MONOTONE-ITERATIVE TECHNIQUES FOR MILD SOLUTIONS OF THE INITIAL VALUE PROBLEM FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES

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ABSTRACT. The main aim of the paper is to suggest some algorithms to approximately solve the initial value problem for scalar nonlinear Caputo fractional differential equations with non-instantaneous impulses. The impulses start abruptly at some points and their action continue on given finite intervals. We study the case when the right hand side of the equations are monotonic functions. Several types of mild lower and mild upper solutions to the problem are defined and used in the algorithms. The convergence of the successive approximations is established. A generalization of the logistic equation is given to illustrate the results.

AMS (MOS) Subject Classification. 34A37, 34E08

1. Introduction

In the real world there are many processes and phenomena that are characterized by rapid changes in their state and these changes are adequately modeled by impulses. In the literature there are two popular types of impulses:

- *instantaneous impulses* – the duration of these changes is relatively short compared to the overall duration of the whole process. The model is given by impulsive differential equations (see, for example, the monograph [20] and the cited references therein);
- *non-instantaneous impulses* – an impulsive action, which starts at an arbitrary fixed point and remains active on a finite time interval (see, for example, [13], [16], [22], [28]).

This paper considers an initial value problem for a nonlinear scalar non-instantaneous impulsive Caputo fractional differential equation on a closed interval. The monotone

iterative technique combined with the method of lower and upper solutions is applied to approximately find the solution of the given problem. Several types of lower and upper solutions are presented. We study the case when the right hand side of the equations are monotone. Several procedures for constructing two monotone functional sequences are given. The elements of these sequences are solutions of suitably chosen initial value problems for scalar linear non-instantaneous impulsive fractional differential equations (for which there are explicit formulas). We prove that both sequences converge and their limits are minimal and maximal solutions of the problem. A non-instantaneous impulsive fractional generalization of the logistic equation is given to illustrate the procedure.

We note that iterative techniques combined with lower and upper solutions are applied in the literature to approximately solve various problems in ordinary differential equations [19], for second order periodic boundary value problems [10], for differential equations with maxima [3], [14], for difference equations with maxima [7], for impulsive differential equations [9], [12], for impulsive integro-differential equations [15], for impulsive differential equations with supremum [17], for differential equations of mixed type [18], for Riemann-Liouville fractional differential equations [8], [25], [27], for Caputo impulsive fractional differential equations [11] and for non-instantaneous impulsive differential equation [6].

2. Preliminary and auxiliary results

In this paper we will assume two increasing finite sequences of points $\{t_i\}_{i=0}^p$ and $\{s_i\}_{i=0}^p$ are given such that $t_0 = 0 < s_i < t_{i+1} < s_{i+1}$, $i = 0, 1, 2, \ldots, p-1$, and $T = s_p$, p is a natural number.

Remark 1. The intervals $(s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, p-1$ are called intervals of non-instantaneous impulses.

Consider the initial value problem (IVP) for the nonlinear *non-instantaneous impulsive Caputo-type fractional differential equation* (NIFrDE)

$$^C D_0^q x(t) = F(t, x) \text{ for } t \in (t_k, s_k], \ k = 0, 1, \ldots, p,$$

$$x(t) = x(s_k - 0) + \Phi_k(t, x(t), x(s_k - 0)),$$

(2.1)
$$\text{for } t \in (s_k, t_{k+1}], \ k = 0, 1, \ldots, p-1,$$

$$x(0) = x_0,$$

where $q \in (0, 1)$, $x, x_0 \in \mathbb{R}$, $F(t, x) = f(t, x) + g(t, x)$, $f, g : \cup_{k=0}^p [t_k, s_k] \times \mathbb{R} \to \mathbb{R}$, $\Phi(t, x, y) = \phi_k(t, x, y) + \psi_k(t, x, y)$, $\phi_k, \psi_k : [s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(k = 0, 1, 2, \ldots, p-1)$, and the fractional derivative $^C D_0^q m(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} m'(s) ds$, $t \geq 0$.

Note the functions $f(t, x)$ and $g(t, x)$ will have different properties, which will be defined later. The same can be said concerning the functions $\phi_k(t, x, y)$ and $\psi_k(t, x, y)$.

Remark 2. If $t_{k+1} = s_k$, $k = 0, 1, 2, \ldots, p$, then the IVP for NIFrDE (2.1) reduces to an IVP for impulsive fractional differential equations.

Consider the corresponding IVP for FrDE

$$(2.2) \qquad {}^{C}D_\tau^q x(t) = F(t, x) \quad \text{for } t \in [\tau, s_k] \quad \text{with } x(\tau) = \tilde{x}_0,$$

where $\tau \in [t_k, s_k)$, $k = 0, 1, 2, \ldots, p$.

The IVP for FrDE (2.2) is equivalent to the following Volterra integral equation

$$(2.3) \qquad x(t) = \tilde{x}_0 + \frac{1}{\Gamma(q)} \int_\tau^t (t-s)^{q-1} F(s, x(s)) ds \quad \text{for } t \in [\tau, s_k].$$

Also consider the special case of a fractional differential equation where the right side does not depend on the unknown function, i.e.

$${}^{C}D_0^q x(t) = G(t) \text{ for } t \in (t_k, s_k], \quad k = 0, 1, \ldots, p,$$

$$(2.4) \qquad x(t) = x(s_k - 0) + g_k(t), \text{ for } t \in (s_k, t_{k+1}], \quad k = 0, 1, \ldots, p-1,$$

$$x(0) = x_0,$$

where $q \in (0, 1)$, $x, x_0 \in \mathbb{R}$, $G : \cup_{k=0}^{p}[t_k, s_k] \to \mathbb{R}$, $g_k(t) : [s_k, t_{k+1}] \to \mathbb{R}$, ($k = 0, 1, 2, \ldots, p-1$).

We introduce the following classes of functions

$$NPC([0, T]) = \{u : [0, T] \to \mathbb{R} : u \in C^1(\cup_{k=0}^{p}[t_k, s_k], \mathbb{R}) :$$

$$u(s_k) = u(s_k - 0) = \lim_{t \uparrow s_k} u(t) < \infty,$$

$$u'(s_k) = \lim_{t \uparrow s_k} u'(t) < \infty, \quad k = 0, 1, 2, \ldots, p+1\},$$

$$u(s_k + 0) = \lim_{t \downarrow s_k} u(t) < \infty, \quad k = 0, 1, 2, \ldots, p\},$$

$$PC([0, T]) = \{u : [0, T] \to \mathbb{R} : u \in NPC([0, T]), u \in C(\cup_{k=0}^{p-1}(s_k, t_{k+1}] \mathbb{R})\}.$$

There are two points of view of the interpretation of the solutions to (2.1).

A. Some authors emphasize the presence of Caputo fractional derivative ${}^{C}D_0^q x(t)$ in (2.1) and the memory property of the fractional derivative. They assume the function $F(t, x)$ is defined on the whole interval of consideration $[0, T]$ including the intervals without non-instantaneous impulses $(s_k, t_{k+1}]$, $k = 0, 1, \ldots, p-1$, and provide an integral presentation to (2.1) with the integral $\int_0^t (t-s)^{q-1} F(s, x(s)) ds$ for $t \in (t_k, s_k]$, $k = 0, 1, \ldots, p$. In [29] a formula for the solution of IVP for NIFrDE (2.4) is given in the case when the impulsive condition is given by $x(t) = g_k(t)$, $t \in (s_k, t_{k+1}]$, $k = 0, 1, \ldots, p-1$:

$$(2.5) \qquad x(t) = \begin{cases} g_k(t), & t \in (s_k, t_{k+1}], \quad k = 0, 1, \ldots, p-1, \\ g_k(t_k) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} G(s) ds - \frac{1}{\Gamma(q)} \int_0^{t_k} (t_k - s)^{q-1} G(s) ds, \\ \qquad t \in (t_k, s_k], \quad k = 0, 1, 2, \ldots p. \end{cases}$$

The function given by (2.5) has a Caputo fractional derivative ${}^{C}D_0^q x(t)$ but the function $G(t)$ has to be at least integrable on the whole interval $[0,T]$. Note formula (2.5) defines a solution depending on the initial value x_0 only on the interval $[0, s_0]$. To obtain a solution depending on the initial value x_0 on the whole interval $[0,T]$ and following the above approach we obtain the solution of the IVP for NIFrDE (2.4) as

(2.6)
$$x(t) = \begin{cases} x(s_k - 0) + g_k(t), & t \in (s_k, t_{k+1}], \quad k = 0, 1, \ldots, p-1, \\ x(s_k - 0) + g_k(t_k) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}G(s)ds - \frac{1}{\Gamma(q)}\int_0^{t_k}(t_k - s)^{q-1}G(s)ds, \\ \quad t \in (t_k, s_k], \quad k = 0, 1, 2, \ldots p. \end{cases}$$

B. Some authors base it on the differential equation in (2.1) which is satisfied only on the intervals (t_k, s_k) without non-instantaneous impulses. Note the right hand side $F(t,x)$ does not have to be defined on the intervals of non-instantaneous impulses. Note there is a fractional derivative ${}^{C}D_0^q x(t)$ which has a memory (i.e. the integral $\int_0^t (t-s)^{q-1}x'(s)ds$ has to exist for any $t \geq 0$). The solution $x(t)$ is defined on the intervals of non-instantaneous impulses by the second equation of (2.1) and in the general case it need not be differentiable on $(s_k, t_{k+1}]$. To avoid the above some authors consider the fractional derivative ${}^{C}D_{t_k}^q x(t)$ of the unknown solution only on the intervals (t_k, s_k) by changing the lower limit of the Caputo derivative at 0 to t_k (see, for example, [1], [2], [4], [5]).

In applications approach A (and (2.6)) and the definition of the function $F(t)$ on the intervals $(s_k, t_{k+1}]$ of non-instantaneous impulses has a huge influence on the formula (2.6) (in spite of the fact that it is not given in the statement of problem (2.4)).

