^{k}\frac{(\delta^2\tau)^j}{j!}\frac{d^j\widetilde{Q}(0)}{dt^j}+R^{(k)}(t), \tag{3.4}$$

where

$$R^{(k)}(t)=\frac{t^k}{k!}\left(\frac{d^k\widetilde{Q}(\xi)}{dt^k}-\frac{d^k\widetilde{Q}(0)}{dt^k}\right),$$

for some $0<\xi<t$. By virtue of assumption (A1), for each $k=0,\dots,n+1$,

$$|R^{(k)}(t)|=O(t^{k+1})\ \text{ uniformly in } t\in[0,T]. \tag{3.5}$$

Denote $\widetilde{Q}^{(j)}(0)=\frac{d^j\widetilde{Q}(0)}{dt^j}$. Drop the term $R^{(n+1)}(t)$ and use the first $(n+2)$ terms in the Taylor expansion up to the order $(n+1)$ for $\widetilde{Q}(t)$, we obtain from (3.3) that

(3.6)
$$\sum_{k=0}^{n+3}\delta^{k+1}\mathcal{D}\psi_k+\sum_{k=0}^{n+3}\delta^k\mathcal{L}\psi_k+\sum_{k=0}^{n+3}\delta^k\psi_k\widehat{Q}(t)=\sum_{k=0}^{n+3}\delta^{k-2}\Big(\frac{\partial\psi_k}{\partial\tau}-\sum_{j=0}^{n+1}\frac{(\delta^2\tau)^j}{j!}\psi_k\widetilde{Q}^{(j)}(0)\Big),$$

that is,

$$
\begin{aligned}
&\sum_{k=0}^{n+3}\delta^{k+1}\mathcal{D}\psi_k+\sum_{k=0}^{n+3}\delta^k\mathcal{L}\psi_k+\sum_{k=0}^{n+3}\delta^k\psi_k\widehat{Q}(t)\\
&\quad=\sum_{k=0}^{n+3}\delta^{k-2}\Big(\frac{\partial\psi_k}{\partial\tau}-\psi_k\widetilde{Q}(0)\Big)-\sum_{k=0}^{n+3}\delta^{k-2}\sum_{j=1}^{n+1}\frac{(\delta^2\tau)^j}{j!}\psi_k\widetilde{Q}^{(j)}(0).
\end{aligned}
$$

Then comparing coefficients of like powers of δ^k for $k = -2, \ldots, n+2$ leads to

$$\begin{aligned}\frac{\partial \psi_0(x,\tau)}{\partial \tau} &= \psi_0(x,\tau)\widetilde{Q}(0),\\ \frac{\partial \psi_1(x,\tau)}{\partial \tau} &= \psi_1(x,\tau)\widetilde{Q}(0),\\ \frac{\partial \psi_k(x,\tau)}{\partial \tau} &= \psi_k(x,\tau)\widetilde{Q}(0) + r_k(x,\tau),\ k = 2,\ldots,n+3,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned} r_2(x,\tau) &= \mathcal{L}\psi_0(x,\tau) + \psi_0(x,\tau)\widehat{Q}(t) + \tau\psi_0(x,\tau)\widetilde{Q}^{(1)}(0),\\ r_k(x,\tau) &= \mathcal{D}\psi_{k-3}(x,\tau) + \mathcal{L}\psi_{k-2}(x,\tau) + \psi_{k-2}(x,\tau)\widehat{Q}(t)\\ &\qquad + \sum_{j=1}^{[k/2]} \frac{\tau^j}{j!}\psi_{k-2j}(x,\tau)\widetilde{Q}^{(j)}(0),\ k = 3,\ldots,n+3.\end{aligned}$$

To ensure the match of the initial conditions, we choose

$$\begin{aligned}\varphi_0(x,0) + \psi_0(x,0) &= g(x),\\ \varphi_k(x,0) + \psi_k(x,0) &= 0,\ k = 1,\ldots,n+3.\end{aligned} \tag{3.8}$$

Our task to follow is to construct the sequences $\{\varphi_k(x,t)\}$ and $\{\psi_k(x,t)\}$.

3.1. **Construction of $\varphi_0(x,t)$ and $\psi_0(x,\tau)$.** By virtue of Lemma 2.2, the first equation in (3.2) implies that $\varphi_0(x,t) = \beta_0(x,t)\widetilde{\nu}(t)$, where $\beta_0(x,t) \in \mathbb{R}$ is a real-valued function to be determined. Consider the third equation of (3.2), namely,

$$\varphi_2(x,t)\widetilde{Q}(t) = \frac{\partial \varphi_0(x,t)}{\partial t} - \mathcal{L}\varphi_0(x,t) - \varphi_0(x,t)\widehat{Q}(t).$$

Plugging in $\varphi_0(x,t) = \beta_0(x,t)\widetilde{\nu}(t)$, we obtain

$$\varphi_2(x,t)\widetilde{Q}(t) = \frac{\partial \beta_0(x,t)}{\partial t}\widetilde{\nu}(t) + \beta_0(x,t)\frac{d\widetilde{\nu}(t)}{dt} - \mathcal{L}\big(\beta_0(x,t)\widetilde{\nu}(t)\big) - \beta_0(x,t)\widetilde{\nu}(t)\widehat{Q}(t).$$

Postmultiplying $\mathbb{1}$ in the above equation with notice that $\widetilde{\nu}(t)\mathbb{1} = 1$, $\frac{d\widetilde{\nu}(t)}{dt}\mathbb{1} = 0$, and $\widetilde{Q}(t)\mathbb{1} = \widehat{Q}(t)\mathbb{1} = 0$, we arrive at

$$\begin{cases} \dfrac{\partial \beta_0(x,t)}{\partial t} - \mathcal{L}\big(\beta_0(x,t)\widetilde{\nu}(t)\big) = 0,\\ \beta_0(x,0) = g(x)\mathbb{1}, \end{cases} \tag{3.9}$$

where the second equation in (3.9) is the initial condition we chose. By virtue of assumption (A3), (3.9) has a unique solution. It follows from (3.7) and (3.8) that

$$\psi_0(x,\tau) = \big(g(x) - \varphi_0(x,0)\big)\exp(\widetilde{Q}(0)\tau). \tag{3.10}$$

We are in a position to derive certain smoothness of $\varphi_0(x,t)$ and the exponential decay property of $\psi_0(x,\tau)$.

Lemma 3.1. *For $\varphi_0(x,t)$ and $\psi_0(x,\tau)$ obtained above, the following assertions hold.*

(a) *The $\varphi_0(\cdot,\cdot) \in C^{2(n+3),n+3}([0,1]\times[0,T])$.*

(b) *There exist positive constants K and κ_0 such that*

$$\sup_{x\in[0,1]}\left|\frac{\partial^j \psi_0(x,\tau)}{\partial x^j}\right| \le K\exp(-\kappa_0\tau),\ \ j=0,1,\dots,2(n+3).$$

Proof. (a) Denote $Q_1(t)=\begin{pmatrix}\widetilde{Q}(t) & \mathbb{1}\end{pmatrix}$. Then $\widetilde{\nu}(t)Q_1(t)=\begin{pmatrix}0_{1\times m} & 1\end{pmatrix}$. Moreover, using the irreducibility of $\widetilde{Q}(t)$, similar to Lemma 2.2, we can show that $\operatorname{rank}\Big(Q_1(t)Q_1'(t)\Big)=m$. As a result, $\widetilde{\nu}(t)=\begin{pmatrix}0_{1\times m} & 1\end{pmatrix}Q_1'(t)\Big(Q_1(t)Q_1'(t)\Big)^{-1}$. Thus $\widetilde{\nu}(t)\in C^{n+3}([0,T])$. Then (a) follows from assumption (A3).

