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THREE SOLUTIONS FOR A CLASS OF NONHOMOGENEOUS NONLOCAL SYSTEMS: AN ORLICZ-SOBOLEV SPACE SETTING

MARTIN BOHNER, GIUSEPPE CARISTI,
SHAPOUR HEIDARKHANI, AND AMJAD SALARI

Missouri S&T, Department of Mathematics and Statistics,
Rolla, MO 65409-0020, USA, bohner@mst.edu
University of Messina, Department of Economics, Messina, Italy,
gcaristi@unime.it
Razi University, Department of Mathematics, Faculty of Sciences, 67149
Kermanshah, Iran, s.heidarkhani@razi.ac.ir, amjads45@yahoo.com

ABSTRACT. In this work, we investigate the existence of multiple solutions for a class of nonhomogeneous nonlocal systems via variational methods and critical point theory. We give a new criteria for guaranteeing that the nonhomogeneous nonlocal systems with a perturbed term have at least three solutions in an appropriate Orlicz-Sobolev space. By presenting two examples we illustrate the results.

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This Paper is Dedicated to Professor Ravi P. Agarwal
on the Occasion of His 70th Birthday

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, ν be the outer unit normal to $\partial\Omega$, $M_i : \mathbb{R}^+ \to \mathbb{R}$ be nondecreasing continuous functions for $i = 1, \ldots, n$, $\alpha_i : (0, \infty) \to \mathbb{R}$ be such that the mappings $\varphi_i : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_i(t) = \begin{cases} \alpha_i(|t|)t & \text{for} \quad t \neq 0, \\ 0 & \text{for} \quad t = 0 \end{cases}$$

are odd and strictly increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, and

$$\Phi_i(t) = \int_0^t \varphi_i(s)\mathrm{d}s \quad \text{for all} \quad t \in \mathbb{R}$$

for $i = 1, \ldots, n$, on which will be imposed some suitable assumptions later.

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In this paper, we study the nonhomogeneous nonlocal system

$$(N_{\lambda,\mu}) \qquad \begin{cases} M_i\left(\int_\Omega \Phi_i(|\nabla u_i|) + \Phi_i(|u_i|)\mathrm{d}x\right) \\ \quad \times (-\operatorname{div}(\alpha_i(|\nabla u_i|)\nabla u_i) + \alpha_i(|u_i|)u_i) \\ \quad = \lambda F_{u_i}(x, u_1, \ldots, u_n) + \mu G_{u_i}(x, u_1, \ldots, u_n) & \text{in } \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

for $i = 1, \ldots, n$, where $F, G : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ are measurable with respect to x, for all $\xi \in \mathbb{R}^N$, continuously differentiable in ξ, for almost every $x \in \overline{\Omega}$ and satisfy the standard summability condition

$$\sup_{|\xi| \le \varrho_1} \left(\max\{|F(\cdot,\xi)|,\ |G(\cdot,\xi)|,\ |F_{\xi_i}(\cdot,\xi)|,\ |G_{\xi_i}(\cdot,\xi)|,\ i = 1,\ldots,n\}\right) \in \mathrm{L}^1(\overline{\Omega}) \tag{1.1}$$

for any $\varrho_1 > 0$ with $\xi = (\xi_1, \ldots, \xi_n)$ and $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$, and

$$F(x, 0, \ldots, 0) = G(x, 0, \ldots, 0) = 0 \quad \text{for a.e.} \quad x \in \overline{\Omega}, \tag{1.2}$$

F_{u_i} and G_{u_i} denote the partial derivatives of F and G with respect to u_i, respectively, $\lambda > 0$ and $\mu \ge 0$ are two parameters.

It should be mentioned that if $\varphi_i(t) = p_i|t|^{p_i-2}t$ for $i = 1, \ldots, n$, then $(N_{\lambda,\mu})$ becomes the well-known $(p_1, \ldots, p_n)$-Kirchhoff-type Neumann system

$$\begin{cases} M_i\Big(\int_\Omega (|\nabla u_i|^{p_i} + |u_i|^{p_i})\mathrm{d}x\Big)\Big(-\Delta_{p_i} u_i + |u_i|^{p_i-2}u_i\Big) \\ \quad = \lambda F_{u_i}(x, u_1, \ldots, u_n) + \mu G_{u_i}(x, u_1, \ldots, u_n) & \text{in } \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

for $i = 1, \ldots, n$.

System (1.3) is related to the stationary problem

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\Big)\frac{\partial^2 u}{\partial x^2} = 0,$$

for $0 < x < L$, $t \ge 0$, where $u = u(x, t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, ρ the mass density, h the cross-section area, L the length and ρ_0 the initial axial tension, proposed by Kirchhoff [16] as an extension of the classical d'Alembert wave equation for free vibrations of elastic strings. Since the equations including the functions M_i depend on integrals over Ω in (1.3), they are no longer pointwise identities, and therefore they are often called nonlocal systems.

Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [2,7]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems, where u describes a process which depends on the average of itself, as for example, the population density. They received great attention only after

Lions [20] proposed an abstract framework for the problem. Solvability of Kirchhoff-type problems was extensively studied by various authors. Some early classical investigations of Kirchhoff equations can be seen in the papers [12, 25] and the references therein.

We point out the fact that if $n = 1$ and $M_1(t) = 1$ for all $t \in \mathbb{R}^+$, then $(N_{\lambda,\mu})$ becomes the nonhomogeneous Neumann problem

$$\begin{cases} -\operatorname{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u = \lambda f(x,u) + \mu g(x,u) & \text{in}\,\Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on}\,\partial\Omega, \end{cases} \tag{1.4}$$

where $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions.