Example 1. Let $q = 0.5$, $t_0 = 0$, $s_0 = 1$, $t_1 = 2$, $s_1 = 3 = T$.

Case 1. Let $G(t) \equiv 1$, $t \in [0,3]$, and $g_0(t) = t$ for $t \in [1,2]$.

Consider the scalar IVP for the NIFrDE (2.4). According to Eq. (2.6) we obtain the solution

(2.7)
$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}ds = x_0 + \frac{\sqrt{t}}{\Gamma(1.5)}, & t \in (0,1], \\ x_0 + \frac{1}{\Gamma(1.5)} + t, & t \in (1,2], \\ x_0 + \frac{1}{\Gamma(1.5)} + 2 + \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}ds - \frac{1}{\Gamma(q)}\int_0^2(2-s)^{q-1}ds, & t \in (2,3]. \end{cases}$$

Case 2. Let $G(t) \equiv 1$ only on the intervals without non-instantaneous impulses, i.e. on $(0,1] \cup (2,3]$, and $g_0(t) = t$ for $t \in [1,2]$.

Define two continuous functions $G_1(t) \equiv 1$, $t \in [0,3]$ and

$$G_2(t) = \begin{cases} 1, & t \in [0,1] \cup (2,3], \\ 4(t-1.5)^2, & t \in (1,2]. \end{cases}$$

Note $G(t) = G_1(t) = G_2(t)$ for $t \in [0,1] \cup [2,3]$. Therefore, both functions G_1 and G_2 could be used in the IVP for NIFrDE (2.4). Apply Eq. (2.6) and we obtain two functions

$$(2.8) \quad x_1(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{-0.5} ds = x_0 + \frac{\sqrt{t}}{\Gamma(1.5)}, & t \in [0,1], \\ x_0 + \frac{1}{\Gamma(1.5)} + t, & t \in (1,2], \\ x_0 + \frac{1}{\Gamma(1.5)} + 1 + \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} ds - \frac{1}{\Gamma(0.5)} \int_0^2 (2-s)^{-0.5} ds \\ \qquad = x_0 + 1 + \frac{\sqrt{t}+1-\sqrt{2}}{\Gamma(1.5)}, & t \in (2,3], \end{cases}$$

(2.9)

$$x_2(t) = \begin{cases} x_0 + \frac{\sqrt{t}}{\Gamma(1.5)}, & t \in [0,1], \\ x_0 + \frac{1}{\Gamma(1.5)} + t, & t \in (1,2], \\ x_0 + \frac{1}{\Gamma(1.5)} + 1 + \frac{1}{\Gamma(0.5)} \int_0^1 ((t-s)^{-0.5} - (2-s)^{-0.5}) ds \\ \qquad + \frac{4}{\Gamma(0.5)} \int_1^2 ((t-s)^{-0.5} - (2-s)^{-0.5})(s-1.5)^2 ds + \frac{1}{\Gamma(0.5)} \int_2^t (t-s)^{-0.5} ds \\ \qquad t \in (2,3]. \end{cases}$$

Note $x_1(t) = x_2(t)$ for $t \in [0,2]$, but $x_1(t) \neq x_2(t)$ for $t \in (2,3]$.

Therefore to use approach A (and Eq. (2.6)) for the solution the right hand side of the fractional differential equation has to be defined initially on the whole interval of consideration. □

Note approach A is used to derive formula (19) in [30] for the solution of (2.1) which depends on an arbitrary constant and it does not guarantee the uniqueness.

To avoid the misunderstanding mentioned in Example 4 we will define a special type of solution of (2.1) combining the equivalent integral presentation (2.3) of the fractional differential equation (2.2) with the requirement concerning the domain of the right side part $F(t,x)$ of the differential equation in the IVP for NIFrDE (2.1). We will assume this function is not necessarily defined on the intervals of non-instantaneous impulses.

Following approach B and the integral presentation (2.3) of the solution of the FRDE (2.2) we present the following definition.

Definition 1. A function $x(t) \in PC([0,T], \mathbb{R})$ is called **a mild solution** of the IVP for NIFrDE (2.1) if

- for any $t \in [t_k, s_k]$, $k = 0, 1, 2, \ldots, p$ the integral equality

(2.10)
$$x(t) = x_0+$$

$$+ \sum_{j=0}^{k-1} \frac{1}{\Gamma(q)} \int_{t_j}^{s_j} (s_j - s)^{q-1} \Big(F(s, x(s)) ds + \Phi_j(t_{j+1}, x(t_{j+1}), x(s_j - 0)) \Big)$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} F(s, x(s)) ds$$

holds;

- for any $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, p - 1$ the equality

$$x(t) = x(s_k - 0) + \Phi_k(t, x(t), x(s_k - 0))$$

holds.

Then from (2.10) we get

$$x(s_{k-1} - 0) = x_0 + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} F(s, x(s)) ds$$

$$+ \sum_{j=0}^{k-2} \Phi_j(t_{j+1}, x(t_{j+1}), x(s_j - 0)).$$

Remark 3. For any $t \in (t_k, s_k]$, $k = 0, 1, 2, \ldots, p$ the mild solution satisfies the Caputo fractional differential equation (2.2) with $\tau = t_k$ and

$$\tilde{x}_0 = x_0 + \sum_{j=0}^{k-1} \frac{1}{\Gamma(q)} \int_{t_j}^{s_j} (s_j - s)^{q-1} \Big(F(s, x(s)) ds + \Phi_j(t_{j+1}, x(t_{j+1}), x(s_j - 0)) \Big)$$

$$= x(s_{k-1} - 0) + \Phi_k(t_k, x(t_k), x(s_{k-1} - 0)).$$

The mild solution given in Definition 1 generalizes many known cases.

1. Let $q = 1$, i.e. consider the non-instantaneous impulsive differential equation with an ordinary first order derivative. Then from (2.10) we obtain the following formula for the solution

(2.11)
$$x(t) = x_0 + \sum_{j=0}^{k-1} \Phi_j(t_{j+1}, x(t_{j+1}), x(s_j - 0))$$

$$\sum_{j=0}^{k-1} \int_{t_j}^{s_j} F(s, x(s)) ds + \int_{t_k}^{t} F(s, x(s)) ds, \quad t \in (t_k, s_k], \ k = 0, 1, 2, \ldots, p.$$

2. Let $q = 1$ and $t_{k+1} = s_k$, $k = 0, 1, 2, \ldots, p$, i.e. consider the impulsive differential equation with an ordinary first order derivative. Then from (2.10) we obtain the following formula for the solution

(2.12)
$$x(t) = x_0 + \int_{t_0}^t F(s, x(s))ds + \sum_{j=0}^{k-1} \Phi_j(s_j, x(s_j + 0), x(s_j - 0)),$$

$$t \in (s_{k-1}, s_k], \quad k = 0, 1, 2, \ldots, p$$

where $s_{-1} = t_0$.

3. Let $q \in (0, 1)$ and $t_{k+1} = s_k$, $k = 0, 1, 2, \ldots, p$, i.e. consider the impulsive fractional differential equation with a Caputo fractional derivative. Then from (2.10) we obtain the following formula for the solution

(2.13)
$$x(t) = x_0 + \sum_{j=0}^{k-1} \Phi_j(s_j, x(s_j + 0), x(s_j - 0))$$
$$+ \sum_{j=0}^{k-1} \frac{1}{\Gamma(q)} \int_{s_{j-1}}^{s_j} F(s, x(s))ds + \frac{1}{\Gamma(q)} \int_{s_{j-1}}^t F(s, x(s))ds,$$

$$t \in (s_{j-1}, s_k], \ k = 0, 1, 2, \ldots, p.$$

where $s_{-1} = t_0$.

Note in both Case 1 and Case 3 for any $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, p - 1$ the equality

$$x(t) = x(s_k - 0) + \Phi_k(t, x(t), x(s_k - 0))$$

holds.

We will apply Definition 1 to find the solution in Example 1.

Example 2. Let $q = 0.5$, $t_0 = 0$, $s_0 = 1$, $t_1 = 2$, $s_1 = 3 = T$. Let $G(t) \equiv 1$ only on the intervals without non-impulses, i.e. let $G(t) \equiv 1$ on $(0, 1] \cup (2, 3]$, and $g_0(t) = t$ for $t \in [1, 2]$.

Consider the scalar IVP for the NIFrDE (2.4). According to Definition 1 the mild solution is given by

(2.14)
$$x(t) = \begin{cases} x_0 + \frac{\sqrt{t}}{\Gamma(1.5)}, & t \in [0, 1], \\ x_0 + \frac{1}{\Gamma(1.5)} + t, & t \in (1, 2], \\ x_0 + \frac{1}{\Gamma(1.5)} + 1 + \frac{1}{\Gamma(0.5)} \int_0^1 (t - s)^{-0.5}ds + \frac{1}{\Gamma(0.5)} \int_2^t (t - s)^{-0.5}ds \\ \quad = x_0 + 1 + \frac{1}{\Gamma(1.5)} + \frac{\sqrt{t-1}+\sqrt{t}}{\Gamma(1.5)} + \frac{\sqrt{t-2}}{\Gamma(1.5)}, & t \in (2, 3]. \end{cases}$$

Note here the function $G(t)$ is not necessarily defined on the interval $[1, 2]$ of non-instantaneous impulse. □

Example 3. Let $q = 0.5$, $t_0 = 0$, $s_0 = 1$, $t_1 = 2$, $s_1 = 3 = T$ and $G(t) \equiv \frac{1}{t-1.5}$, $t \in (0, 1] \cup (2, 3]$, and $g_0(t) = t$ for $t \in [1, 2]$. Consider the scalar IVP for the NIFrDE 2.4.

The function $G(t)$ is not defined on the whole interval $[0,3]$ and the integral $\int_0^t (t-s)^{-0.5} \frac{1}{s-1.5} ds$ is not convergent on $[0,3]$. Therefore, formula (2.6) could not be applied and the IVP for NIFrDE (2.4) has no solution using approach A.