(b) Since $\psi_0(x,0)=g(x)-\varphi_0(x,0)$, then $\psi_0(\cdot,0)\in C^{2(n+3)}([0,1])$. We deduce that $\psi_0(\cdot,\cdot)\in C^{2(n+3),n+3}([0,1]\times[0,T])$. Since $\varphi_0(x,0)=g(x)\mathbb{1}\widetilde{\nu}(0)$,

$$\psi_0(x,0)\mathbb{1}=g(x)\mathbb{1}-g(x)\mathbb{1}\widetilde{\nu}(0)\mathbb{1}=0.$$

It follows that $\psi_0(x,0)\overline{P}=0$, where $\overline{P}=\mathbb{1}\widetilde{\nu}(0)$. By virtue of Lemma 2.2, we have

$$\begin{aligned}|\psi_0(x,\tau)| &= \big|\psi_0(x,0)\overline{P}+\psi_0(x,0)\big(\exp(\widetilde{Q}(0)\tau)-\overline{P}\big)\big|\\ &\le \big|\psi_0(x,0)\big|\big|\exp(\widetilde{Q}(0)\tau)-\overline{P}\big|\\ &\le K\exp(-\kappa_0\tau).\end{aligned}$$

Moreover, it follows from (3.7) and $\psi_0(x,0)\overline{P}=0$ that for each $j=1,\dots,2(n+3)$,

$$\frac{\partial^j\psi_0(x,\tau)}{\partial x^j}=\frac{\partial^j\psi_0(x,0)}{\partial x^j}\exp(\widetilde{Q}(0)\tau),\ \ \frac{\partial^j\psi_0(x,0)}{\partial x^j}\overline{P}=0.$$

A similar argument as that for $\psi_0(x,\tau)$ yields that

$$\left|\frac{\partial^j\psi_0(x,\tau)}{\partial x^j}\right|\le K\exp(-\kappa_0\tau),\ \ j=1,\dots,2(n+3).$$

Furthermore, it is readily seen that the above estimate holds uniformly for $x\in[0,1]$. The conclusion follows. □

3.2. **Construction of Higher-Order Terms.** We proceed to obtain the asymptotic expansions. Using the second equation in (3.2),

$$\varphi_1(x,t)=\beta_1(x,t)\widetilde{\nu}(t),$$

with $\beta_1(x,t)\in\mathbb{R}$ being a real-valued function to be determined, we obtain that $\varphi_1(x,i,t)$ is a function independent of i. By substituting $\varphi_1(x,t)=\beta_1(x,t)\widetilde{\nu}(t)$ into the fourth equation of (3.2) and postmultiplying it by $\mathbb{1}$, we obtain

$$\frac{\partial\beta_1(x,t)}{\partial t}-\mathcal{L}\big(\beta_1(x,t)\widetilde{\nu}(t)\big)=\mathcal{D}\varphi_0(x,t)\mathbb{1}. \tag{3.11}$$

We need to determine the initial value $\beta_1(x,0)$. It follows from the second equation in (3.7) that

$$\begin{aligned}\psi_1(x,\tau)&=-\varphi_1(x,0)\exp(\widetilde{Q}(0)\tau)\\ &=-\beta_1(x,0)\widetilde{\nu}(0)\exp(\widetilde{Q}(0)\tau).\end{aligned} \tag{3.12}$$

We require the initial layer terms go to 0 for large τ leading to $\psi_1(x,\tau) \to 0$ as $\tau \to \infty$. Letting $\tau \to \infty$ in (3.12) and noting that $\exp(\widetilde{Q}(0)\tau) \to \overline{P}$ with $\overline{P} = \mathbb{1}\widetilde{\nu}(0)$, we obtain $0 = -\beta_1(x,0)\widetilde{\nu}(0)$, that is,

$$\beta_1(x,0) = 0. \tag{3.13}$$

As a result, $\psi_1(x,\tau) = 0$. Moreover, $\beta_1(x,t)$ can be determined from (3.11) and (3.13). Since $\varphi_0(\cdot,\cdot) \in C^{2(n+3),n+3}([0,1]\times[0,T])$, we have $\beta_1(\cdot,\cdot) \in C^{2(n+2),n+2}([0,1]\times[0,T])$.

Next, for $k = 2,\dots,n+1$, we construct $\varphi_k(x,t)$ and $\psi_k(x,\tau)$ by induction. Suppose that the terms $\varphi_j(x,t)$ and $\psi_j(x,\tau)$ for $j < k$ have been constructed such that for each $j < k$, $\varphi_j(\cdot,\cdot) \in C^{2(n+3-j),n+3-j}([0,1]\times[0,T])$, $\psi_j(x,\tau)$ and its partial derivatives in x up to order $2(n+3-j)$ decay exponentially fast. Moreover, assume that

$$c_j(x,t)\mathbb{1} = 0, \; j \le k-1,$$

where

$$\begin{aligned} c_0(x,t) &= \frac{\partial\varphi_0(x,t)}{\partial t} - \mathcal{L}\varphi_0(x,t) - \varphi_0(x,t)\widehat{Q}(t),\\ c_j(x,t) &= \frac{\partial\varphi_j(x,t)}{\partial t} - \mathcal{L}\varphi_j(x,t) - \varphi_j(x,t)\widehat{Q}(t) - \mathcal{D}\varphi_{j-1}(x,t), \; j = 1,2,\dots,k-1. \end{aligned}$$

We proceed to construct $\varphi_k(x,t)$ and $\psi_k(x,\tau)$. Using (3.2), we have

$$\varphi_k(x,t)\widetilde{Q}(t) = c_{k-2}(x,t). \tag{3.14}$$

The right-hand side above, namely $c_{k-2}(x,t)$ is a known function since $\varphi_{k-2}(x,t)$ and $\varphi_{k-3}(x,t)$ have been constructed. Because $c_{k-2}(x,t)\mathbb{1} = 0$, by Lemma 2.2, $\varphi_k(x,t)$ is the sum of solutions to the homogeneous equation and a particular solution $\widehat{\varphi}_k(x,t)$ of the nonhomogeneous equation with $\widehat{\varphi}_k(x,t)\mathbb{1} = 0$. It is of the form

$$\varphi_k(x,t) = \beta_k(x,t)\widetilde{\nu}(t) + \widehat{\varphi}_k(x,t). \tag{3.15}$$

Define

$$\widetilde{Q}_c(t) = \Big(\mathbb{1}\ \widetilde{Q}(t)\Big) \in \mathbb{R}^{m\times(m+1)},\ \widehat{c}_{k-2}(x,t) = \Big(0\ c_{k-2}(x,t)\Big) \in \mathbb{R}^{1\times(m+1)}.$$