It is well known that quasilinear elliptic partial differential equations involving nonhomogeneous differential operators are important in applications in many fields, such as elasticity, fluid dynamics, calculus of variations, nonlinear potential theory, the theory of quasi-conformal mappings, differential geometry, geometric function theory, probability theory and image processing (for instance, see [13, 22, 26]). The study of nonlinear elliptic equations involving quasilinear homogeneous-type operators is based on the theory of Sobolev spaces $\mathrm{W}^{m,p}(\Omega)$ in order to find weak solutions. In the case of nonhomogeneous differential operators, the natural setting for this approach is the use of Orlicz-Sobolev spaces. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions. Many properties of Orlicz-Sobolev spaces are given in [1]. Existence of solutions for problems associated to nonhomogeneous differential operators in Orlicz-Sobolev space has been studied by means of variational techniques, monotone operator methods, fixed point theory and degree theory (see [3–6, 8, 9, 11, 14, 15, 18, 23, 28]). Clément et al., in [11], discussed the existence of weak solutions in an Orlicz-Sobolev space to the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\alpha(|\nabla u(x)|)\nabla u(x)) = g(x,u(x)) & \text{in}\,\Omega, \\ u = 0 & \text{on}\,\partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $g \in \mathrm{C}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, and the function $\varphi(s) = sa(|s|)$ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Under appropriate conditions on φ, g and the Orlicz-Sobolev conjugate Φ^* of $\Phi(s) = \int_0^s \varphi(t)\mathrm{d}t$, they investigated the existence of nontrivial solutions of mountain pass type. Kristály et al., in [18], by using a recent variational principle of Ricceri, ensured the existence of at least two nontrivial solutions for (1.4) in the case $\mu = 0$ in the Orlicz-Sobolev space $\mathrm{W}^1\mathrm{L}_\Phi(\Omega)$. In [3–5], Bonanno et al., based on variational methods, discussed the existence of multiple solutions in the Orlicz-Sobolev space $\mathrm{W}^1\mathrm{L}_\Phi(\Omega)$ for (1.4) in the case $\mu = 0$. In [6], Cammaroto and Vilasi continued within the framework of Orlicz-Sobolev spaces and guaranteed through variational arguments the existence of three weak solutions to the nonhomogeneous boundary value problem

$$\begin{cases} \operatorname{div}(\alpha(|\nabla u|)\nabla u) = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, λ and μ are two positive parameters and the function $t \to \alpha(|t|)t$ is an odd and increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. They also presented applications and comparisons. Yang, in [28], by using variational methods and three critical point theorems due to Ricceri, investigated the existence of multiple solutions for (1.4) in an appropriate Orlicz-Sobolev space. Chung, in [8], using variational methods, studied the existence of multiple solutions for nonhomogeneous nonlocal problems. In [15], based on variational methods for smooth functionals defined on Orlicz-Sobolev spaces, the existence of three distinct weak solutions for perturbed Kirchhoff-type nonhomogeneous Neumann problems was established under suitable assumptions on the nonlinear terms.

To the best of our knowledge, for nonhomogeneous Neumann problems, there has so far been few papers concerning their multiple solutions.

Motivated by the above facts, in this paper, we establish a new criterion for guaranteeing that the nonhomogeneous nonlocal system $(N_{\lambda,\mu})$ has at least three weak solutions in an Orlicz-Sobolev space for appropriate values of the parameters λ and μ belonging to real intervals. It is clear that this is a natural extension of the earlier studies on Kirchhoff-type problems in classical Sobolev spaces and on nonlinear nonhomogeneous problems in Orlicz-Sobolev spaces. Our approach is based on variational methods and a three critical points theorem due to Ricceri [24].

2. **PRELIMINARIES**

We first recall some basic facts about Orlicz-Sobolev spaces. Let φ_i and Φ_i for $i = 1, \ldots, n$ be as introduced at the beginning of the paper. Set

$$\Phi_i^\star(t) = \int_0^t \varphi_i^{-1}(s)\mathrm{d}s \quad \text{for all} \quad t \in \mathbb{R},\ i = 1, \ldots, n.$$

We note that Φ_i is a *Young function*, that is, $\Phi_i(0) = 0$, Φ_i is convex, and

$$\lim_{t\to\infty} \Phi_i(t) = \infty$$

for $i = 1, \ldots, n$. Furthermore, since $\Phi_i(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0} \frac{\Phi_i(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi_i(t)}{t} = \infty,$$

and Φ_i is called an N-function for $i = 1, \ldots, n$. The function $\Phi_i^\star$ is called the *complementary* function of Φ_i, and it satisfies

$$\Phi_i^\star(t) = \sup\{st - \Phi_i(s) :\ s \geq 0\} \quad \text{for all} \quad t \geq 0, \quad i = 1, \ldots, n.$$

We see that $\Phi_i^\star$ is also an N-function satisfying the Young inequality

$$st \le \Phi_i(s) + \Phi_i^\star(t) \quad \text{for all} \quad s,t \ge 0, \quad i = 1, \dots, n.$$

Throughout this article, we assume

$$(\Phi_0) \qquad 1 < \liminf_{t\to\infty} \frac{t\varphi_i(t)}{\Phi_i(t)} \le (p_i)^0 := \sup_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} < \infty,$$

and

$$(\Phi_1) \qquad N < (p_i)_0 := \inf_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} < \liminf_{t\to\infty} \frac{\log(\Phi_i(t))}{\log(t)}$$

for $i = 1, \dots, n$.

The Orlicz space $\mathrm{L}_{\Phi_i}(\Omega)$ defined by the N-function Φ_i (see for instance [1,17]) is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{\mathrm{L}_{\Phi_i}} := \sup\left\{ \int_\Omega u(x)v(x)\mathrm{d}x : \int_\Omega \Phi_i^\star(|v(x)|)\mathrm{d}x \le 1 \right\} < \infty$$

for $i = 1, \dots, n$. Then $(\mathrm{L}_{\Phi_i}(\Omega), \|\cdot\|_{\mathrm{L}_{\Phi_i}})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_{\Phi_i} := \inf\left\{ k > 0 : \int_\Omega \Phi_i\left(\frac{u(x)}{k}\right)\mathrm{d}x \le 1 \right\}$$

for $i = 1, \dots, n$.