The mild solution presented in Definition 1 exists and it is given by

$$
x(t) = \begin{cases}
x_0 + \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} \frac{1}{s-1.5} ds, & t \in [0,1], \\
x_0 + \frac{1}{\Gamma(0.5)} \int_0^1 (1-s)^{-0.5} \frac{1}{s-1.5} ds + t, & t \in (1,2], \\
x_0 + 1 + \frac{1}{\Gamma(0.5)} \int_0^1 (1-s)^{-0.5} \frac{1}{s-1.5} ds \\
\quad + \frac{1}{\Gamma(0.5)} \int_2^t (t-s)^{-0.5} \frac{1}{s-1.5} ds, & t \in (2,3].
\end{cases}
$$

$\square$

Consider the IVP for the NIFrDE (2.4). The mild solution of (2.4) is given by

$$
(2.15) \quad x(t) = \begin{cases}
x(s_k - 0) + g_k(t), & t \in (s_k, t_{k+1}], \quad k = 0,1,2,\ldots,p-1, \\
x_0 + \sum_{j=0}^{k-1} g_j(t_{j+1}) + \sum_{j=0}^{k-1} \frac{1}{\Gamma(q)} \int_{t_j}^{s_j} (t-s)^{q-1} G(s) ds \\
\quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} G(s) ds, & t \in (t_k, s_k], \quad k = 0,1,2,\ldots,p.
\end{cases}
$$

For any pair of functions $v, w \in PC[0,T])$ such that $v(t) \leq w(t)$ for $t \in [0,T]$, we define the sets

$$
\begin{aligned}
S(v,w) &= \{ u \in PC^1 : v(t) \leq u(t) \leq w(t), \quad t \in [0,T] \}, \\
\Omega_k(t,v,w) &= \{ x \in \mathbb{R} : v(t) \leq x \leq w(t) \} \text{ for } t \in [t_k, s_k] \}, \quad k = 0,1,\ldots,p, \\
\Lambda_k(t,v,w) &= \{ x \in \mathbb{R} : v(t) \leq x \leq w(t) \text{ for } t \in (s_k, t_{k+1}] \}, \quad k = 0,1,\ldots,p-1, \\
\Gamma_k(v,w) &= \{ x \in \mathbb{R} : v(s_k - 0) \leq x \leq w(s_k - 0) \}, \quad k = 0,1,2\ldots,p-1.
\end{aligned}
$$

3. Mild lower and mild upper solutions of NIFrDE

Following the ideas in [23] we present various type of lower/upper solutions of the NIFrDE.

Definition 2. We say that the function $v(t) \in PC([0,T])$ is a mild minimal (mild maximal) solution of the IVP for NIFrDE (2.1) if it is a mild solution of (2.1) and for any mild solution $u(t) \in PC([0,T])$ of (2.1) the inequality $v(t) \leq u(t)$ ($v(t) \geq u(t)$) holds on $[0,T]$.

For any functions $\xi, \eta \in PC([0,T])$ and vector $a \in \mathbb{R}^p$, $a = (a_1, a_2, \ldots, a_p)$ we define the operator $\Delta : \mathbb{R}^p \times PC([0,T]) \times PC([0,T]) \to PC([0,T])$ by the equalities

$$(3.1) \qquad \Delta(a, \xi, \eta)(t) = \begin{cases} x_0 + \sum_{j=0}^{k-1} \phi_j(t_{j+1}, \xi(t_{j+1}), \xi(s_j - 0)) \\ \quad + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} f(s, \xi(s)) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, \xi(s)) ds, \\ \quad + \sum_{j=0}^{k-1} \psi_j(t_{j+1}, \eta(t_{j+1}), \eta(s_j - 0)) \\ \quad + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} g(s, \eta(s)) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} g(s, \eta(s)) ds, \\ \qquad t \in (t_k, s_k], \ k = 0, 1, 2, \ldots, p, \\ a_{k-1} + \phi_k(t, \xi(t), \xi(s_k - 0)) + \psi_k(t, \eta(t), \eta(s_k - 0)), \\ \qquad t \in (s_k, t_{k+1}], k = 0, 1, 2, \ldots, p - 1. \end{cases}$$

Remark 4. Any mild solution $u(t)$ of the IVP for NIFrDE (2.1) satisfies the equality $u(t) = \Delta(u, u)(t)$ on $[0, T]$.

Apply Definition 1 and similar to Definition 2.7 [23] we present the following definition.

Definition 3. We say that the function $v(t) \in PC([0, T])$ is a mild lower (mild upper) solution of the IVP for NIFrDE (2.1) if $v(t) \leq (\geq)\Delta(v, v)(t)$ on $[0, T]$.

Definition 4. We say that the functions $v, w \in PC([0, T])$ form a couple of mild solutions of the IVP for NIFrDE (2.1) if

$$v(t) = \Delta(a, v, w)(t), w(t) = \Delta(b, w, v)(t) \quad \text{for } t \in [0, T],$$

where $a = (a_1, a_2, \ldots, a_p)$, $a_k = v(s_{k-1}-0)$, $k = 1, 2, \ldots, p$ and $b = (b_1, b_2, \ldots, b_p)$, $b_k = w(s_{k-1} - 0)$, $k = 1, 2, \ldots, p$.

Definition 5. We say that the functions $v, w \in PC([0, T])$ form a couple of mild minimal and maximal solutions of the IVP for NIFrDE (2.1) if v, w form a couple of mild solutions and for any couple of mild solutions $\xi, \eta \in PC([0, T])$ of (2.1) the inequalities $v(t) \leq \xi(t) \leq w(t)$, $v(t) \leq \eta(t) \leq w(t)$ hold on $[0, T]$.

Definition 6. We say that the functions $v(t), w(t) \in PC([0, T])$ form a *couple of mild lower and upper solutions of type I* of the IVP for NIFrDE (2.1) if

$$v(t) \leq \Delta(a, v, w)(t), \quad w(t) \geq \Delta(b, w, v)(t), \quad \text{for } t \in [0, T],$$

where $a = (a_1, a_2, \ldots, a_p)$, $a_k = v(s_{k-1} - 0)$, $k = 1, 2, \ldots, p$ and $b = (b_1, b_2, \ldots, b_p)$, $b_k = w(s_{k-1} - 0)$, $k = 1, 2, \ldots, p$.

Definition 7. We say that the functions $v(t), w(t) \in PC([0, T])$ form a *couple of mild lower and upper solutions of type II* of the IVP for NIFrDE (2.1) if

$$v(t) \leq \Delta(a, w, v)(t), \quad w(t) \geq \Delta(b, v, w)(t), \quad \text{for } t \in [0, T],$$

where $a = (a_1, a_2, \ldots, a_p)$, $a_k = v(s_{k-1} - 0)$, $k = 1, 2, \ldots, p$ and $b = (b_1, b_2, \ldots, b_p)$, $b_k = w(s_{k-1} - 0)$, $k = 1, 2, \ldots, p$.

4. Main result

In the case when the right hand side of the NIFrDE are monotonic we present several algorithms for constructing successive approximations to mild solutions of the IVP for NIFrDE (2.1).

4.1. Couple of mild lower and upper solution of type I.

Theorem 4.1 (first iteration scheme). *Let the following conditions be fulfilled:*

1. *The functions $v, w \in PC([0, T])$ form a couple of mild lower and upper solutions of type I of the IVP for NIFrDE (2.1) and $v(t) \le w(t)$ for $t \in [0, T]$.*
2. *The functions $f, g \in C(\cup_{k=0}^{p} [t_k, s_k], \mathbb{R})$ and for any $x, y \in \Omega_k(t, v, w) : x \le y$ and any fixed $t \in [t_k, s_k]$ the inequalities $f(t, x) \le f(t, y)$ and $g(t, x) \ge g(t, y)$ hold.*
3. *The functions $\phi_k \in C([s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $k = 0, 1, 2, \ldots, p-1$, and for any fixed $t \in [s_k, t_{k+1}]$ and $x_1, x_2 \in \Lambda_k(t, v, w) : x_1 \le x_2$, $y_1, y_2 \in \Gamma_k(v, w) : y_1 \le y_2$ the inequality $\phi_k(t, x_1, y_1) \le \phi_k(t, x_2, y_2)$ holds.*
4. *The functions $\Phi_k \in C([s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $k = 0, 1, 2, \ldots, p-1$, are such that*
   (i) *for any $y \in \Gamma_k(v, w)$, $t \in [s_k, t_{k+1}]$ there exists exactly one function $u \in C([s_k, t_{k+1}], \mathbb{R})$ such that $u(t) = y + \Phi_k(t, u(t), y)$*
   (ii) *for any $x_1, x_2 \in \Lambda_k(t, v, w) : x_1 \le x_2$, $y_1, y_2 \in \Gamma_k(v, w) : y_1 \le y_2$ the inequalities $\phi_k(t, x_1, y_1) \le \phi_k(t, x_2, y_2)$ and $\psi_k(t, x_1, y_1) \ge \psi_k(t, x_2, y_2)$ hold.*