By virtue of Lemma 2.2, it follows from (3.14) that

$$\widehat{\varphi}_k(x,t) = \widehat{c}_{k-2}(x,t)\widetilde{Q}_c(t)'\Big(\widetilde{Q}_c(t)\widetilde{Q}_c'(t)\Big)^{-1}. \tag{3.16}$$

Note that $\widehat{\varphi}_k(x,t)\mathbb{1} = 0$. We proceed to find $\beta_k(x,t)$. Using (3.2), we obtain

$$\varphi_{k+2}(x,t)\widetilde{Q}(t) = \frac{\partial\varphi_k(x,t)}{\partial t} - \mathcal{L}\varphi_k(x,t) - \varphi_k(x,t)\widehat{Q}(t) - \mathcal{D}\varphi_{k-1}(x,t) = c_k(x,t). \tag{3.17}$$

Multiplying both sides by $\mathbb{1}$ from the right and noting the form of $\varphi_k(x,t)$ in (3.15), we arrive at

$$\frac{\partial\beta_k(x,t)}{\partial t} - \mathcal{L}\big(\beta_k(x,t)\widetilde{\nu}(t)\big) = -\frac{\partial\widehat{\varphi}_k(x,t)}{\partial t}\mathbb{1} + \mathcal{L}\widehat{\varphi}_k(x,t)\mathbb{1} + \mathcal{D}\varphi_{k-1}(x,t)\mathbb{1}. \tag{3.18}$$

Equation (3.18) is solvable if the initial condition is specified. We need to use the initial layer term to determine the initial condition. In view of (3.7),

(3.19)
$$\psi_k(x,\tau) = -\varphi_k(x,0)\exp(\widetilde{Q}(0)\tau) + \int_0^\tau r_k(x,s)\exp(\widetilde{Q}(0)(\tau-s))ds,\ k=2,\dots,n+1.$$

We demand that $\psi_k(x,\tau)\to 0$ as $\tau\to\infty$. Letting $\tau\to\infty$ in (3.19) and noting that $\exp(\widetilde{Q}(0)\tau)\to\overline{P}$ with $\overline{P}=\mathbb{1}\widetilde{\nu}(0)$, we obtain

$$0 = -\varphi_k(x,0)\overline{P} + \int_0^\infty r_k(x,s)\overline{P}ds, \tag{3.20}$$

By multiplying both sides from the right by $\mathbb{1}$, the above equation is equivalent to

$$0 = -\varphi_k(x,0)\mathbb{1} + \int_0^\infty r_k(x,s)\mathbb{1}ds. \tag{3.21}$$

On the other hand, we have

(3.22)
$$\begin{aligned}-\varphi_k(x,0)\mathbb{1} + \int_0^\infty r_k(x,s)\mathbb{1}ds &= -\Big(\beta_k(x,0)\widetilde{\nu}(0)\mathbb{1} + \widehat{\varphi}_k(x,0)\mathbb{1}\Big) + \int_0^\infty r_k(x,s)\mathbb{1}ds\\ &= -\beta_k(x,0) + \int_0^\infty r_k(x,s)\mathbb{1}ds,\end{aligned}$$

where $\widehat{\varphi}_k(x,0)\mathbb{1}=0$ by our construction. Note that the integral involving $r_k(x,s)$ is well defined since $|r_k(x,s)|\le K\exp(-\kappa_0 s)$ for some $K>0$ and $\kappa_0>0$ by the induction hypothesis. By virtue of (3.21) and (3.22),

$$\beta_k(x,0) = \int_0^\infty r_k(x,s)\mathbb{1}ds. \tag{3.23}$$

Conversely, with $\beta_k(x,0)$ given in (3.23), $\psi_k(x,\tau)\to 0$ as $\tau\to\infty$ and (3.20) holds as desired. The initial condition for $\beta_k(x,t)$ has thus been found. Then it follows from (3.15) that

$$\varphi_k(x,t) = \beta_k(x,t)\widetilde{\nu}(t) + \widehat{c}_{k-2}(x,t)\widetilde{Q}_c(t)'\big(\widetilde{Q}_c(t)\widetilde{Q}_c'(t)\big)^{-1}, \tag{3.24}$$

with $\beta_k(x,t)$ determined by (3.18) and the initial condition (3.23). By our construction, we have $c_k(x,t)\mathbb{1}=0$. Then we can establish the following lemma.

Lemma 3.2. *For $\varphi_k(x,t)$ and $\psi_k(x,\tau)$ obtained above, the following assertions hold.*

(a) *The $\varphi_k(\cdot,\cdot)\in C^{2(n+3-k),n+3-k}([0,1]\times[0,T])$.*
(b) *There exist positive constants K and κ_0 such that*

$$\sup_{x\in[0,1]}\left|\frac{\partial^j\psi_k(x,\tau)}{\partial x^j}\right| \le K\exp(-\kappa_0\tau),\ j=0,1,\dots,2(n+3-k).$$

Proof. (a) Recall from Lemma 3.1 that we have $\varphi_j(\cdot,\cdot)\in C^{2(n+3-j),n+3-j}([0,1]\times[0,T])$ for $0\le j<k$ and $\widetilde{\nu}(\cdot)\in C^{n+3}([0,T])$. Then $c_{k-2}(\cdot,\cdot)\in C^{2(n+4-k),n+4-k}([0,1]\times[0,T])$. Then we derive from (3.16) and (3.18) that $\widehat{\varphi}_k(\cdot,\cdot)\in C^{2(n+4-k),n+4-k}([0,1]\times[0,T])$

and $\beta_k(\cdot,\cdot) \in C^{2(n+3-k),n+3-k}([0,1]\times[0,T])$. Thus, (3.24) implies that $\varphi_k(\cdot,\cdot) \in C^{2(n+3-k),n+3-k}([0,1]\times[0,T])$ as desired.

(b) Since $\psi_k(\cdot,0) = -\varphi_k(\cdot,0) \in C^{2(n+3-k)}([0,1])$, $\psi_k(\cdot,0) \in C^{2(n+3-k)}([0,1])$. We deduce that $\psi_k(\cdot,\cdot) \in C^{2(n+3-k),n+3-k}([0,1]\times[0,T])$. Recall that for $l = 0,1,\dots,k-1$, we have

$$\Big|\frac{\partial^j \psi_l(x,\tau)}{\partial x^j}\Big| \le K\exp(-\kappa_0\tau),\ j = 0,1,\dots,2(n+3-l),$$

It follows that

$$\Big|\frac{\partial^j r_k(x,\tau)}{\partial x^j}\Big| \le K\exp(-\kappa_0\tau),\ j = 0,1,\dots,2(n+4-k).$$

We proceed to show that $\psi_k(x,\tau)$ and its partial derivatives in x also decays exponentially fast. In view of (3.20), we have

$$\begin{aligned}\psi_k(x,\tau) = \Big|&\psi_k(x,0)\big(\exp(\widetilde{Q}(0)\tau) - \overline{P}\big)\\ &+\int_0^\tau r_k(x,s)\big(\exp\big(\widetilde{Q}(0)(\tau-s)\big) - \overline{P}\big)ds + \int_\tau^\infty -r_k(x,s)\overline{P}ds\Big|\\ \le{}& K\exp(-\kappa_0\tau) + K\int_0^\tau \exp(-\kappa_0(\tau-s))\exp(-\kappa_0 s)ds\\ &+K\int_\tau^\infty \exp(-\kappa_0 s)ds.\end{aligned}$$