The Orlicz-Sobolev space $\mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega)$ building upon the Orlicz space $\mathrm{L}_{\Phi_i}(\Omega)$ is the space defined by

$$\mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega) = \left\{ u \in \mathrm{L}_{\Phi_i}(\Omega) : \frac{\partial u}{\partial x_j} \in \mathrm{L}_{\Phi_i}(\Omega),\ j = 1, \dots, N \right\}$$

for $i = 1, \dots, n$, and it is a Banach space with respect to the norm

$$\|u\|_{1,\Phi_i} = \||\nabla u|\|_{\Phi_i} + \|u\|_{\Phi_i}, \quad i = 1, \dots, n$$

(see [1,11]).

Hypothesis (Φ_0) is equivalent to the fact that Φ_i and Φ_i^* both satisfy the Δ_2-condition (at infinity), i.e.,

$$(2.1) \qquad \Phi_i(2t) \le K\Phi_i(t) \quad \text{for all} \quad t \ge 0, \quad i = 1, \dots, n,$$

where K is a positive constant (see [1, page 232] and [23, Proposition 2.3]). In particular, (Φ_i, Ω) and (Φ_i^*, Ω) for $i = 1, \dots, n$ are Δ-regular [1, page 232]. Consequently, the spaces $\mathrm{L}_{\Phi_i}(\Omega)$ and $\mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega)$ for $i = 1, \dots, n$ are separable and reflexive Banach spaces [1, pages 241, 247].

Furthermore, we assume that Φ_i satisfies the condition

$$(\Phi_2) \qquad \text{the function } [0,\infty) \ni t \to \Phi_i(\sqrt{t}) \text{ is convex}$$

for $i = 1, \dots, n$.

Remark 2.1. Using [11, Lemma D.2], it follows that $\mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega)$ is continuously embedded in $\mathrm{W}^{1,(p_i)_0}(\Omega)$ for $i = 1, \ldots, n$. On the other hand, since we assume that $(p_i)_0 > N$, we deduce that $\mathrm{W}^{1,(p_i)_0}(\Omega)$ is compactly embedded in $\mathrm{C}(\overline{\Omega})$ for $i = 1, \ldots, n$. Thus, we deduce that there exist constants $c_i > 0$ such that

$$\|u\|_\infty \leq c_i \, \|u\|_{1,\Phi_i} \quad \text{for all} \quad u \in \mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega), \tag{2.2}$$

where $\|u\|_\infty := \sup_{x\in\overline{\Omega}} |u(x)|$ for $i = 1, \ldots, n$. A concrete estimation of a concrete upper bound for the constants c_i remains an open question.

We recall the following useful properties regarding the norms on Orlicz-Sobolev spaces.

Lemma 2.2 (See [18, Lemma 2.2]). *On* $\mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega)$, *the norms*

$$\|u\|_{1,\Phi_i} = \||\nabla u|\|_{\Phi_i} + \|u\|_{\Phi_i},$$
$$\|u\|_{2,\Phi_i} = \max\{\||\nabla u|\|_{\Phi_i}, \|u\|_{\Phi_i}\},$$
$$\|u\|_i = \inf\left\{\mu > 0 : \int_\Omega \left[\Phi_i\left(\frac{|u(x)|}{\mu}\right) + \Phi_i\left(\frac{|\nabla u(x)|}{\mu}\right)\right] \mathrm{d}x \leq 1\right\}$$

are equivalent. More precisely, for every $u \in \mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega)$, *we have*

$$\|u\|_i \leq 2\|u\|_{2,\Phi_i} \leq 2\|u\|_{1,\Phi_i} \leq 4\|u\|_i. \tag{2.3}$$

Lemma 2.3 (See [18, Lemma 2.3] and [15, Lemma 2.4]). *If* $u \in \mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega)$, *then*

$$\int_\Omega [\Phi_i(|u(x)|) + \Phi_i(|\nabla u(x)|)]\mathrm{d}x \geq \|u\|_i^{(p_i)^0} \quad \text{if } \|u\|_i < 1, \quad i = 1, \ldots, n,$$
$$\int_\Omega [\Phi_i(|u(x)|) + \Phi_i(|\nabla u(x)|)]\mathrm{d}x \geq \|u\|_i^{(p_i)_0} \quad \text{if } \|u\|_i > 1, \quad i = 1, \ldots, n,$$
$$\int_\Omega [\Phi_i(|u(x)|) + \Phi_i(|\nabla u(x)|)]\mathrm{d}x \leq \|u\|_i^{(p_i)_0} \quad \text{if } \|u\|_i < 1, \quad i = 1, \ldots, n,$$
$$\int_\Omega [\Phi_i(|u(x)|) + \Phi_i(|\nabla u(x)|)]\mathrm{d}x \leq \|u\|_i^{(p_i)^0} \quad \text{if } \|u\|_i > 1, \quad i = 1, \ldots, n.$$

Lemma 2.4 (See [3, Lemma 2.2]). *Let* $u \in \mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega)$. *If*

$$\int_\Omega [\Phi_i(|u(x)|) + \Phi_i(|\nabla u(x)|)]\mathrm{d}x \leq r$$

for some $0 < r < 1$, *then* $\|u\|_i < 1$.