Then there exist two sequences of functions $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$ such that:

a. *The sequences are defined by $v^{(0)}(t) = v(t)$, $w^{(0)}(t) = w(t)$ and for $n \ge 1$*

$$
(4.1) \quad v^{(n)}(t) = \begin{cases}
x_0 + \sum_{j=0}^{k-1} \phi_j(t_{j+1}, v^{(n-1)}(t_{j+1}), v^{(n-1)}(s_j - 0)) \\
\quad + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} f(s, v^{(n-1)}(s)) ds \\
\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, v^{(n-1)}(s)) ds, \\
\quad + \sum_{j=0}^{k-1} \psi_j(t_{j+1}, w^{(n-1)}(t_{j+1}), w^{(n-1)}(s_j - 0)) \\
\quad + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} g(s, w^{(n-1)}(s)) ds \\
\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} g(s, w^{(n-1)}(s)) ds, \\
\qquad for \ t \in (t_k, s_k], \ k = 0, 1, 2, \ldots, p, \\
v^{(n)}(s_k - 0) + \phi_k(t, v^{(n-1)}(t), v^{(n-1)}(s_k - 0)) \\
\quad + \psi_k(t, w^{(n-1)}(t), w^{(n-1)}(s_k - 0)), \\
\qquad for \ t \in (s_k, t_{k+1}], \quad k = 0, 1, 2, \ldots, p-1,
\end{cases}
$$

and

$$(4.2) \quad w^{(n)}(t) = \begin{cases} x_0 + \sum_{j=0}^{k-1} \phi_j(t_{j+1}, w^{(n-1)}(t_{j+1}), w^{(n-1)}(s_j - 0)) \\ \quad + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} f(s, w^{(n-1)}(s)) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, w^{(n-1)}(s)) ds, \\ \quad + \sum_{j=0}^{k-1} \psi_j(t_{j+1}, v^{(n-1)}(t_{j+1}), v^{(n-1)}(s_j - 0)) \\ \quad + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} g(s, v^{(n-1)}(s)) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} g(s, v^{(n-1)}(s)) ds, \\ \quad for \ t \in (t_k, s_k], \ k = 0, 1, 2, \dots, p, \\ w^{(n)}(s_k - 0) + \phi_k(t, w^{(n-1)}(t), w^{(n-1)}(s_k - 0)) \\ \quad + \psi_k(t, v^{(n-1)}(t), v^{(n-1)}(s_k - 0)), \\ \quad for \ t \in (s_k, t_{k+1}], \quad k = 0, 1, 2, \dots, p - 1. \end{cases}$$

b. *The sequence $\{v^{(n)}(t)\}$ is increasing on $[0, T]$ and the sequence $\{w^{(n)}(t)\}$ is decreasing on $[0, T]$ and*

$$v^{(n)}(t) \leq v^{(n+1)}(t)$$
$$(4.3) \qquad \leq \cdots \leq w^{(n+1)}(t) \leq w^{(n)}(t) \quad for \ t \in [0, T], \quad n = 0, 1, 2, \dots.$$

c. *Both sequences converge on $[0, T]$ and $V(t) = \lim_{k \to \infty} v^{(n)}(t)$, $W(t) = \lim_{k \to \infty} w^{(n)}(t)$ on $[0, T]$.*

d. *The limit functions $(V(t), W(t))$ form a couple of mild minimal and maximal solution of IVP for NIFrDE (2.1) in $S(v, w)$.*

Proof. We use induction to prove properties of the sequences of successive approximations.

Let $n = 1$. Define $\mu(t) = v^{(0)}(t) - v^{(1)}(t)$ for $t \in [0, T]$. For any $t \in (t_k, s_k]$, $k = 0, 1, \dots, p$, from Eq. (4.1), Condition 1, and the monotonicity of the functions f, g, ϕ_k, ψ_k we obtain $\mu(t) \leq 0$. Therefore $\mu(s_k - 0) \leq 0$, $k = 0, 1, \dots, p$.

For any $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \dots, p - 1$, from Eq. (4.1) and the monotonicity of the functions ϕ_k, ψ_k we get the inequality

$$(4.4) \quad \mu(t) \leq v^{(0)}(s_k - 0) - v^{(1)}(s_k - 0) \leq 0, \quad t \in (s_k, t_{k+1}], \ k = 0, 1, 2, \dots, p - 1.$$

Therefore $v^{(0)}(t) \leq v^{(1)}(t)$, $t \in [0, T]$.

Similarly, we prove $w^{(0)}(t) \geq w^{(1)}(t)$, $t \in [0, T]$.

Define $\mu(t) = v^{(1)}(t) - w^{(1)}(t)$ for $t \in [0, T]$. From Eq. (4.1) and Eq. (4.2) for any $t \in (t_k, s_k]$, $k = 0, 1, \ldots, p$, we obtain

(4.5)

$$
\mu(t) = \sum_{j=0}^{k-1} \Big(\phi_j(t_{j+1}, v^{(0)}(t_{j+1}), v^{(0)}(s_j - 0)) - \phi_j(t_{j+1}, w^{(0)}(t_{j+1}), w^{(0)}(s_j - 0)) \Big)
$$

$$
+ \sum_{j=0}^{k-1} \Big(\psi_j(t_{j+1}, w^{(0)}(t_{j+1}), w^{(0)}(s_j - 0)) - \psi_j(t_{j+1}, v^{(0)}(t_{j+1}), v^{(0)}(s_j - 0)) \Big)
$$

$$
+ \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} \Big(f(s, v^{(0)}(s)) - f(s, w^{(0)}(s)) \Big) ds
$$

$$
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \Big(f(s, v^{(0)}(s)) - f(s, w^{(0)}(s)) \Big) ds,
$$

$$
+ \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} \Big(g(s, w^{(0)}(s)) - g(s, v^{(0)}(s)) \Big) ds
$$

$$
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \Big(g(s, w^{(0)}(s)) - g(s, v^{(0)}(s)) \Big) ds \leq 0.
$$

For any $t \in (s_k, t_{k+1}]$, $k = 0, 1, \ldots, p-1$, from Eq. (4.1) and Eq. (4.2), inequalities $v^{(1)}(s_k - 0) \leq w^{(1)}(s_k - 0)$, $k = 0, 1, 2, \ldots, p$, and the monotonicity of the functions ϕ_k, ψ_k we obtain

(4.6)
$$
\begin{aligned}
\mu(t) = \; & v^{(1)}(s_k - 0) - w^{(1)}(s_k - 0) \\
& + \phi_k(t, v^{(0)}(t), v^{(0)}(s_k - 0)) - \phi_k(t, w^{(0)}(t), w^{(0)}(s_k - 0)) \\
& + \psi_k(t, w^{(0)}(t), w^{(0)}(s_k - 0)) - \psi_k(t, v^{(0)}(t), v^{(0)}(s_k - 0)) \leq 0.
\end{aligned}
$$

Inequalities (4.5) and (4.6) prove $v^{(1)}(t) \leq w^{(1)}(t)$, $t \in [0, T]$.

We proved $v^{(0)}(t) \leq v^{(1)}(t) \leq w^{(1)}(t) \leq w^{(0)}(t)$, $t \in [0, T]$.

Let $n = 1$. Define $\mu(t) = v^{(1)}(t) - v^{(2)}(t)$ for $t \in [0, T]$. For any $t \in (t_k, s_k]$, $k = 0, 1, \ldots, p$, from Eq. (4.1), Condition 1 and the monotonicity of the functions

f, g, ϕ_k, ψ_k we obtain

(4.7)

$$
\mu(t) = \sum_{j=0}^{k-1} \Big(\phi_j(t_{j+1}, v^{(0)}(t_{j+1}), v^{(0)}(s_j - 0)) - \phi_j(t_{j+1}, v^{(1)}(t_{j+1}), v^{(1)}(s_j - 0)) \Big)
$$

$$
+ \sum_{j=0}^{k-1} \Big(\psi_j(t_{j+1}, w^{(0)}(t_{j+1}), w^{(0)}(s_j - 0)) - \psi_j(t_{j+1}, w^{(1)}(t_{j+1}), w^{(1)}(s_j - 0)) \Big)
$$

$$
+ \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} \Big(f(s, v^{(0)}(s)) - f(s, v^{(1)}(s)) \Big) ds
$$

$$
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \Big(f(s, v^{(0)}(s)) - f(s, v^{(1)}(s)) \Big) ds,
$$

$$
+ \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} \Big(g(s, w^{(0)}(s)) - g(s, w^{(1)}(s)) \Big) ds
$$

$$
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \Big(g(s, w^{(0)}(s)) - g(s, w^{(1)}(s)) \Big) ds \le 0.
$$

Inequality (4.7) proves $v^{(1)}(s_k - 0) \le v^{(2)}(s_k - 0)$, $k = 0, 1, \ldots, p$.

For any $t \in (s_k, t_{k+1}]$, $k = 0, 1, \ldots, p - 1$, from Eq. (4.1) and Eq. (4.2) and the monotonicity of the functions ϕ_k, ψ_k we obtain

(4.8)
$$
\begin{aligned}
\mu(t) = & \, v^{(1)}(s_k - 0) - v^{(2)}(s_k - 0) \\
& + \phi_k(t, v^{(0)}(t), v^{(0)}(s_k - 0)) - \phi_k(t, v^{(1)}(t), v^{(1)}(s_k - 0)) \\
& + \psi_k(t, w^{(0)}(t), w^{(0)}(s_k - 0)) - \psi_k(t, w^{(1)}(t), w^{(1)}(s_k - 0)) \le 0.
\end{aligned}
$$

Inequalities (4.7) and (4.8) prove $v^{(1)}(t) \le v^{(2)}(t)$, $t \in [0, T]$.

By induction we note

(4.9) $v^{(0)}(t) \le v^{(1)}(t) \le \cdots \le v^{(n)}(t) \le w^{(n)}(t) \le \cdots \le w^{(1)}(t) \le w^{(0)}(t)$, $\quad t \in [0, T]$

hold.

We will prove the convergence of the sequences of functions $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$ on $[0, T]$.