Thus $|\psi_k(x,\tau)| \le K\exp(-\kappa_0\tau)$ as desired. In view of (3.7), we have

$$\frac{\partial^j\psi_k(x,\tau)}{\partial x^j} = \frac{\partial^j\psi_k(x,0)}{\partial x^j}\exp(\widetilde{Q}(0)\tau) + \int_0^\tau \frac{\partial^j r_k(x,s)}{\partial x^j}\exp\big(\widetilde{Q}(0)(\tau-s)\big)ds, \tag{3.25}$$

for $j = 1,\dots,2(n+3-k)$. Moreover, it follows from (3.20) that

$$\frac{\partial^j\psi_k(x,0)}{\partial x^j}\overline{P} + \int_0^\infty \frac{\partial^j r_k(x,s)}{\partial x^j}\overline{P}ds = 0. \tag{3.26}$$

By combining (3.25) and (3.26), we arrive at

$$\begin{aligned}\frac{\partial^j\psi_k(x,\tau)}{\partial x^j} = \Big|&\frac{\partial^j\psi_k(x,0)}{\partial x^j}\big(\exp(\widetilde{Q}(0)\tau) - \overline{P}\big)\\ &+\int_0^\tau \frac{\partial^j r_k(x,s)}{\partial x^j}\big(\exp(\widetilde{Q}(0)(\tau-s)) - \overline{P}\big)ds + \int_\tau^\infty -\frac{\partial^j r_k(x,s)}{\partial x^j}\overline{P}ds\Big|\\ \le{}& K\exp(-\kappa_0\tau) + K\int_0^\tau \exp(-\kappa_0(\tau-s))\exp(-\kappa_0 s)ds\\ &+K\int_\tau^\infty \exp(-\kappa_0 s)ds.\end{aligned}$$

Detail computations lead to

$$\Big|\frac{\partial^j\psi_k(x,\tau)}{\partial x^j}\Big| \le K\exp(-\kappa_0\tau),\ j = 0,1,\dots,2(n+3-k).$$

Furthermore, it is readily seen that the above estimates hold uniformly for $x \in [0,1]$. The desired result thus follows. $\square$

Up to this point, $\varphi_k(x,t)$ and $\psi_k(x,\tau)$ have been completely specified for $k = 0,1,\dots,n+1$. By virtue of Lemma 2.2, the last two equations in (3.2) have solutions and we choose

(3.27)

$$\varphi_k(x,t) = \widehat{c}_{k-2}(x,t)\widetilde{Q}_c(t)'\big(\widetilde{Q}_c(t)\widetilde{Q}_c'(t)\big)^{-1} + \Big(\int_0^\infty r_k(x,s)1\!\!1 ds\Big)\widetilde{\nu}(t),\ k = n+2, n+3.$$

Finally, using (3.7),

$$\psi_k(x,\tau) = -\varphi_k(x,0)\exp(\widetilde{Q}(0)\tau) + \int_0^\tau r_k(x,s)\exp\big(\widetilde{Q}(0)(\tau - s)\big)ds,\ k = n+2, n+3.$$

With $\varphi_{n+2}(x,t)$ and $\varphi_{n+3}(x,t)$ defined above, (3.20) holds for $k = n+2, n+3$. We have $\varphi_{n+2}(\cdot,\cdot) \in C^{4,2}([0,1]\times[0,T])$ and $\varphi_{n+3}(\cdot,\cdot) \in C^{2,1}([0,1]\times[0,T])$. The exponential decay property of $\psi_{n+2}(x,\tau)$, $\psi_{n+3}(x,\tau)$, and their partial derivatives with respect to x can be obtained as in Lemma 3.2. We summarize what we have obtained thus far. It is given in the following theorem.

Theorem 3.3. *Under conditions* (A1), (A2), *and* (A3), *we can construct sequences* $\{\varphi_k(x,t) : k = 0,\dots,n+3\}$ *and* $\{\psi_k(x,t) : k = 0,\dots,n+3\}$ *satisfying* (3.2), (3.7), *and* (3.8) *as follows.*

(a) *The* $\varphi_0(x,t) = \beta_0(x,t)\widetilde{\nu}(t)$ *with* $\beta_0(x,t)$ *given by* (3.9); $\psi_0(x,\tau)$ *is given by* (3.10).

(b) *The* $\varphi_1(x,t) = \beta_1(x,t)\widetilde{\nu}(t)$ *with* $\beta_1(x,t)$ *given by* (3.11) *and* (3.13); $\psi_1(x,\tau) = 0$.

(c) *For* $k = 2,\dots,n+1$, $\varphi_k(x,t) = \beta_k(x,t)\widetilde{\nu}(t) + \widehat{\varphi}_k(x,t)$, *where* $\beta_k(x,t)$ *is given by* (3.18) *and* $\widehat{\varphi}_k(x,t)$ *is given by* (3.16); $\psi_k(x,\tau)$ *is given by* (3.19).

(d) $\varphi_k(x,t) \in C^{2(n+3-k),n+3-k}([0,1]\times[0,T])$ *for* $k = 0,\dots,n+1$; $\varphi_{n+2}(x,t) \in C^{4,2}([0,1]\times[0,T])$ *and* $\varphi_{n+3}(x,t) \in C^{2,1}([0,1]\times[0,T])$.

(e) $\psi_k(x,\tau)$ *decays exponentially fast in that*

$$\sup_{x\in[0,1]}\left|\frac{\partial^j\psi_k(x,\tau)}{\partial x^j}\right| \le K\exp(-\kappa_0\tau),$$

with $j = 0,1,\dots,2(n+3-k)$ *for each* $k = 0,\dots,n+1$, $j = 0,1,\dots,4$ *for* $k = n+2$, *and* $j = 0,1,2$ *for* $k = n+3$.

3.3. **Error Bounds.** We have constructed the formal asymptotic expansions of $p^\varepsilon(x,t)$. We need to prove the validity of the expansions by deriving the error bounds. We aim to show that

$$\sup_{(x,t)\in[0,1]\times[0,T]}\left|p^\varepsilon(x,t) - \sum_{k=0}^n \delta^k\varphi_k(x,t) - \sum_{k=0}^n \delta^k\psi_k\Big(x,\frac{t}{\varepsilon}\Big)\right| = O(\delta^{n+1}) = O(\varepsilon^{(n+1)/2}).$$

We first recall a lemma from [7, Propposition 4.1]. Let $V : [0,1]\times\mathcal{M}\times[0,T] \mapsto \mathbb{R}$ be a sufficiently smooth function. We define

$$\mathcal{G}^\varepsilon V(x,t) = -\frac{\partial V(x,t)}{\partial t} + \delta\mathcal{D}V(x,t) + \mathcal{L}V(x,t) + V(x,t)Q^\varepsilon(t),\ (x,t)\in[0,1]\times[0,T].$$

Lemma 3.4. *Suppose that $v^\varepsilon(x,t)$ is a solution of the following system*

$$\begin{cases} \mathcal{G}^\varepsilon v^\varepsilon(x,t) = f(x,t), & (x,t) \in [0,1]\times(0,T], \\ v^\varepsilon(x,0) = 0, & x \in [0,1], \end{cases}$$

where $\sup\limits_{(x,t)} |f(x,t)| = O(\delta^{n+2})$. *Then*

$$\sup_{(x,t)} |v^\varepsilon(x,t)| = O(\delta^{n+1}).$$

With the preparation above, we proceed to obtain the desired upper bounds on the approximation errors. For $k = 0, \dots, n+3$, define a sequence of approximation errors

$$e_k^\varepsilon(x,t) = p^\varepsilon(x,t) - \Phi_k^\varepsilon(x,t) - \Psi_k^\varepsilon(x,\tau),$$

where $p^\varepsilon(x,t)$ is the solution of (2.1), and $\Phi_k^\varepsilon(x,t) + \Psi_k^\varepsilon(x,\tau)$ is the kth-order approximation to $p^\varepsilon(x,t)$. We proceed to obtain the order of magnitude estimates of $e_n^\varepsilon(x,t)$.