Lemma 2.5 (See [15, Lemma 2.6]). *Let* $u \in \mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega)$. *If* $\|u\|_i = 1$, *then*

$$\int_\Omega [\Phi_i(|u(x)|) + \Phi_i(|\nabla u(x)|)]\mathrm{d}x = 1.$$

In what follows, E will denote the Cartesian product of the Orlicz-Sobolev spaces $\mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega)$, ..., $\mathrm{W}^1\mathrm{L}_{\Phi_n}(\Omega)$, i.e., $E = \prod_{i=1}^n \mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega)$, endowed with the norm

$$\|u\| = \sum_{i=1}^n \|u_i\|_i,$$

where $u = (u_1, \dots, u_n)$ and $\|u_i\|_i$ is the norm of $\mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega)$ for $i = 1, \dots, n$.

Now we assume that M_i satisfies the condition

(M_0) there exist $m_i > 0$ and $1 < a_i < \infty$ with $M_i(t) \geq m_i t^{a_i - 1}$ for all $t \geq 0$

for $i = 1, \dots, n$.

In the sequel, we set

$$\underline{m} := \min\{m_i,\ i = 1, \dots, n\}, \quad \overline{m} := \max\{m_i,\ i = 1, \dots, n\},$$
$$\underline{a} = \min\{a_i,\ i = 1, \dots, n\}, \quad \overline{a} := \max\{a_i,\ i = 1, \dots, n\}$$

and

$$\underline{p_0} := \min\{(p_i)_0,\ i = 1, \dots, n\}, \quad \overline{p_0} := \max\{(p_i)_0,\ i = 1, \dots, n\}.$$

For a real Banach space X, denote by $\mathcal{W}_X$ the class of all functionals $J : X \to \mathbb{R}$ possessing the following property: If $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty} J(u_n) \leq J(u)$, then $\{u_n\}$ has a subsequence converging strongly to u.

For example, if X is uniformly convex and $h : [0, \infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then, by a classical result, the functional $u \to h(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Our main tool is the following result obtained by Ricceri (see [24, Theorem 2]).

Theorem 2.6. *Let X be a separable and reflexive real Banach space, $J : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1-functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*, $I : X \to \mathbb{R}$ be a C^1-functional with compact derivative. Assume that J has a strict local minimum u_0 with $J(u_0) = I(u_0) = 0$. Finally, setting*

$$\rho = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{I(u)}{J(u)}, \limsup_{u\to u_0} \frac{I(u)}{J(u)}\right\},$$
$$\sigma = \sup_{u\in J^{-1}((0,\infty[)} \frac{I(u)}{J(u)},$$

assume that $\rho < \sigma$. Then for each compact interval $[c, d] \subset (\frac{1}{\sigma}, \frac{1}{\rho})$ (with the conventions $\frac{1}{0} = +\infty$, $\frac{1}{+\infty} = 0$), there exists $\Lambda > 0$ with the following property: For every

$\lambda \in [c,d]$ and every C^1-functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$,

$$J'(u) = \lambda I'(u) + \mu \Psi'(u)$$

has at least three solutions in X whose norms are less than Λ.

We refer the reader to the papers [6,27,28] in which Theorem 2.6 was successfully employed to ensure the existence of at least three solutions for nonhomogeneous problems.

Put

$$\widehat{M_i}(t) = \int_0^t M_i(s)\mathrm{d}s, \quad t \geq 0, \quad i = 1, \ldots, n. \tag{2.4}$$

For every $u = (u_1, \ldots, u_n) \in E$, we define the functionals $\omega_i, J, I, \Psi : E \to \mathbb{R}$ by

$$\omega_i(u_i) = \int_\Omega \left[\Phi_i(|u_i(x)|) + \Phi_i(|\nabla u_i(x)|)\right] \mathrm{d}x, \quad i = 1, \ldots, n, \tag{2.5}$$

$$J(u) = \sum_{i=1}^n \widehat{M_i}\left(\omega_i(u_i)\right), \tag{2.6}$$

$$I(u) = \int_\Omega F(x, u_1(x), \ldots, u_n(x))\mathrm{d}x \tag{2.7}$$

and

$$\Psi(u) = \int_\Omega G(x, u_1(x), \ldots, u_n(x))\mathrm{d}x. \tag{2.8}$$

For every $u \in E$, set

$$\Gamma_{\lambda,\mu}(u) := J(u) - \lambda I(u) - \mu \Psi(u).$$

Standard arguments show that $\Gamma_\lambda \in \mathrm{C}^1(E, \mathbb{R})$. In fact, one has

$$\begin{aligned}
\Gamma'_{\lambda,\mu}(u)(v) &= \lim_{h \to 0} \frac{\Gamma_{\lambda,\mu}(u + hv) - \Gamma_{\lambda,\mu}(u)}{h} \\
&= \sum_{i=1}^n M_i\left(\omega_i(u_i)\right) \int_\Omega \left(\alpha_i(|\nabla u_i(x)|)\nabla u_i(x) \cdot \nabla v_i(x) + \alpha_i(|u_i(x)|)u_i(x)v_i(x)\right) \mathrm{d}x \\
&\quad -\lambda \sum_{i=1}^n \int_\Omega F_{u_i}(x, u_1(x), \ldots, u_n(x))v_i(x)\mathrm{d}x \\
&\quad -\mu \sum_{i=1}^n \int_\Omega G_{u_i}(x, u_1(x), \ldots, u_n(x))v_i(x)\mathrm{d}x
\end{aligned}$$

for all $u, v \in E$ (see [18] for more details).