Let $t \in [0, s_0]$. Then from definition (3.1) of the operator Δ any element of the sequences $v^{(n)}, w^{(n)} \in C^1([0, s_0], \mathbb{R})$ and

(4.10) $\qquad v^{(n)}(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \Big(f(s, v^{(n-1)}(s)) + g(s, w^{(n-1)}(s)) \Big) ds$

and

(4.11) $\qquad w^{(n)}(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \Big(f(s, w^{(n-1)}(s)) + g(s, v^{(n-1)}(s)) \Big) ds.$

The sequences of functions $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$ being monotonic and bounded are uniformly convergent on $[0, s_0]$. Let $V_1(t) = \lim_{n\to\infty} v^{(n)}(t)$, $W_1(t) = \lim_{n\to\infty} w^{(n)}(t)$, $t \in [0, s_0]$. According to (4.9) the inequalities

$$(4.12) \qquad v(t) \leq V_1(t) \leq W_1(t) \leq w(t), \quad t \in [0, s_0]$$

hold. Take the limit in (4.10) and (4.11) and we obtain the Volterra fractional integral equations

$$(4.13)$$
$$V_1(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\Big(f(s, V_1(s)) + g(s, W_1(s))\Big)ds, \quad t \in [0, s_0]$$
$$W_1(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\Big(f(s, W_1(s)) + g(s, V_1(s))\Big)ds, \quad t \in [0, s_0].$$

Let $t \in (s_0, t_1]$. Then according to definition (3.1) of the operator Δ note $v^{(n)}, w^{(n)} \in C((s_0, t_1], \mathbb{R})$ and we have

$$(4.14)$$
$$v^n(t) = v^{(n)}(s_0 - 0) + \phi_0(t, v^{(n-1)}(t), v^{(n-1)}(s_0 - 0)) + \psi_0(t, w^{(n-1)}(t), w^{(n-1)}(s_0 - 0))$$
$$w^n(t) = w^{(n)}(s_0 - 0) + \phi_0(t, w^{(n-1)}(t), w^{(n-1)}(s_0 - 0)) + \psi_0(t, v^{(n-1)}(t), v^{(n-1)}(s_0 - 0)).$$

From $v^{(n)}(t), w^{(n)}(t) \in PC([0, T])$ it follows that $v^{(n)}(s_0+0) < \infty$ and $w^{(n)}(s_0+0) < \infty$ exist. For any $n = 1, 2, \ldots$ we define the functions

$$\tilde{v}^{(n)}(t) = \begin{cases} v^{(n)}(s_0 + 0) & \text{for } t = s_0, \\ v^{(n)}(t) & \text{for } t \in (s_0, t_1]. \end{cases} \qquad \tilde{w}^{(n)}(t) = \begin{cases} w^{(n)}(s_0 + 0) & \text{for } t = s_0, \\ w^{(n)}(t) & \text{for } t \in (s_0, t_1]. \end{cases}$$

Then $\tilde{v}^{(n)}, \tilde{w}^{(n)} \in C([s_0, t_1], \mathbb{R})$. The sequences of functions $\{\tilde{v}^{(n)}(t)\}_0^\infty$ and $\{\tilde{w}^{(n)}(t)\}_0^\infty$ being monotonic and bounded are uniformly convergent on $[s_0, t_1]$. Let $V_2(t) = \lim_{n\to\infty} \tilde{v}^{(n)}(t)$, $W_2(t) = \lim_{n\to\infty} \tilde{v}^{(n)}(t)$, $t \in [s_0, t_1]$. According to (4.9) the inequality

$$(4.15) \qquad v(t) \leq V_2(t) \leq W_2(t) \leq w(t), \quad t \in (s_0, t_1]$$

holds. Take the limit in (4.14) and obtain for $t \in [s_0, t_1]$,

$$(4.16)$$
$$V_2(t) = V_1(s_0 - 0) + \phi_0(t, V_2(t), V_1(s_0 - 0)) + \psi_0(t, W_2(t), W_1(s_0 - 0)),$$
$$W_2(t) = W_1(s_0 - 0) + \phi_0(t, W_2(t), W_1(s_0 - 0)) + \psi_0(t, V_2(t), V_1(s_0 - 0)).$$

Let $t \in [t_1, s_1]$. Then according to definition (3.1) of the operator Δ note $v^{(n)}, w^{(n)} \in C^1([t_1, s_1], \mathbb{R})$ and

$$(4.17)$$
$$v^{(n)}(t) = x_0 + \phi_0(t_1, v^{(n-1)}(t_1), v^{(n-1)}(s_0 - 0)) + \psi_0(t_1, w^{(n-1)}(t_1), w^{(n-1)}(s_0 - 0))$$
$$+ \frac{1}{\Gamma(q)} \int_0^{s_0} (s_0 - s)^{q-1} f(s, v^{(n-1)}(s))ds + \frac{1}{\Gamma(q)} \int_0^{s_0} (s_0 - s)^{q-1} g(s, w^{(n-1)}(t))ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_1}^t (t - s)^{q-1} f(s, v^{(n-1)}(s))ds + \frac{1}{\Gamma(q)} \int_{t_1}^t (t - s)^{q-1} g(s, w^{(n-1)}(s))ds$$

and

(4.18)
$$w^{(n)}(t) = x_0 + \phi_0(t_1, w^{(n-1)}(t_1), w^{(n-1)}(s_0 - 0)) + \psi_0(t_1, v^{(n-1)}(t_1), v^{(n-1)}(s_0 - 0))$$

$$+ \frac{1}{\Gamma(q)} \int_0^{s_0} (s_0 - s)^{q-1} f(s, w^{(n-1)}(s)) ds + \frac{1}{\Gamma(q)} \int_0^{s_0} (s_0 - s)^{q-1} g(s, v^{(n-1)}(t)) ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t - s)^{q-1} f(s, w^{(n-1)}(s)) ds + \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t - s)^{q-1} g(s, v^{(n-1)}(s)) ds.$$

The sequences of functions $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$ being monotonic and bounded are uniformly convergent on $[t_1, s_1]$. Let $V_3(t) = \lim_{n\to\infty} v^{(n)}(t)$, $W_3(t) = \lim_{n\to\infty} w^{(n)}(t)$, $t \in [t_1, s_1]$. According to (4.9) the inequality

(4.19)
$$v(t) \le V_3(t) \le W_3(t) \le w(t), \quad t \in [t_1, s_1]$$

holds. Take the limit in (4.17) and (4.18) and obtain

(4.20)
$$V_3(t) = x_0 + \phi_0(t_1, V_2(t_1), V_1(s_0 - 0)) + \psi_0(t_1, W_2(t_1), W_1(s_0 - 0))$$

$$+ \frac{1}{\Gamma(q)} \int_0^{s_0} (s_0 - s)^{q-1} f(s, V_2(s)) ds + \frac{1}{\Gamma(q)} \int_0^{s_0} (s_0 - s)^{q-1} g(s, W_2(t)) ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t - s)^{q-1} f(s, V_2(s)) ds + \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t - s)^{q-1} g(s, W_2(s)) ds,$$

$$W_3(t) = x_0 + \phi_0(t_1, W_2(t_1), W_1(s_0 - 0)) + \psi_0(t_1, V_2(t_1), V_1(s_0 - 0))$$

$$+ \frac{1}{\Gamma(q)} \int_0^{s_0} (s_0 - s)^{q-1} f(s, W_2(s)) ds + \frac{1}{\Gamma(q)} \int_0^{s_0} (s_0 - s)^{q-1} g(s, V_2(t)) ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t - s)^{q-1} f(s, W_2(s)) ds + \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t - s)^{q-1} g(s, V_2(s)) ds$$

for $t \in [t_1, s_1]$.

By induction we can construct limit functions $V_{2k+2}(t), W_{2k+2}(t) \in C([s_k, t_{k+1}], \mathbb{R})$, $k = 0, 1, 2, \ldots, p$, and $V_{2k+1}(t), W_{2k+1}(t) \in C^1([t_k, s_k], \mathbb{R})$, $k = 0, 1, 2, \ldots, p + 1$, which similar to (4.16) satisfy the equations

$$V_{2k+2}(t) = V_{2k+1}(s_k - 0) + \phi_k(t, V_{2k+2}(t), V_{2k+1}(s_k - 0))$$

$$+ \psi_k(t, W_{2k+2}(t), W_{2k+1}(s_k - 0)),$$

(4.21)
$$W_{2k+2}(t) = W_{2k+1}(s_k - 0) + \phi_k(t, W_{2k+2}(t), W_{2k+1}(s_k - 0))$$

$$+ \psi_k(t, V_{2k+2}(t), V_{2k+1}(s_k - 0)),$$

$$\text{for } t \in [s_k, t_{k+1}], \ k = 0, 1, 2, \ldots, p - 1,$$

and similar to (4.13), (4.20) we have

(4.22)

$$V_{2k+1}(t) = x_0 + \sum_{j=0}^{k-1} \phi_j(t_{j+1}, V_{2j+2}(t_{j+1}), V_{2j+1}(s_j - 0))$$

$$+ \sum_{j=0}^{k-1} +\psi_j(t_{j+1}, W_{2j+2}(t_{j+1}), W_{2j+1}(s_j - 0))$$

$$+ \sum_{j=0}^{k-1} \left(\frac{1}{\Gamma(q)} \int_{t_j}^{s_j} (s_j - s)^{q-1} f(s, V_{2j+2}(s)) ds \right.$$

$$\left. + \frac{1}{\Gamma(q)} \int_{t_j}^{s_j} (s_j - s)^{q-1} g(s, W_{2j+2}(s)) \right) ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, V_{2k}(s)) ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} g(s, W_{2k}(s)) ds$$

$$\text{for } t \in [t_k, s_k], \ k = 0, 1, 2, \ldots, p,$$

and

(4.23)

$$W_{2k+1}(t) = x_0 + \sum_{j=0}^{k-1} \phi_j(t_{j+1}, W_{2j+2}(t_{j+1}), W_{2j+1}(s_j - 0))$$

$$+ \sum_{j=0}^{k-1} \psi_j(t_{j+1}, V_{2j+2}(t_{j+1}), V_{2j+1}(s_j - 0))$$

$$+ \sum_{j=0}^{k-1} \left(\frac{1}{\Gamma(q)} \int_{t_j}^{s_j} (s_j - s)^{q-1} f(s, W_{2j+2}(s)) ds \right.$$

$$\left. + \frac{1}{\Gamma(q)} \int_{t_j}^{s_j} (s_j - s)^{q-1} g(s, V_{2j+2}(s)) \right) ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, W_{2k}(s)) ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} g(s, V_{2k}(s)) ds$$

$$\text{for } t \in [t_k, s_k], \ k = 0, 1, 2, \ldots, p.$$

Define the functions $V(t), W(t) \in PC([0, T], \mathbb{R})$ by

$$V(t) = \begin{cases} V_{2k+2}(t) & \text{for } t \in (s_k, t_{k+1}], \quad k = 0, 1, 2, \ldots, p - 1, \\ V_{2k+1}(t) & \text{for } t \in [t_k, s_k], \quad k = 0, 1, 2, \ldots, p, \end{cases}$$

and

$$W(t) = \begin{cases} W_{2k+2}(t) & \text{for } t \in (s_k, t_{k+1}], \quad k = 0, 1, 2, \ldots, p - 1, \\ W_{2k+1}(t) & \text{for } t \in [t_k, s_k], \quad k = 0, 1, 2, \ldots, p. \end{cases}$$

Similar to (4.12), (4.15), (4.19) it follows that $V(t) \in S(v, w)$. From Eq. (4.21), (4.22) and (4.23) and Definition 4 it follows that the functions $V(t), W(t)$ form a couple of mild solutions of IVP for NIFrDE (2.1).