Theorem 3.5. *Assume* (A1), (A2), *and* (A3). *Then for the asymptotic expansions constructed in* Theorem 3.3, *there exists a positive constant K such that*

$$\sup_{(x,t)\in[0,1]\times[0,T]} \left| e_n^\varepsilon(x,t) \right| \le K\delta^{n+1}.$$

Proof. First, we obtain an estimate on $\mathcal{G}^\varepsilon e_{n+3}^\varepsilon(x,t)$. Then we derive the desired order estimate. Since $p^\varepsilon(x,t)$ is a solution of (2.1), $\mathcal{G}^\varepsilon p^\varepsilon(x,t) = 0$. Therefore,

$$\mathcal{G}^\varepsilon e_{n+3}^\varepsilon(x,t) = -\mathcal{G}^\varepsilon \Phi_{n+3}^\varepsilon(x,t) - \mathcal{G}^\varepsilon \Psi_{n+3}^\varepsilon(x,\tau).$$

In view of (3.2), we have

$$\begin{aligned} \mathcal{G}^\varepsilon \Phi_{n+3}^\varepsilon(x,t) &= \sum_{k=0}^{n+3} \delta^k \Big(-\frac{\partial \varphi_k}{\partial t} + \delta \mathcal{D}\varphi_k + \mathcal{L}\varphi_k + \varphi_k \widehat{Q}(t) + \frac{\varphi_k \widetilde{Q}(t)}{\delta^2} \Big) \\ &= -\frac{\partial \varphi_0}{\partial t} + \mathcal{L}\varphi_0 + \varphi_0 \widehat{Q}(t) + \frac{1}{\delta^2}\varphi_0 \widetilde{Q}(t) + \frac{1}{\delta}\varphi_1 \widetilde{Q}(t) + \varphi_2 \widetilde{Q}(t) \\ &\quad + \sum_{k=1}^{n+1} \delta^k \Big(-\frac{\partial \varphi_k}{\partial t} + \mathcal{L}\varphi_k + \varphi_k \widehat{Q}(t) + \mathcal{D}\varphi_{k-1} + \varphi_{k+2}\widetilde{Q}(t) \Big) \\ &\quad + \sum_{k=n+2}^{n+3} \delta^k \Big(-\frac{\partial \varphi_k}{\partial t} + \mathcal{L}\varphi_k + \varphi_k \widehat{Q}(t) \Big) + \sum_{k=n+2}^{n+4} \delta^k \mathcal{D}\varphi_{k-1} \\ &= \sum_{k=n+2}^{n+3} \delta^k \Big(-\frac{\partial \varphi_k}{\partial t} + \mathcal{L}\varphi_k + \varphi_k \widehat{Q}(t) \Big) + \sum_{k=n+2}^{n+4} \delta^k \mathcal{D}\varphi_{k-1}. \end{aligned}$$

The smoothness of $\varphi_k(x,t)$ then yields

$$|\mathcal{G}^\varepsilon \Phi_{n+3}^\varepsilon(x,t)| \le K\delta^{n+2}, \quad (x,t) \in [0,1]\times[0,T].$$

Using the definition $\tau = t/\delta^2$,

$$\frac{d\psi_k(x,\tau)}{dt} = \frac{1}{\delta^2}\frac{d\psi_k(x,\tau)}{d\tau},$$

which yields

$$
\begin{aligned}
\mathcal{G}^\varepsilon \Psi^\varepsilon_{n+3}(x,\tau) &= \sum_{k=0}^{n+3} \delta^k \Big(-\frac{1}{\delta^2}\frac{\partial \psi_k}{\partial \tau} + \delta \mathcal{D}\psi_k + \mathcal{L}\psi_k + \psi_k \widehat{Q}(t) + \frac{\psi_k \widetilde{Q}(t)}{\delta^2} \Big) \\
&= -\sum_{k=0}^{n+3} \delta^{k-2} \big(\psi_k \widetilde{Q}(0) + r_k(x,\tau) \big) \\
&\quad + \sum_{k=0}^{n+3} \delta^k \Big(\delta \mathcal{D}\psi_k + \mathcal{L}\psi_k(x,\tau) + \psi_k \widehat{Q}(t) + \frac{\psi_k \widetilde{Q}(t)}{\delta^2} \Big),
\end{aligned}
$$

where we set $r_0(x,\tau) = r_1(x,\tau) = 0$. Note that

$$
\begin{aligned}
\sum_{k=0}^{n+3} \delta^{k-2} \ \big(\psi_k(x,\tau)\widetilde{Q}(0) + r_k(x,\tau) \big) &= \sum_{k=0}^{n+3} \delta^{k-2} \psi_k \widetilde{Q}(0) + \mathcal{L}\psi_0 + \psi_0 \widehat{Q}(t) + \tau \psi_0 \widetilde{Q}^{(1)}(0) \\
&\quad + \sum_{k=3}^{n+3} \delta^{k-2} \Big(\mathcal{D}\psi_{k-3} + \mathcal{L}\psi_{k-2} + \psi_{k-2}\widehat{Q}(t) + \sum_{j=1}^{[k/2]} \frac{\tau^j}{j!} \psi_{k-2j} \widetilde{Q}^{(j)}(0) \Big) \\
&= \sum_{k=0}^{n+1} \delta^k \Big(\mathcal{L}\psi_k + \psi_k \widehat{Q}(t) \Big) + \sum_{k=0}^{n} \delta^{k+1} \mathcal{D}\psi_k + \sum_{k=0}^{n+3} \delta^{k-2} \sum_{j=0}^{[k/2]} \frac{\tau^j}{j!} \psi_{k-2j} \widetilde{Q}^{(j)}(0).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{G}^\varepsilon \Psi^\varepsilon_{n+3}(x,\tau) &= \delta^{n+2}\mathcal{L}\psi_{n+2} + \delta^{n+2}\psi_{n+2}\widehat{Q}(t) + \delta^{n+3}\mathcal{L}\psi_{n+3} + \delta^{n+3}\psi_{n+3}\widehat{Q}(t) \\
&\quad + \delta^{n+2}\mathcal{D}\psi_{n+1} + \delta^{n+3}\mathcal{D}\psi_{n+2} + \delta^{n+4}\mathcal{D}\psi_{n+3} \\
&\quad - \sum_{k=0}^{n+3} \delta^{k-2} \sum_{j=0}^{[k/2]} \frac{\tau^j}{j!} \psi_{k-2j} \widetilde{Q}^{(j)}(0) + \sum_{k=0}^{n+3} \delta^{k-2} \psi_k \widetilde{Q}(t).
\end{aligned} \tag{3.28}
$$