A function $u = (u_1, \dots, u_n) \in E$ is a weak solution for $(N_{\lambda,\mu})$ if

$$\begin{aligned}
\sum_{i=1}^{n} M_i & \left(\int_\Omega [\Phi_i(|u_i(x)|) + \Phi_i(|\nabla u_i(x)|)]\, \mathrm{d}x \right) \\
& \times \int_\Omega (\alpha_i(|\nabla u_i(x)|)\nabla u_i(x) \cdot \nabla v_i(x) + \alpha_i(|u_i(x)|)u_i(x)v_i(x))\, \mathrm{d}x \\
& = \lambda \sum_{i=1}^{n} \int_\Omega F_{u_i}(x, u_1(x), \dots, u_n(x))v_i(x)\mathrm{d}x \\
& \quad + \mu \sum_{i=1}^{n} \int_\Omega G_{u_i}(x, u_1(x), \dots, u_n(x))v_i(x)\mathrm{d}x
\end{aligned}$$

for every $v = (v_1, \dots, v_n) \in E$.

We use the following proposition in the proof of our main result.

Proposition 2.7. *Let $S : E \to E^*$ be the operator defined by*

$$S(u)(v) = \sum_{i=1}^{n} M_i\left(\omega_i(u_i)\right) \int_\Omega (\alpha_i(|\nabla u_i(x)|)\nabla u_i(x) \cdot \nabla v_i(x) + \alpha_i(|u_i(x)|)u_i(x)v_i(x))\, \mathrm{d}x$$

for every $u, v \in E$. Then, S admits a continuous inverse on E^*.

Proof. For any $u = (u_1, \dots, u_n) \in E$ with $\|u_i\|_i > 1$, $i = 1, \dots, n$, by (M_0) and Lemma 2.3, one has

$$\begin{aligned}
S(u)(u) &= \sum_{i=1}^{n} M_i\left(\omega_i(u_i)\right) \omega_i(u_i) \geq \sum_{i=1}^{n} m_i\, \omega_i(u_i)^{a_i} \\
&\geq \sum_{i=1}^{n} m_i \|u_i\|_i^{(p_i)_0 a_i} \geq \underline{m} \sum_{i=1}^{n} \|u_i\|_i^{(p_i)_0 a_i}.
\end{aligned}$$

It follows that S is coercive. Now let $u, v \in E$ with $u \neq v$ and $t_1, t_2 \in [0, 1]$ with $t_1 + t_2 = 1$. Note that, since the function φ_i is increasing in $\mathbb{R}$, we have

$$(\varphi_i(|\xi|) - \varphi_i(|\eta|))(|\xi| - |\eta|) \geq 0 \quad \text{for all} \quad \xi, \eta \in \mathbb{R},$$

with equality if and only if $\xi = \eta$ for $i = 1, \dots, n$. Thus, for all $\xi, \eta \in \mathbb{R}$,

$$(\alpha_i(|\xi|)|\xi| - \alpha_i(|\eta|)|\eta|)(|\xi| - |\eta|) \geq 0 \quad \text{for all} \quad \xi, \eta \in \mathbb{R},$$

with equality if and only if $\xi = \eta$ for $i = 1, \dots, n$. On the other hand, simple calculations show that for all $\xi, \eta \in \mathbb{R}$,

$$(\alpha_i(|\xi|)|\xi| - \alpha_i(|\eta|)|\eta|) \cdot (\xi - \eta) \geq (\alpha_i(|\xi|)|\xi| - \alpha_i(|\eta|)|\eta|)(|\xi| - |\eta|)$$

for $i = 1, \dots, n$. Consequently, we conclude that for all $\xi, \eta \in \mathbb{R}$,

$$(\alpha_i(|\xi|)|\xi| - \alpha_i(|\eta|)|\eta|) \cdot (\xi - \eta) \geq 0$$

with equality if and only if $\xi = \eta$ for $i = 1, \dots, n$. This shows that the operator $\omega_i' : \mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega) \to (\mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega))^*$ given by

$$\omega_i'(u_i)v_i = \int_\Omega (\alpha_i(|\nabla u_i(x)|)\nabla u_i(x) \cdot \nabla v_i(x) + \alpha_i(|u_i(x)|)u_i(x)v_i(x))\,\mathrm{d}x$$

is strictly monotone, so by [29, Proposition 25.10], ω_i is strictly convex for $i = 1, \dots, n$. Moreover, since M_i is nondecreasing, the function $\widehat{M_i}$ is convex in $[0, \infty)$ for $i = 1, \dots, n$. Thus, we have

$$\widehat{M_i}(\omega_i(t_1u_i + t_2v_i)) < \widehat{M_i}(t_1\omega_i(u_i) + t_2\omega_i(u_i)) \leq t_1\widehat{M_i}(\omega_i(u_i)) + t_2\widehat{M_i}(\omega_i(u_i))$$

for $i = 1, \dots, n$. This shows that the operator $S_i : \mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega) \to (\mathrm{W}^1\mathrm{L}_{\Phi_i}(\Omega))^*$ defined by

$$S_i(u_i) = \widehat{M_i}\left(\omega_i(u_i)\right)$$

is strictly convex and so S_i' is strictly monotone for $i = 1, \dots, n$. Thus, since $S(u) = \sum_{i=1}^n S_i'(u_i)$, S is strictly monotone. Moreover, since E is reflexive, for $u_n \to u$ strongly in E as $n \to \infty$, one has $S(u_n) \to S(u)$ weakly in E^* as $n \to \infty$. Hence, S is hemicontinuous, so by [29, Theorem 26.A(d)], the inverse operator S^{-1} of S exists and it is bounded. Now we prove that S^{-1} is continuous by showing that it is sequentially continuous. Let $\{e_m\}$ be a sequence in E^* such that $e_m \to e$ strongly in E^* as $n \to \infty$. Let $\{u_m\} = \{(u_{1m}, \dots, u_{nm})\}$ and $u = (u_1, \dots, u_n)$ in E such that $S^{-1}(e_m) = u_m$ and $S^{-1}(e) = u$. Taking into account that S is coercive, one has that the sequence $\{u_m\}$ is bounded in the reflexive space E. For a suitable subsequence, we have $u_m \to \hat{u} = (\hat{u}_1, \dots, \hat{u}_n)$ weakly in E as $m \to \infty$, which yields

$$\lim_{m\to\infty} S(u_m)(u_m - \hat{u}) = \lim_{m\to\infty} e_m(u_m - \hat{u}) = 0,$$

so

$$\lim_{m\to\infty} \sum_{i=1}^n M_i\left(\omega_i(u_{im})\right) \int_\Omega \Big(\alpha_i(|\nabla u_{im}|)\nabla u_{im} \cdot (\nabla u_{im} - \nabla \hat{u}_i) + \alpha_i(|u_{im}|)u_{im}(u_{im} - \hat{u}_i)\Big)\mathrm{d}x = 0. \tag{2.9}$$