We now prove that the functions $V(t)$ and $W(t)$ form a couple of mild minimal and maximal solutions of IVP for NIFrDE (2.1) in $S(v, w)$.

Let $(\xi, \eta) \in S(v, w) \times S(v, w)$ be a couple of mild solutions of IVP for NIFrDE (2.1). From inequality (4.9) it follows that there exists a natural number N such that $v^{(N)}(t) \leq \xi(t) \leq w^{(N)}(t)$, $v^{(N)}(t) \leq \eta(t) \leq w^{(N)}(t)$ for $t \in [0, T]$. Denote $\mu(t) = v^{(N+1)}(t) - \xi(t)$, $\nu(t) = \eta(t) - w^{(N+1)}(t)$ for $t \in [0, T]$.

Let $t \in (t_k, s_k]$, $k = 0, 1, \ldots, p$. Then applying the monotonicity property of the functions f, g, ϕ_j, ψ_j $j = 0, 1, \ldots, m$ and the choice of N we obtain

$$
\begin{aligned}
\mu(t) = {}& \Delta(v^{(N)}, w^{(N)})(t) - \Delta(\xi, \eta)(t) \\
= {}& \sum_{j=0}^{k-1} \Big(\phi_j(t_{j+1}, v^{(N)}(t_{j+1}), v^{(N)}(s_j - 0)) - \phi_j(t_{j+1}, \xi(t_{j+1}), \xi(s_j - 0)) \Big) \\
& + \sum_{j=0}^{k-1} \Big(\psi_j(t_{j+1}, w^{(N)}(t_{j+1}), w^{(N)}(s_j - 0)) - \psi_j(t_{j+1}, \eta(t_{j+1}), \eta(s_j - 0)) \Big) \\
& + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} \Big(f(s, v^{(N)}(s)) - f(s, \xi(s)) \Big) ds \\
& + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \Big(f(s, v^{(N)}(s)) - f(s, \xi(s)) \Big) ds \\
& + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} \Big(g(s, w^{(N)}(s)) - g(s, \eta(s)) \Big) ds \\
& + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \Big(g(s, w^{(N)}(s)) - g(s, \eta(s)) \Big) ds \leq 0.
\end{aligned}
$$

Similarly, we prove $\xi(t) \leq w^{(N+1)}(t)$, $v^{(N+1)}(t) \leq \eta(t) \leq w^{(N+1)}(t)$ $t \in (t_k, s_k]$.

For any $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, p-1$, from the monotonicity of the functions $f, g, \phi_j, \psi_j, j = 0, 1, \ldots, p-1$ and the choice of N we get the inequality

$$
\begin{aligned}
\mu(t) = {}& v^{(N)}(s_k - 0) - \xi(s_k - 0) \\
& + \phi_k(t, v^{(N)}(t), v^{(N)}(s_k - 0)) - \phi_k(t, \xi(t), \xi(s_k - 0)) \\
& + \psi_k(t, w^{(N)}(t), v^{(N)} w(s_k - 0)) - \psi_k(t, \eta(t), \eta(s_k - 0)) \leq 0.
\end{aligned}
$$

Similarly, we prove $\xi(t) \leq w^{(N+1)}(t)$, $v^{(N+1)}(t) \leq \eta(t) \leq w^{(N+1)}(t)$ $t \in (s_k, t_{k+1}]$.

The inequalities $v^{(n)}(t) \leq \xi(t) \leq w^{(n)}(t)$, $v^{(n)}(t) \leq \eta(t) \leq w^{(n)}(t)$ for $t \in [0, T]$ and $n = N, N+1, \ldots$ prove $V_{2k+1}(t) \leq \xi(t) \leq W_{2k+1}(t)$, $V_{2k+1}(t) \leq \eta(t) \leq W_{2k+1}(t)$ for $t \in [t_k, s_k]$, $k = 0, 1, 2, \ldots, p$, and $V_{2k+2}(t) \leq \xi(t) \leq W_{2k+2}(t)$, $V_{2k+2}(t) \leq \eta(t) \leq W_{2k+2}(t)$ for $t \in [s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, p-1$.

Therefore, according to Definition 5 the functions $V(t)$ and $W(t)$ form a couple of mild minimal and maximal solutions of IVP for NIFrDE (2.1) in $S(v, w)$. $\qquad\square$

Theorem 4.2 (second iteration scheme). *Let the conditions of Theorem 4.1 be satisfied.*

Then there exist two sequences of functions $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$ such that:

a. *The sequences are defined by $v^{(0)}(t) = v(t)$, $w^{(0)}(t) = w(t)$ and for $n \geq 1$*

$$(4.24) \qquad v^{(n)}(t) = \begin{cases} x_0 + \sum_{j=0}^{k-1} \phi_j(t_{j+1}, w^{(n-1)}(t_{j+1}), w^{(n-1)}(s_j - 0)) \\ \quad + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} f(s, w^{(n-1)}(s)) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, w^{(n-1)}(s)) ds, \\ \quad + \sum_{j=0}^{k-1} \psi_j(t_{j+1}, v^{(n-1)}(t_{j+1}), v^{(n-1)}(s_j - 0)) \\ \quad + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} g(s, v^{(n-1)}(s)) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} g(s, v^{(n-1)}(s)) ds, \\ \qquad \text{for } t \in (t_k, s_k], \ k = 0, 1, 2, \ldots, p, \\ v^{(n)}(s_k - 0) + \phi_k(t, w^{(n-1)}(t), w^{(n-1)}(s_k - 0)) \\ \quad + \psi_k(t, v^{(n-1)}(t), v^{(n-1)}(s_k - 0)), \\ \qquad \text{for } t \in (s_k, t_{k+1}], k = 0, 1, 2, \ldots, p - 1, \end{cases}$$

and

$$(4.25) \qquad w^{(n)}(t) = \begin{cases} x_0 + \sum_{j=0}^{k-1} \phi_j(t_{j+1}, v^{(n-1)}(t_{j+1}), v^{(n-1)}(s_j - 0)) \\ \quad + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} f(s, v^{(n-1)}(s)) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, v^{(n-1)}(s)) ds, \\ \quad + \sum_{j=0}^{k-1} \psi_j(t_{j+1}, w^{(n-1)}(t_{j+1}), w^{(n-1)}(s_j - 0)) \\ \quad + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} g(s, w^{(n-1)}(s)) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} g(s, w^{(n-1)}(s)) ds, \\ \qquad \text{for } t \in (t_k, s_k], \ k = 0, 1, 2, \ldots, p, \\ w^{(n)}(s_k - 0) + \phi_k(t, v^{(n-1)}(t), v^{(n-1)}(s_k - 0)) \\ \quad + \psi_k(t, w^{(n-1)}(t), w^{(n-1)}(s_k - 0)), \\ \qquad \text{for } t \in (s_k, t_{k+1}], k = 0, 1, 2, \ldots, p - 1. \end{cases}$$

b. *The sequence $\{v^{(n)}(t)\}$ is increasing on $[0, T]$ and the sequence $\{w^{(n)}(t)\}$ is decreasing on $[0, T]$ and the inequalities*

$$v^{(0)}(t) \leq w^{(1)}(t) \leq v^{(2)}(t) \cdots \leq w^{(2n-1)}(t) \leq v^{(2n)}(t)$$

$$(4.26) \qquad \leq \cdots \leq w^{(2n)}(t) \leq v^{(2n-1)}(t)$$

$$\leq \cdots \leq w^{(2)}(t) \leq v^{(1)}(t) \leq w^{(0)}(t), \quad t \in [0, T]$$

hold.

c. *Both sequences converge on $[0, T]$ and $V(t) = \lim_{k \to \infty} v^{(n)}(t)$, $W(t) = \lim_{k \to \infty} w^{(n)}(t)$ on $[0, T]$.*

d. *The limit functions $(V(t), W(t))$ form a couple of mild minimal and maximal solution of IVP for NIFrDE (2.1) in $S(v, w)$.*

Proof. The proof is similar to that in Theorem 4.1 so we omit it. □

Theorem 4.3 (Existence of a mild solution). *Let:*

1. *The conditions of Theorem 4.1 are satisfied.*
2. *There exist positive constants L_f, L_g such that for $x, y \in \Omega_k(t, v, w) : x \le y$ the inequalities*

$$f(t, y) - f(t, x) \le L_f(y - x), \quad t \in \cup_{k=0}^{p}[t_k, s_k],$$

$$g(t, y) - g(t, x) \ge -L_g(y - x), \quad t \in \cup_{k=0}^{p}[t_k, s_k]$$

   hold.
3. *There exist positive constants M_k^{ϕ} and $M_k^{\psi} : M_k^{\phi} + M_k^{\psi} < 1, \ k = 0, 1, \ldots, p - 1$, such that for $x_1, y_1 \in \Lambda_k(t, v, w) : x_1 \le y_1$ and $u \in \Gamma_k(v, w)$ the inequalities*

$$\phi_k(t, y_1, u) - \phi_k(t, x_1, u) \le M_k^{\phi}(y_1 - x_1), \quad t \in [s_k, t_{k+1}],$$

$$\psi_k(t, y_1, u) - \psi_k(t, x_1, u) \ge -M_k^{\psi}(y_1 - x_1), \quad t \in [s_k, t_{k+1}]$$

   hold.