By the smoothness of $\psi_{n+1}(x,\tau)$, $\psi_{n+2}(x,\tau)$, and $\psi_{n+3}(x,\tau)$,

$$
\begin{aligned}
&\Big| \delta^{n+2}\mathcal{L}\psi_{n+2} + \delta^{n+2}\psi_{n+2}\widehat{Q}(t) + \delta^{n+3}\mathcal{L}\psi_{n+3} + \delta^{n+3}\psi_{n+3}\widehat{Q}(t) \Big| \le K\delta^{n+2}, \\
&\Big| \delta^{n+2}\mathcal{D}\psi_{n+1} + \delta^{n+3}\mathcal{D}\psi_{n+2} + \delta^{n+4}\mathcal{D}\psi_{n+3} \Big| \le K\delta^{n+2}.
\end{aligned} \tag{3.29}
$$

By virtue of (3.4) and (3.5),

$$
\Big| \widetilde{Q}(t) - \sum_{j=0}^{k} \frac{t^j}{j!} \widetilde{Q}^{(j)}(0) \Big| = |R^{(k)}(t)| \le K t^{k+1}, \quad t \in [0,T].
$$

Using this estimate and the exponential decay property of $\psi_k(x,\tau)$, we obtain

$$
\begin{aligned}
&\Big| \sum_{k=0}^{n+3} \delta^{k-2} \psi_k \widetilde{Q}(t) - \sum_{k=0}^{n+3} \delta^{k-2} \sum_{j=0}^{[k/2]} \frac{\tau^j}{j!} \psi_{k-2j} \widetilde{Q}^{(j)}(0) \Big| \\
&= \Big| \sum_{k=0}^{n+3} \delta^{k-2} \psi_k \Big(\widetilde{Q}(t) - \sum_{j=0}^{[(n+3-k)/2]} \frac{t^j}{j!} \widetilde{Q}^{(j)}(0) \Big) \Big| \\
&\le K \sum_{k=0}^{n+3} \delta^{k-2} (\delta^2 \tau)^{[(n+3-k)/2]+1} \exp(-\kappa_0 \tau) \\
&\le K \delta^{n+2}.
\end{aligned} \tag{3.30}
$$

It follows from (3.28), (3.29), and (3.30) that

$$\left|\mathcal{G}^\varepsilon \Psi^\varepsilon_{n+3}(x,\tau)\right| \le K\delta^{n+2}.$$

Putting these together with the estimates on $\mathcal{G}^\varepsilon \Phi^\varepsilon_{n+3}(x,t)$, we have shown that

$$|\mathcal{G}^\varepsilon e^\varepsilon_{n+3}(x,t)| \le K\delta^{n+2}, \quad \text{for any} \quad (x,t) \in [0,1]\times[0,T].$$

Note that the initial condition $e^\varepsilon_{n+3}(x,0) = 0$ for $x \in [0,1]$. By virtue of Lemma 3.4,

$$\sup_{(x,t)\in[0,1]\times[0,T]} |e^\varepsilon_{n+3}(x,t)| \le K\delta^{n+1}.$$

Finally,

$$e^\varepsilon_{n+3}(x,t) = e^\varepsilon_n(x,t) - \sum_{k=n+1}^{n+3} \delta^k \Big(\varphi_k(x,t) + \psi_k(x,\tau)\Big). \tag{3.31}$$

The boundedness of $\varphi_k(x,t)$ and $\psi_k(x,\tau)$ yields that

$$\sup_{(x,t)\in[0,1]\times[0,T]} \sum_{k=n+1}^{n+3} \delta^k \Big|\varphi_k(x,t) + \psi_k(x,\tau)\Big| \le K\delta^{n+1}.$$

Substituting this into (3.31), we obtain the order estimate in terms of δ. Finally, note that $\varepsilon = \delta^2$. The desired result follows. □

4. ASYMPTOTIC EXPANSION: $\delta = \varepsilon^2$

In this section, we consider the case $\varepsilon = \delta^\gamma$ satisfying $1/\gamma > 1$. That is, δ goes to 0 much faster than ε. There are many different choices. To fix the notation for discussion, we work with $\gamma = 1/2$ that is, $\delta = \varepsilon^2$. The other cases can be handled in the same way.

We use the same method as in Section 3. However, some important modifications are required. We will only discuss such modifications and skip the details. First, instead of (3.1), we now seek asymptotic expansions of $p^\varepsilon(x,t)$ of the form

$$\Phi^\varepsilon_n(x,t) + \Psi^\varepsilon_n(x,\tau) = \sum_{k=0}^{n} \varepsilon^k \varphi_k(x,t) + \sum_{k=0}^{n} \varepsilon^k \psi_k(x,\tau), \tag{4.1}$$

where $\tau = t/\varepsilon$ is the stretched-time variables. Moreover, we only need four extra terms $\varphi_k(x,t)$, $\psi_k(x,\tau)$ with $k = n+1, n+2$. Substituting the outer expansions $\sum_{k=0}^{n+2} \varepsilon^k \varphi_k(x,t)$ into (2.2) and comparing coefficients of like powers of ε^k for $k =$

$-1, \ldots, n+1$, we obtain

(4.2)
$$\begin{aligned}
&\varphi_0(x,t)\widetilde{Q}(t) = 0,\\
&\varphi_1(x,t)\widetilde{Q}(t) = \frac{\partial \varphi_0(x,t)}{\partial t} - \mathcal{L}\varphi_0(x,t) - \varphi_0(x,t)\widehat{Q}(t),\\
&\varphi_2(x,t)\widetilde{Q}(t) = \frac{\partial \varphi_1(x,t)}{\partial t} - \mathcal{L}\varphi_1(x,t) - \varphi_1(x,t)\widehat{Q}(t),\\
&\varphi_{k+1}(x,t)\widetilde{Q}(t) = \frac{\partial \varphi_k(x,t)}{\partial t} - \mathcal{L}\varphi_k(x,t) - \varphi_k(x,t)\widehat{Q}(t) - \mathcal{D}\varphi_{k-2}(x,t),\\
&\hspace{24em} k = 2, \ldots, n+1.
\end{aligned}$$

Likewise, substituting $\Psi^{\varepsilon}_{n+2}(x,\tau) = \sum_{k=0}^{n+2} \varepsilon^k \psi_k(x,\tau)$ into (2.2) and using the Taylor expansion $\sum_{j=0}^{n+1} \frac{(\varepsilon\tau)^j}{j!} \widetilde{Q}^{(j)}(0)$ as an approximation for $\widetilde{Q}(t)$, we obtain

$$\sum_{k=0}^{n+2} \varepsilon^{k+2}\mathcal{D}\varphi_k + \sum_{k=0}^{n+2} \varepsilon^k \Big(\mathcal{L}\psi_k + \psi_k\widehat{Q}(t)\Big) = \sum_{k=0}^{n+2} \varepsilon^{k-1}\Big(\frac{\partial \psi_k}{\partial \tau} - \sum_{j=0}^{n+1} \frac{(\varepsilon\tau)^j}{j!}\psi_k \widetilde{Q}^{(j)}(0)\Big),$$

i.e.,

$$\begin{aligned}
\sum_{k=0}^{n+2} \varepsilon^{k+2}\mathcal{D}\varphi_k + \sum_{k=0}^{n+2} \varepsilon^k \Big(\mathcal{L}\psi_k + \psi_k\widehat{Q}(t)\Big) &= \sum_{k=0}^{n+2} \varepsilon^{k-1}\Big(\frac{\partial \psi_k}{\partial \tau} - \psi_k \widetilde{Q}(0)\Big)\\
&\quad - \sum_{k=0}^{n+2} \varepsilon^{k-1} \sum_{j=1}^{n+1} \frac{(\varepsilon\tau)^j}{j!}\psi_k \widetilde{Q}^{(j)}(0),
\end{aligned}$$