Using Lemma 2.3, since $\{u_m\}$ is bounded in E, by passing to a subsequence if necessary, we may assume that

$$\omega_i(u_{im}) \to s_i \geq 0 \quad \text{as} \quad m \to \infty, \quad i = 1, \dots, n.$$

If $s_i = 0$, $i = 1, \dots, n$, then, by Lemma 2.3, $\{u_m\}$ converges strongly to $\hat{u} = (0, \dots, 0)$ in E. Hence, taking into account that S is a continuous injection, we have $u = (0, \dots, 0)$, and the proof is finished. If there exists $i \in \{1, \dots, n\}$ such that $s_i > 0$, then, by continuity of the functions M_i, $i = 1, \dots, n$, we have

$$\sum_{i=1}^n M_i\left(\omega_i(u_{im})\right) \to \sum_{i=1}^n M_i\left(s_i\right) \quad \text{as} \quad m \to \infty.$$

Thus, by (M_0), there exists a constant D such that

$$\sum_{i=1}^{n} M_i\left(\omega_i(u_{im})\right) \geq D > 0. \tag{2.10}$$

From (2.9) and (2.10), it follows that

$$\lim_{m\to\infty} \sum_{i=1}^{n} \int_\Omega \Big(\alpha_i(|\nabla u_{im}|)\nabla u_{im} \cdot (\nabla u_{im} - \nabla \hat{u}_i) + \alpha_i(|u_{im}|)u_{im}(u_{im} - \hat{u}_i)\Big)\mathrm{d}x = 0. \tag{2.11}$$

From (2.11) and the fact that $\{u_m\}$ converges weakly to $\hat{u}$ in E, we can apply [21, Lemma 5] in order to infer that $\{u_m\}$ converges strongly to $\hat{u}$ in E. Hence, taking into account that S is a continuous injection, we have $u = \hat{u}$. □

3. MAIN RESULTS

In this section, we formulate our main results. Let us denote by $\mathcal{F}$ the class of all functions $F : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ that are measurable with respect to x, for all $\xi \in \mathbb{R}^N$, continuously differentiable in ξ, for almost every $x \in \overline{\Omega}$, and satisfy (1.1) and (1.2). Put

$$\lambda_1 = \inf\left\{ \frac{\sum_{i=1}^n \widehat{M}_i\left(\omega_i(u_i)\right)}{2\int_\Omega F(x,u(x))\mathrm{d}x} : \ u \in E,\ \int_\Omega F(x,u(x))\mathrm{d}x > 0 \right\}$$

and

$$\lambda_2 = \left(\max\left\{ 0, \limsup_{|u|\to 0} \frac{2\int_\Omega F(x,u(x))\mathrm{d}x}{\sum_{i=1}^n \widehat{M}_i\left(\omega_i(u_i)\right)}, \limsup_{\|u\|\to\infty} \frac{2\int_\Omega F(x,u(x))\mathrm{d}x}{\sum_{i=1}^n \widehat{M}_i\left(\omega_i(u_i)\right)} \right\} \right)^{-1},$$

where $u = (u_1, \ldots, u_n)$.

Theorem 3.1. *Suppose that $F \in \mathcal{F}$. Assume that the following conditions hold:*

$(\mathcal{A}_1)$ *There exists a constant $\varepsilon > 0$ such that*

$$\max\left\{ \limsup_{\xi\to(0,\ldots,0)} \frac{\sup_{x\in\Omega} F(x,\xi)}{\sum_{i=1}^n |\xi_i|^{a_i\,(p_i)_0}}, \limsup_{|\xi|\to\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{\sum_{i=1}^n |\xi_i|^{a_i\,(p_i)_0}} \right\} < \varepsilon,$$

where $\xi = (\xi_1, \ldots, \xi_n)$ with $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$.

$(\mathcal{A}_2)$ *There exists a function $w = (w_1, \ldots, w_n) \in E$ such that $\sum_{i=1}^n \widehat{M}_i\left(\omega_i(w_i)\right) \neq 0$ and*

$$2\overline{a}\varepsilon \max\{c_1^{a_1(p_1)_0}, \ldots, c_n^{a_n(p_n)_0}\} < \frac{\underline{m} \int_\Omega F(x,w(x))\mathrm{d}x}{\mathrm{meas}(\Omega)\sum_{i=1}^n \widehat{M}_i\left(\omega_i(w_i)\right)}.$$

Then, for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$, there exists $\Lambda > 0$ with the following property: For every $\lambda \in [c,d]$ and every $G \in \mathcal{F}$, there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the system $(N_{\lambda,\mu})$ has at least three weak solutions whose norms in E are less than Λ.