Then $V(t) = W(t)$ on $[0, T]$ and the function $V(t)$ is a mild solution of IVP for NIFrDE (2.1) on the set $S(v, w)$ (the limit functions $V(t)$ and $W(t)$ are defined in Theorem 4.1).

Proof. Define $\mu(t) = W(t) - V(t), t \in [0, T]$. According to the proof of Theorem 4.1 we have $V(t) \le W(t)$ on $[0, T]$ and the functions $V(t), W(t)$ form a couple of mild minimal and maximal solution of type I of the IVP for NIFrDE (2.1), i.e. $V(t) = \Delta(a, V, W)(t)$, $W(t) = \Delta(b, W, V)(t)$ and $\mu(t) \ge 0, \ t \in [0, T]$ where $a = (a_1, a_2, \ldots, a_p)$, $a_k = V(s_{k-1} - 0), k = 1, 2, \ldots, p$ and $b = (b_1, b_2, \ldots, b_p), b_k = W(s_{k-1} - 0), k = 1, 2, \ldots, p$.

Let $t \in [0, s_0]$. Then from condition 2 we obtain

$$
\begin{aligned}
\mu(t) &= \frac{1}{\Gamma(q)} \int_0^{s_0} (t - s)^{q-1} \Big(f(s, W(s)) - f(s, V(s)) \Big) ds \\
&+ \frac{1}{\Gamma(q)} \int_0^{s_0} (t - s)^{q-1} \Big(g(s, V(s)) - g(s, W(s)) \Big) ds \\
&\le \frac{L_f + L_g}{\Gamma(q)} \int_0^{s_0} (t - s)^{q-1} \mu(s) ds.
\end{aligned}
$$

(4.27)

The inequality (4.27) proves $\mu(t) \le 0, \ t \in [0, s_0]$.

Let $t \in (s_0, t_1]$. Then applying the monotonicity property of the function ϕ_0, ψ_0, the inequality $\mu(s_0 - 0) \leq 0$ and Condition 3 with $k = 0$ we get

(4.28)

$$
\begin{aligned}
\mu(t) &= W(s_0 - 0) - V(s_0 - 0) + \phi_0(t, W(t), W(s_0 - 0)) - \phi_0(t, V(t), V(s_0 - 0)) \\
&\quad + \psi_0(t, V(t), V(s_0 - 0)) - \psi_0(t, W(t), W(s_0 - 0)) \\
&\leq \phi_0(t, W(t), V(s_0 - 0)) - \phi_0(t, V(t), V(s_0 - 0)) \\
&\quad + \psi_0(t, V(t), W(s_0 - 0)) - \psi_0(t, W(t), W(s_0 - 0)) \\
&\leq (M_0^\phi + M_0^\psi)\mu(t).
\end{aligned}
$$

Since $M_0^\phi + M_0^\psi < 1$ inequality (4.28) proves $\mu(t) \leq 0$, $t \in (s_0, t_1]$.

Assume $\mu(t) \leq 0$ for $t \in [0, t_m]$, $m < p$ is a natural number.

Let $t \in (t_m, s_m]$. Then applying the monotonicity property of the functions ϕ_k, ψ_k, the inequality $\mu(s_j - 0)) \leq 0$, $\mu(t_{j+1})) \leq 0$, $j = 0, 1, \ldots, m-1$ and Conditions 2 and 3 we obtain

(4.29)

$$
\begin{aligned}
\mu(t) &= \sum_{j=0}^{m-1} \Big(\phi_j(t_{j+1}, W(t_{j+1}), W(s_j - 0)) - \phi_j(t_{j+1}, V(t_{j+1}), V(s_j - 0)) \Big) \\
&\quad + \sum_{j=0}^{m-1} \Big(\psi_j(t_{j+1}, V(t_{j+1}), V(s_j - 0)) - \psi_j(t_{j+1}, W(t_{j+1}), W(s_j - 0)) \Big) \\
&\quad + \sum_{j=0}^{k-1} \frac{1}{\Gamma(q)} \int_{t_j}^{s_j} (s_j - s)^{q-1} \Big(f(s, W(s)) - f(s, V(s)) \Big) ds \\
&\quad + \sum_{j=0}^{k-1} \frac{1}{\Gamma(q)} \int_{t_j}^{s_j} (s_j - s)^{q-1} \Big(g(s, V(t)) - g(s, W(t)) \Big) ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \Big(f(s, W(s)) - f(s, V(s)) \Big) ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \Big(g(s, V(s)) - g(s, W(s)) \Big) ds \\
&\leq \sum_{j=0}^{m-1} \Big(\phi_j(t_{j+1}, W(t_{j+1}), V(s_j - 0)) - \phi_j(t_{j+1}, V(t_{j+1}), V(s_j - 0)) \Big) \\
&\quad + \sum_{j=0}^{m-1} \Big(\psi_j(t_{j+1}, V(t_{j+1}), W(s_j - 0)) - \psi_j(t_{j+1}, W(t_{j+1}), W(s_j - 0)) \Big) \\
&\quad + \sum_{j=0}^{k-1} \frac{L_f}{\Gamma(q)} \int_{t_j}^{s_j} (s_j - s)^{q-1} \mu(s) ds + \sum_{j=0}^{k-1} \frac{L_g}{\Gamma(q)} \int_{t_j}^{s_j} (s_j - s)^{q-1} \mu(s) ds \\
&\quad + \frac{L_f}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \mu(s) ds + \frac{L_g}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \mu(s) ds
\end{aligned}
$$

or

$$\begin{aligned}
\mu(t) &\leq \sum_{j=0}^{m-1}(M_j^\phi + M_j^\psi)\mu(t_{j+1}) + \frac{L_f + L_g}{\Gamma(q)} \int_{t_j}^{s_j}(s_j - s)^{q-1}\mu(s)ds \\
&\leq \frac{L_f + L_g}{\Gamma(q)} \int_{t_j}^{s_j}(s_j - s)^{q-1}\mu(s)ds.
\end{aligned}$$
(4.30)

Inequality (4.29) proves $\mu(t) \leq 0$ on $t \in (t_m, s_m]$.

Let $(s_m, t_{m+1}]$. Then

$$\begin{aligned}
\mu(t) &= W(s_m - 0) - V(s_m - 0) \\
&\quad + \phi_m(t, W(t), W(s_m - 0)) - \phi_m(t, V(t), V(s_m - 0)) \\
&\quad + \psi_m(t, V(t), V(s_m - 0)) - \psi_m(t, W(t), W(s_m - 0)) \\
&\leq (M_m^\phi + M_m^\psi)\mu(t).
\end{aligned}$$
(4.31)

Since $M_m^\phi + M_m^\psi < 1$ inequality (4.31) proves $\mu(t) \leq 0$, $t \in (s_m, t_{m+1}]$.

Therefore, $\mu(t) \leq 0$, $t \in [0, T]$. The proof is complete. $\square$

4.2. **Couple of mild lower and upper solution of type II.** In the case when the IVP for NIFrDE (2.1) has a couple of mild lower and upper solutions of type II we present two iteration schemes for approximately obtaining the solution.

Theorem 4.4 (first iteration scheme). *Let the following conditions be fulfilled:*

1. *The functions $v, w \in PC^1([0, T])$ form a couple of mild lower and upper solutions of type II of the IVP for NIFrDE (2.1) and $v(t) \leq w(t)$ for $t \in [0, T]$.*
2. *The conditions 2, 3, 4 of Theorem 4.1 are satisfied.*
3. *The inequalities $v(t) \leq v^{(1)}(t)$ and $w(t) \geq w^{(1)}(t)$ hold on $[0, T]$ where the functions $v^{(1)}(t)$ and $w^{(1)}(t)$ are obtained from the first iteration scheme (4.1) and (4.2) with $n = 1$.*

Then the claim of Theorem 4.1 is true.

Remark 5. Note the case without impulses and fractional differential equations is studied in [23] and the ordinary case when $q = 1$ is studied in [26]. A brief overview of the application of the monotone method in the case without impulses is given in [27].

Remark 6. Note that if the functions $f(t, x), \phi_k(t, x, y)$ are increasing w.r.t. their arguments x and x, y respectively, and the functions $g(t, x), \psi_k(t, x, y)$ are decreasing w.r.t. their arguments x and x, y respectively, then

  - if the inequality $v \leq w$ holds and (v, w) form a couple of mild solutions of type I, then the functions (v, w) form a couple of mild solutions of type II and $v \leq v^{(1)}$, $w \geq w^{(1)}$ hold;

- the inequalities $v \leq v^{(1)}$, $w \geq w^{(1)}$ are not enough to guarantee the functions (v, w) form a couple of mild solutions of type II.

Theorem 4.5 (second iteration scheme). *Let:*

1. *The conditions 2,3,4 of Theorem 4.1 are satisfied.*
2. *The functions $v, w \in PC^1([0, T])$ form a couple of mild lower and upper solutions of type II of the IVP for NIFrDE (2.1) and $v(t) \leq w(t)$ for $t \in [0, T]$.*
3. *The inequalities $v(t) \leq v^{(1)}(t)$ and $w(t) \geq w^{(1)}(t)$ hold on $[0, T]$ where the functions $v^{(1)}(t)$ and $w^{(1)}(t)$ are obtained from the second iteration scheme (4.24) and (4.25) with $n = 1$.*

   Then the claim of Theorem 4.2 is true.

The proofs of Theorem 4.4 and Theorem 4.5 are similar to the ones in Theorem 4.1 and Theorem 4.2 respectively, so we omit them.