Then comparing the coefficients of ε^k for $k = -1, \ldots, n+1$ leads to

$$\begin{aligned}
&\frac{\partial \psi_0(x,\tau)}{\partial \tau} = \psi_0(x,\tau)\widetilde{Q}(0),\\
&\frac{\partial \psi_k(x,\tau)}{\partial \tau} = \psi_k(x,\tau)\widetilde{Q}(0) + r_k(x,\tau), \quad k = 1, \ldots, n+2,
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
&r_1(x,\tau) = \mathcal{L}\psi_0 + \psi_0\widehat{Q}(t) + \tau\psi_0\widetilde{Q}^{(1)}(0),\\
&r_2(x,\tau) = \mathcal{L}\psi_1 + \psi_1\widehat{Q}(t) + \frac{\tau^2}{2!}\psi_0\widetilde{Q}^{(2)}(0) + \tau\psi_1\widetilde{Q}^{(1)(0)},\\
&r_k(x,\tau) = \mathcal{D}\psi_{k-3} + \mathcal{L}\psi_{k-1} + \psi_{k-1}\widehat{Q}(t) + \sum_{j=1}^{k} \frac{\tau^j}{j!}\psi_{k-j}\widetilde{Q}^{(j)}(0),\ k = 3, \ldots, n+2.
\end{aligned}$$

To ensure the match of the initial conditions, we choose

$$\begin{aligned}
&\varphi_0(x,0) + \psi_0(x,0) = g(x),\\
&\varphi_k(x,0) + \psi_k(x,0) = 0, \quad k = 1, \ldots, n+2.
\end{aligned} \tag{4.4}$$

Similar to the previous case, we then proceed to construct the asymptotic expansions. In view of (4.3),

(4.5)
$$\begin{aligned}
&\psi_0(x,\tau) = \psi_0(x,0)\exp(\widetilde{Q}(0)\tau),\\
&\psi_k(x,\tau) = \psi_k(x,0)\exp(\widetilde{Q}(0)\tau) + \int_0^{\tau} r_k(x,s)\exp(\widetilde{Q}(0)(\tau - s))ds,\ k = 1, \ldots, n+2.
\end{aligned}$$

Using the first two equations in (4.2), we have $\varphi_0(x,t) = \beta_0(x,t)\widetilde{\nu}(t)$, where $\beta_0(x,t)$ is a real-valued function satisfying

$$\begin{cases} \dfrac{\partial \beta_0(x,t)}{\partial t} - \mathcal{L}\big(\beta_0(x,t)\widetilde{\nu}(t)\big) = 0, \\ \beta_0(x,0) = g(x)\mathbb{1}. \end{cases} \tag{4.6}$$

Then $\psi_0(x,0) = g(x) - \varphi_0(x,0)$ and $\psi_0(x,\tau)$ is specified by (4.5). Moreover, we can also prove that $\psi_0(x,\tau)$ and its partial derivative in x decay exponentially fast. Next,

$$\varphi_1(x,t) = \beta_1(x,t)\widetilde{\nu}(t) + \widehat{\varphi}_1(x,t), \tag{4.7}$$

where

$$\widehat{\varphi}_1(x,t) = \widehat{d}_0(x,t)\widetilde{Q}_c'(t)\Big(\widetilde{Q}_c(t)\widetilde{Q}_c'(t)\Big)^{-1}, \tag{4.8}$$

with

$$d_0(x,t) = \frac{\partial \varphi_0(x,t)}{\partial t} - \mathcal{L}\varphi_0(x,t) - \varphi_0(x,t)\widehat{Q}(t) \quad \text{and} \quad \widehat{d}_0(x,t) = \Big(0 \quad d_0(x,t)\Big).$$

Moreover, $\beta_1(x,t)$ is a real-valued function satisfying

$$\begin{cases} \dfrac{\partial \beta_1(x,t)}{\partial t} - \mathcal{L}\big(\beta_1(x,t)\widetilde{\nu}(t)\big) = -\dfrac{\partial \widehat{\varphi}_1(x,t)}{\partial t}\mathbb{1} + \mathcal{L}\widehat{\varphi}_1(x,t)\mathbb{1}, \\ \beta_1(x,0) = \displaystyle\int_0^\infty r_1(x,s)\mathbb{1}ds. \end{cases} \tag{4.9}$$

Then $\psi_1(x,0) = -\varphi_1(x,0)$ and $\psi_1(x,\tau)$ is specified by (4.5). Moreover, we can also prove that $\psi_1(x,\tau)$ and its partial derivatives in x decay exponentially fast. Proceeding in a similar way, for $k = 2, \ldots, n+1$,

$$\varphi_k(x,t) = \beta_k(x,t)\widetilde{\nu}(t) + \widehat{\varphi}_k(x,t), \tag{4.10}$$

where

$$\widehat{\varphi}_k(x,t) = \widehat{d}_{k-1}(x,t)\widetilde{Q}_c'(t)\Big(\widetilde{Q}_c(t)\widetilde{Q}_c'(t)\Big)^{-1}, \tag{4.11}$$

with

$$\begin{aligned} d_1(x,t) &= \frac{\partial \varphi_1(x,t)}{\partial t} - \mathcal{L}\varphi_1(x,t) - \varphi_1(x,t)\widehat{Q}(t), \\ d_{k-1}(x,t) &= \frac{\partial \varphi_{k-1}(x,t)}{\partial t} - \mathcal{L}\varphi_{k-1}(x,t) - \varphi_{k-1}(x,t)\widehat{Q}(t) - \mathcal{D}\varphi_{k-3}(x,t), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 3, \ldots, n+1, \\ \widehat{d}_{k-1}(x,t) &= \Big(0 \quad d_{k-1}(x,t)\Big). \end{aligned}$$

Moreover, $\beta_k(x,t)$ is a real-valued function satisfying

$$\begin{cases} \dfrac{\partial \beta_k(x,t)}{\partial t} - \mathcal{L}\big(\beta_k(x,t)\widetilde{\nu}(t)\big) = -\dfrac{\partial \widehat{\varphi}_k(x,t)}{\partial t}\mathbb{1} + \mathcal{L}\widehat{\varphi}_k(x,t)\mathbb{1} + \mathcal{D}\varphi_{k-2}(x,t)\mathbb{1}, \\ \beta_k(x,0) = \displaystyle\int_0^\infty r_k(x,s)\mathbb{1}ds. \end{cases} \tag{4.12}$$