Proof. Take $X = E$. Clearly, X is a separable and reflexive Banach space. Let the functionals J, I and Ψ be as given in (2.6), (2.7) and (2.8), respectively. The functional J is C^1, and due to Proposition 2.7, its derivative admits a continuous inverse on X^*. Moreover, J is sequentially weakly lower semicontinuous in X. Indeed, let $\{u_m\} = \{(u_{1m}, \ldots, u_{nm})\} \subset X$ be a sequence that converges weakly to $u = (u_1, \ldots, u_n)$ in X. By [23, Lemma 4.3], we conclude that the functionals

$$u_i \to \omega_i(u_i) = \int_\Omega [\Phi_i(|u_i(x)|) + \Phi_i(|\nabla u_i(x)|)]\mathrm{d}x, \quad i = 1, \ldots, n$$

are weakly lower semi-continuous, i.e.,

$$\begin{aligned} &\int_\Omega [\Phi_i(|u_i(x)|) + \Phi_i(|\nabla u_i(x)|)]\mathrm{d}x \\ &\qquad \le \liminf_{m\to\infty} \int_\Omega [\Phi_i(|u_{im}(x)|) + \Phi_i(|\nabla u_{im}(x)|)]\mathrm{d}x, \quad i = 1, \ldots, n. \end{aligned} \tag{3.1}$$

Thus, by (3.1) and continuity and monotonicity of the functions $t \mapsto \widehat{M}_i(t)$, $i = 1, \ldots, n$, we get

$$\begin{aligned} \liminf_{m\to\infty} J(u_m) &= \liminf_{m\to\infty} \sum_{i=1}^n \widehat{M}_i \left(\int_\Omega [\Phi_i(|u_{im}(x)|) + \Phi_i(|\nabla u_{im}(x)|)]\mathrm{d}x \right) \\ &\ge \sum_{i=1}^n \liminf_{m\to\infty} \widehat{M}_i \left(\int_\Omega [\Phi_i(|u_{im}(x)|) + \Phi_i(|\nabla u_{im}(x)|)]\mathrm{d}x \right) \\ &\ge \sum_{i=1}^n \widehat{M}_i \left(\liminf_{m\to\infty} \int_\Omega [\Phi_i(|u_{im}(x)|) + \Phi_i(|\nabla u_{im}(x)|)]\mathrm{d}x \right) \\ &\ge \sum_{i=1}^n \widehat{M}_i \left(\int_\Omega [\Phi_i(|u_i(x)|) + \Phi_i(|\nabla u_i(x)|)]\mathrm{d}x \right) \\ &= J(u). \end{aligned}$$

Thus, the functional J is sequentially weakly lower semicontinuous. On the other hand, if $u \in X$ and $\|u_i\|_i > 1$, $i = 1, \ldots, n$, then, by Lemma 2.3 and (M_0), we have

$$\begin{aligned} J(u) &= \sum_{i=1}^n \widehat{M}_i \left(\int_\Omega [\Phi_i(|u_i(x)|) + \Phi_i(|\nabla u_i(x)|)]\mathrm{d}x \right) \\ &\ge \sum_{i=1}^n \frac{m_i}{a_i} \left(\int_\Omega [\Phi_i(|u_i(x)|) + \Phi_i(|\nabla u_i(x)|)]\mathrm{d}x \right)^{a_i} \qquad (3.2) \\ &\ge \sum_{i=1}^n \frac{m_i}{a_i} \|u_i\|_i^{a_i (p_i)_0} \ge \frac{\underline{m}}{\overline{a}} \sum_{i=1}^n \|u_i\|_i^{a_i (p_i)_0}. \end{aligned}$$

Hence, J is coercive. Moreover, let A be a bounded subset of X. That is, there exists a constant $l_i > 0$ such that $\|u\|_i \le l_i$ for each $u \in A$ for $i = 1, \ldots, n$. Then, we have

$$|J(u)| = \left| \sum_{i=1}^n \widehat{M}_i\left(\omega_i(u)\right) \right| \le \sum_{i=1}^n \begin{cases} |\widehat{M}_i\big(l_i^{(p_i)_0}\big)| & \text{if } \|u\|_i \le 1, \\ |\widehat{M}_i\big(l_i^{(p_i)^0}\big)| & \text{if } \|u\|_i > 1. \end{cases}$$

Hence, J is bounded on each bounded subset of X. Furthermore, $J \in \mathcal{W}_X$. Indeed, since

$$\sum_{i=1}^{n} \liminf_{m\to\infty} \widehat{M}_i(\omega_i(u_{im})) \le \liminf_{m\to\infty} \sum_{i=1}^{n} \widehat{M}_i(\omega_i(u_{im})),$$

$\widehat{M}_i$ is continuous and strictly increasing, so it suffices to show that $\omega_i \in \mathcal{W}_X$ for $i = 1, \ldots, n$. So, let $\{u_m\} = \{(u_{1m}, \ldots, u_{nm})\}$ be a sequence weakly converging to $u = (u_1, \ldots, u_n)$ in X and let $\liminf_{m\to\infty} \omega_i(u_{im}) \le \omega_i(u_i)$ for $i = 1, \ldots, n$. Since the functional ω_i is sequentially weakly lower semicontinuous, there exists a subsequence of $\{u_{im}\}$, still denoted by $\{u_{im}\}$, such that

$$\lim_{m\to\infty} \omega_i(u_{im}) = \omega_i(u_i) \tag{3.3}$$

for $i = 1, \ldots, n$. Since $\{u_m\}$ converges weakly to u, also $\left\{\frac{u_m+u}{2}\right\}$ converges weakly to u in X. Since the functionals w_i are sequentially weakly lower semicontinuous, we have

$$\liminf_{m\to\infty} \omega_i\left(\frac{u_{im}+u_i}{2}\right) \ge \omega_i(u_i) \tag{3.4}$$

for $i = 1, \ldots, n$. Now we assume by contradiction that $\{u_m\}$ does not converge to u in X. Hence, there exist $\varepsilon_i > 0$, $i = 1, \ldots, n$, such that $\|u_{im} - u_i\|_i \ge \varepsilon_i$, so $|\frac{u_{im}-u_i}{2}| \ge \frac{\varepsilon_i}{2}$. By Lemma 2.3, we have