4.3. **Mild lower and mild upper solutions.** For the case $g(t, x) \equiv 0$ for $t \in [0, T]$, $x \in \mathbb{R}$ the IVP for NIFrDE (2.1) is reduced to

(4.32)
$$^{C}D_t^q \, x(t) = f(t, x) \text{ for } t \in (t_k, s_k], \ k = 0, 1, \ldots, p,$$
$$x(t) = x(s_k - 0) + \phi_k(t, x(t), x(s_k - 0)), \quad t \in (s_k, t_{k+1}], \ k = 0, 1, \ldots, p - 1,$$
$$x(0) = x_0,$$

For any function $\xi \in PC([0, T])$ we define the operator $\Xi : PC([0, T]) \times PC([0, T]) \rightarrow PC([0, T])$ by the equalities

(4.33)
$$\Xi(\xi, \eta)(t) = \begin{cases} x_0 + \sum_{j=0}^{k-1} \phi_j(t_{j+1}, \xi(t_{j+1}), \xi(s_j - 0)) \\ \quad + \frac{1}{\Gamma(q)} \sum_{j=0}^{k-1} \int_{t_j}^{s_j} (s_j - s)^{q-1} f(s, \xi(s)) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, \xi(s)) ds, \\ \qquad t \in (t_k, s_k], \ k = 0, 1, 2, \ldots, p, \\ \xi(s_k - 0) + \phi_k(t, \xi(t), \xi(s_k - 0)), \\ \qquad t \in (s_k, t_{k+1}], k = 0, 1, 2, \ldots, p - 1. \end{cases}$$

In this case as a special case of Theorem 4.1 we obtain the following result:

Theorem 4.6. *Let the following conditions be fulfilled:*

1. *The functions $v, w \in PC^1([0, T])$ are a mild lower and a mild upper solutions of the IVP for NIFrDE (4.32), respectively, and $v(t) \leq w(t)$ for $t \in [0, T]$.*
2. *The function $f \in C(\cup_{k=0}^{p+1}[t_k, s_k], \mathbb{R})$ and for any $x, y \in \Omega_k(t, v, w) : \ x \leq y$ and any fixed $t \in [t_k, s_k]$ the inequality $f(t, x) \leq f(t, y)$ holds.*
3. *The functions $\phi_k \in C([s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $k = 0, 1, 2, \ldots, p - 1$, and for any fixed $t \in [s_k, t_{k+1}]$ and $x_1, x_2 \in \Lambda_k(t, v, w) : x_1 \leq x_2$, $y_1, y_2 \in \Gamma_k(v, w) : y_1 \leq y_2$ the inequality $\phi_k(t, x_1, y_1) \leq \phi_k(t, x_2, y_2)$ holds.*

4. *For any $y \in \Gamma_k(v, w)$, $t \in [s_k, t_{k+1}]$ there exists exactly one function $u \in C([s_k, t_{k+1}], \mathbb{R})$ such that $u(t) = y + \phi_k(t, u(t), y)$.*

Then there exist two sequences of functions $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$ such that:

a. *The sequences are defined by $v^{(0)}(t) = v(t)$, $w^{(0)}(t) = w(t)$ and for $n \geq 1$*

$$v^{(n)}(t) = \Xi(v^{(n-1)})(t), \quad w^{(n)}(t) = \Xi(w^{(n-1)})(t), \quad \text{for } t \in [0, T].$$

b. *The sequence $\{v^{(n)}(t)\}$ is increasing on $[0, T]$ and the sequence $\{w^{(n)}(t)\}$ is decreasing on $[0, T]$ i.e.*

$$v^{(n)}(t) \leq v^{(n+1)}(t) \leq \cdots \leq w^{(n+1)}(t) \leq w^{(n)}(t) \quad \text{for } t \in [0, T], \ n = 0, 1, 2, \ldots.$$

c. *Both sequences converge on $[0, T]$ and $V(t) = \lim\limits_{k \to \infty} v^{(n)}(t)$, $W(t) = \lim\limits_{k \to \infty} w^{(n)}(t)$ on $[0, T]$.*

d. *The limit functions $V(t)$ and $W(t)$) are a mild minimal solution and a mild maximal solution of IVP for NIFrDE (2.1) in $S(v, w)$, respectively.*

5. Applications

Consider the non-instantaneous impulsive fractional generalization of a special case of the logistic model

(5.1)
$$^{c}D_0^{0.5}x(t) = 0.4x(1 - 2x) \text{ for } t \in [0, 0.45] \cup (0.5, 1],$$
$$x(t) = x(s_0 - 0) + (t - 0.5)x(t) \text{ for } t \in (0.45, 0.5],$$
$$x(0) = 0.5,$$

where $f(t, x) = 0.4x$, $g(t, x) = -0.8x^2$, $\phi_0 = tx$, $\psi_0 = -0.5x$.

The IVP for NIFrDE (5.1) has an exact constant solution $x(t) \equiv 0.5$.

The functions $v(t) \equiv 0.49$, $w(t) \equiv 0.51$, $t \in [0, 4]$ form a couple of mild lower and upper solutions of type II for (5.1) because the following inequalities

$$v(t) \equiv 0.49 \leq \begin{cases} 0.5 + \frac{1}{\Gamma(0.5)} \int_0^t (t - s)^{-0.5}(0.4(0.51) - 0.8(0.49)^2)ds \\ \quad = 0.5 - 2(0.01192)\sqrt{\frac{t}{\pi}}, \quad t \in [0, 0.45], \\ 0.49 + 1.1t(0.51) - 0.5(0.49) = 0.245 + 0.567t, \quad t \in (0.45, 0.5], \\ 0.5 - 2(0.01192)\sqrt{\frac{0.45}{\pi}} + 1.1(0.5)(0.51) - 0.5(0.49) \\ \quad + \frac{1}{\Gamma(0.5)} \int_{0.5}^t (t - s)^{-0.5}(0.4(0.51) - 0.8(0.49)^2)ds \\ \quad = 0.5 - 2(0.01192)\sqrt{\frac{0.45}{\pi}} + 1.1(0.5)(0.51) - 0.5(0.49) \\ \quad - 2(0.01192)\sqrt{\frac{t-0.5}{\pi}} = 0.526477 - 0.02384\sqrt{\frac{t-0.5}{\pi}}, \quad t \in [0.5, 1], \end{cases}$$

and

$$w(t) \equiv 0.51 \geq \begin{cases} 0.5 + \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5}(0.4(0.49) - 0.8(0.51)^2)ds \\ \quad = 0.5 - 2(0.01208)\sqrt{\frac{t}{\pi}}, \quad t \in [0, 0.45], \\ 0.51 + 1.1t(0.49) - 0.5(0.51) = 0.255 + 0.539t, \quad t \in (0.45, 0.5], \\ 0.5 - 2(0.01192)\sqrt{\frac{0.45}{\pi}} + 1.1(0.5)(0.49) - 0.5(0.51) \\ \quad + \frac{1}{\Gamma(0.5)} \int_{0.5}^t (t-s)^{-0.5}(0.4(0.49) - 0.8(0.51)^2)ds \\ \quad = 0.5 - 2(0.01192)\sqrt{\frac{0.45}{\pi}} + 1.1(0.5)(0.49) - 0.5(0.51) \\ \quad -2(0.01192)\sqrt{\frac{t-0.5}{\pi}} = 0.526477 - 0.02384\sqrt{\frac{t-0.5}{\pi}}, \quad t \in [0.5, 1]. \end{cases}$$

hold.

Applying the first iteration scheme we obtain the first approximations to the mild solution of (5.1):

$$v^{(1)}(t) = \begin{cases} 0.5 + \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5}(0.4(0.49) - 0.8(0.51)^2)ds = 0.5 - 0.02416\sqrt{\frac{t}{\pi}}, \\ \quad t \in [0, 0.45], \\ 0.5 - 2(0.01208)\sqrt{\frac{0.45}{\pi}} + 1.1t(0.49) - 0.5(0.51) = 0.235856 + 0.539t, \\ \quad t \in (0.45, 0.5], \\ 0.5 - 2(0.01208)\sqrt{\frac{0.45}{\pi}} + 1.1(0.5)(0.49) - 0.5(0.51) \\ \quad + \frac{1}{\Gamma(0.5)} \int_{0.5}^t (t-s)^{-0.5}(0.4(0.49) - 0.8(0.51)^2)ds \\ \quad = 0.505356 - 0.02416\sqrt{\frac{t}{\pi}}, \quad t \in [0.5, 1]. \end{cases}$$

and

$$w^{(1)}(t) = \begin{cases} 0.5 + \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5}(0.4(0.51) - 0.8(0.49)^2)ds = 0.5 - 0.02384\sqrt{\frac{t}{\pi}}, \\ \quad t \in [0, 0.45], \\ 0.5 - 2(0.01192)\sqrt{\frac{0.45}{\pi}} + 1.1t(0.51) - 0.5(0.49) = 0.245977 + 0.561t, \\ \quad t \in (0.45, 0.5], \\ 0.5 - 2(0.01192)\sqrt{\frac{0.45}{\pi}} + 1.1(0.5)(0.51) - 0.5(0.49) \\ \quad + \frac{1}{\Gamma(0.5)} \int_{0.5}^t (t-s)^{-0.5}(0.4(0.51) - 0.8(0.49)^2)ds, \quad t \in [1, 1]. \end{cases}$$

The inequalities $v(t) \leq v^{(1)}(t)$ and $w(t) \leq w^{(1)}(t)$ for $t \in [0, T]$, hold. All conditions of Theorem 4.4 are satisfied and the algorithm (first iteration scheme) could be applied to obtain approximately the solution of the IVP for NIFrDE (5.1). $\square$

6. Conclusions

This paper considers initial value problem for a nonlinear scalar Caputo fractional differential equation with non-instantaneous impulses. The mild solution for the non-instantaneous impulsive fractional equation is defined. This definition is based on

the values of the right hand side of the fractional differential equation only over the intervals without non-instantaneous impulses. We show this definition generalizes the case of instantaneous impulses as well as the case without impulses. Several types of lower and upper solutions to the initial value problem of the equation are presented. Several iterative techniques combined with the method of lower and upper solutions are applied to construct approximately the solution of the problem. In all of these schemes the elements of the sequences of successive approximations are mild solutions of suitably chosen initial value problems for scalar linear non-instantaneous impulsive Caputo fractional differential equations (whose solutions can be obtained in an explicit form). The convergence is proved. A non-instantaneous impulsive fractional generalization of the logistic model is given to illustrate the procedure.

Acknowledgements

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