Then $\psi_k(x,0) = -\varphi_k(x,0)$ and $\psi_k(x,\tau)$ is specified by (4.5). Using the same argument as in 3.2, we can also prove that $\psi_k(x,\tau)$ and its partial derivatives in x decay exponentially fast. Finally, we choose

$$\varphi_k(x,t) = \widehat{d}_{k-1}(x,t)\widetilde{Q}'_c(t)\Big(\widetilde{Q}_c(t)\widetilde{Q}'_c(t)\Big)^{-1} + \Big(\int_0^\infty r_k(x,s)\mathbb{1}ds\Big)\widetilde{\nu}(t),\ k = n+1, n+2,$$

where

$$d_{k-1}(x,t) = \frac{\partial\varphi_{k-1}(x,t)}{\partial t} - \mathcal{L}\varphi_{k-1}(x,t) - \varphi_{k-1}(x,t)\widehat{Q}(t) - \mathcal{D}\varphi_{k-3}(x,t),$$

and $\widehat{d}_{k-1}(x,t) = \Big(0 \quad d_{k-1}(x,t)\Big)$. For $k = n+1, n+2$, $\psi_k(x,0) = -\varphi_k(x,0)$ and $\psi_k(x,\tau)$ is specified by (4.5). With $\varphi_{n+1}(x,t)$ and $\varphi_{n+2}(x,t)$ defined above, we can also prove that $\psi_{n+1}(x,\tau)$, $\psi_{n+2}(x,\tau)$ and its partial derivatives in x decay exponentially fast. We summarize the results in the following theorem. It provides a detailed construction of the asymptotic series as well as the error bounds. The details are omitted for brevity.

Theorem 4.1. *Under conditions* (A1), (A2), *and* (A3), *we can construct sequences* $\{\varphi_k(x,t) : k = 0,\dots,n\}$ *and* $\{\psi_k(x,t) : k = 0,\dots,n\}$ *as follows.*

(a) $\varphi_0(x,t) = \beta_0(x,t)\widetilde{\nu}(t)$ *with* $\beta_0(x,t)$ *is given by* (4.6)*;* $\psi_0(x,\tau)$ *is given by* (4.5) *with* $\psi_0(x,0) = g(x) - \varphi_0(x,0)$.

(b) $\varphi_1(x,t) = \beta_1(x,t)\widetilde{\nu}(t) + \widehat{\varphi}_1(x,t)$, *where* $\beta_1(x,t)$ *is given by* (4.9) *and* $\widehat{\varphi}_1(x,t)$ *is given by* (4.8)*;* $\psi_1(x,\tau)$ *is given by* (4.5) *with* $\psi_1(x,0) = -\varphi_1(x,0)$.

(c) *For* $k = 2,\dots,n$, $\varphi_k(x,t) = \beta_k(x,t)\widetilde{\nu}(t) + \widehat{\varphi}_k(x,t)$, *where* $\beta_k(x,t)$ *is given by* (4.12) *and* $\widehat{\varphi}_k(x,t)$ *is given by* (4.11)*;* $\psi_k(x,\tau)$ *is given by* (4.5) *with* $\psi_k(x,0) = -\varphi_k(x,0)$.

(d) $\varphi_k(x,t) \in C^{2(n+3-k),n+3-k}([0,1]\times[0,T])$.

(e) $\psi_k(x,\tau)$ *decays exponentially fast in that*

$$\sup_{x\in[0,1]}\left|\frac{\partial^j\psi_k(x,\tau)}{\partial x^j}\right| \le K\exp(-\kappa_0\tau),\ j = 0,1,\dots,2(n+3-k).$$

(f) *The following error bound holds:*

$$\sup_{(x,t)\in[0,1]\times[0,T]}\left|p^\varepsilon(x,t) - \sum_{k=0}^n \varepsilon^k\varphi_k(x,t) - \sum_{k=0}^n \varepsilon^k\psi_k\Big(x,\frac{t}{\varepsilon}\Big)\right| = O(\varepsilon^{n+1}).$$

To illustrate the utility of the results, let us take $\delta = \varepsilon^2$ and revisit the optimization problem for $J^{\varepsilon,\varepsilon^2}(\theta)$ given by

$$J^{\varepsilon,\varepsilon^2}(\theta) = E\int_0^T G(x^\varepsilon(t),\alpha^\varepsilon(t),t,\theta)dt = \int_0^T\int\sum_{i=1}^m G(x,i,t,\theta)p_i^\varepsilon(x,t)dxdt, \tag{4.13}$$

for a given $T\in(0,\infty)$, where θ is a parameter, $p_i^\varepsilon(x,t)$ is the probability density of $(x^\varepsilon(t),\alpha^\varepsilon(t))$ with a given initial probability density, and $G(\cdot)$ is a suitable function.

We have shown that, under appropriate conditions,

$$p^{\varepsilon}(x,t)=\sum_{k=0}^{n}\varepsilon^k\varphi_k(x,t)+\sum_{k=0}^{n}\varepsilon^k\psi_k\Big(x,\frac{t}{\varepsilon}\Big)+O(\varepsilon^{n+1}).$$

Thus, we obtain asymptotic expansions for $J^{\varepsilon,\varepsilon^2}(\theta)$ as follows

$$J^{\varepsilon,\varepsilon^2}(\theta)=\sum_{k=0}^{n}\varepsilon^k J_k^{\varphi}(\theta)+\sum_{k=0}^{n}\varepsilon^k J_k^{\psi}(\theta)+O(\varepsilon^{n+1}),$$

where

$$\begin{aligned}J_k^{\varphi}(\theta)&=\int_0^T\int\sum_{i=1}^{m}G(x,i,t,\theta)\varphi_{k,i}(x,t)dxdt,\\ J_k^{\psi}(\theta)&=\int_0^T\int\sum_{i=1}^{m}G(x,i,t,\theta)\psi_{k,i}(x,t)dxdt.\end{aligned}$$

As a result, we get the leading term in the approximation of the objective function as

$$J_0^{\varphi}(\theta)=\int_0^T\int\sum_{i=1}^{m}G(x,i,t,\theta)\varphi_{0,i}(x,t)dxdt.$$

Therefore, in lieu of (4.13), we can use $J_0^{\varphi}(\theta)$ for an approximation with error $O(\varepsilon)$. The resulting problem then is much simpler than the original one and is easier to analyze. Note that the expansions obtained shows not only $J^{\varepsilon,\varepsilon^2}(\theta)\to J_0^{\varphi}(\theta)$, but also the rate of convergence.

5. FURTHER REMARKS

In this paper, we have developed asymptotic expansions for probability densities of multi-scale switching diffusions with rapid switching and slow diffusion. For definiteness, to specify $\varepsilon\ll\delta$ and $\delta\ll\varepsilon$, we used $\varepsilon=\delta^2$ and $\delta=\varepsilon^2$ for convenience. Our approach works for other scalings as well. Our results indicate that the limiting behavior of the underlying system depends on how fast small parameters ε and δ go to zero. Our work focuses on the case of scalar systems, although the main methodology of this paper can be directly carried over to multi-dimensional cases, with perhaps more complicated notation. Similar to the study on pure jump processes in [17], one may study alternative forms of switching processes with multiple weakly irreducible classes, including absorbing states and transient states. A worthwhile effort is to consider singularly perturbed multi-scale systems in which the processes can jump from a state to another state; see [6].

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