$$\omega_i\Big(\frac{u_{im}-u_i}{2}\Big) \ge \max\left\{\varepsilon_i^{(p_i)_0}, \varepsilon_i^{(p_i)^0}\right\}$$

for $i = 1, \ldots, n$. On the other hand, by (2.1) and (Φ_2), we can apply [19, Lemma 2.1] in order to obtain

$$\frac{1}{2}\omega_i(u_{im}) + \frac{1}{2}\omega_i(u_i) - \omega_i\left(\frac{u_{im}+u_i}{2}\right) \ge \omega_i\left(\frac{u_{im}-u_i}{2}\right) \ge \max\left\{\varepsilon_i^{(p_i)_0}, \varepsilon_i^{(p_i)^0}\right\} \tag{3.5}$$

for $i = 1, \ldots, n$. From (3.3) and (3.5), we get

$$\omega_i(u_i) - \max\left\{\varepsilon_i^{(p_i)_0}, \varepsilon_i^{(p_i)^0}\right\} \ge \limsup_{m\to\infty} \omega_i\left(\frac{u_{im}+u_i}{2}\right) \tag{3.6}$$

for $i = 1, \ldots, n$. From (3.4) and (3.6), we obtain a contradiction. This shows that $\{u_m\}$ converges strongly to u and the functional J belongs to the class $\mathcal{W}_X$. The functionals I and Ψ are C^1 with compact derivatives. Moreover, J has a strict local minimum 0 with $J(0) = I(0) = 0$. In view of $(\mathcal{A}_1)$, there exist two constants τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(x,\xi) \le \varepsilon \sum_{i=1}^{n} |\xi_i|^{a_i\,(p_i)_0} \tag{3.7}$$

for every $x \in \Omega$ and every $\xi = (\xi_1, \ldots, \xi_n)$ with $|\xi| \in [0, \tau_1) \cup (\tau_2, \infty)$. By (1.1), $F(x,\xi)$ is bounded on $x \in \Omega$ and $|\xi| \in [\tau_1, \tau_2]$. So we can choose $\delta > 0$ and $\upsilon_i > a_i\,(p_i)_0$,

$i = 1, \dots, n$, in such a manner that

$$F(x, \xi) \le \varepsilon \sum_{i=1}^{n} |\xi_i|^{a_i\,(p_i)_0} + \delta \sum_{i=1}^{n} |\xi_i|^{\upsilon_i}$$

for all $(x, \xi) \in \Omega \times \mathbb{R}^n$. So, by (2.2) and (2.3), we have

$$\begin{aligned} I(u) &\le 2\,\mathrm{meas}(\Omega)\varepsilon \sum_{i=1}^{n} c_i^{a_i\,(p_i)_0} \|u_i\|_i^{a_i\,(p_i)_0} + 2\,\mathrm{meas}(\Omega)\delta \sum_{i=1}^{n} c_i^{\upsilon_i} \|u_i\|_i^{\upsilon_i} \\ &\le 2\,\mathrm{meas}(\Omega)\varepsilon \max\{c_1^{a_1\,(p_1)_0}, \dots, c_n^{a_n\,(p_n)_0}\} \sum_{i=1}^{n} \|u_i\|_i^{a_i\,(p_i)_0} \\ &\quad + 2\,\mathrm{meas}(\Omega)\delta \max\{c_1^{\upsilon_1}, \dots, c_n^{\upsilon_n}\} \sum_{i=1}^{n} \|u_i\|_i^{\upsilon_i} \end{aligned} \tag{3.8}$$

for all $u \in X$. Hence, from (3.2) and (3.8), we have

$$\limsup_{u \to (0,\dots,0)} \frac{I(u)}{J(u)} \le \frac{2\overline{a}\,\mathrm{meas}(\Omega)\varepsilon \max\{c_1^{a_1(p_1)_0}, \dots, c_n^{a_n(p_n)_0}\}}{\underline{m}}. \tag{3.9}$$

Moreover, by using (3.7), for each $u \in X \setminus \{0\}$, we obtain

$$\begin{aligned} \frac{I(u)}{J(u)} &= \frac{\int_{|u| \le \tau_2} F(x,u)\mathrm{d}x}{J(u)} + \frac{\int_{|u| > \tau_2} F(x,u)\mathrm{d}x}{J(u)} \\ &\le \frac{\mathrm{meas}(\Omega) \sup_{x \in \Omega, |u| \in [0,\tau_2]} F(x,u)}{J(u)} \\ &\quad + \frac{2\,\mathrm{meas}(\Omega)\varepsilon \max\{c_1^{a_1(p_1)_0}, \dots, c_n^{a_n(p_n)_0}\} \sum_{i=1}^{n} \|u_i\|_i^{a_i\,(p_i)_0}}{J(u)} \\ &\le \frac{\overline{a}\,\mathrm{meas}(\Omega) \sup_{x \in \Omega, |u| \in [0,\tau_2]} F(x,u)}{\underline{m} \sum_{i=1}^{n} \|u_i\|_i^{a_i(p_i)_0}} + \frac{2\overline{a}\,\mathrm{meas}(\Omega)\varepsilon \max\{c_1^{a_1(p_1)_0}, \dots, c_n^{a_n(p_n)_0}\}}{\underline{m}}. \end{aligned}$$

So, we get

$$\limsup_{\|u\| \to \infty} \frac{I(u)}{J(u)} \le \frac{2\overline{a}\,\mathrm{meas}(\Omega)\varepsilon \max\{c_1^{a_1(p_1)_0}, \dots, c_n^{a_n(p_n)_0}\}}{\underline{m}}. \tag{3.10}$$

In view of (3.9) and (3.10), we have

$$\begin{aligned} \rho &= \max\left\{0, \limsup_{\|u\| \to \infty} \frac{I(u)}{J(u)}, \limsup_{u \to (0,\dots,0)} \frac{I(u)}{J(u)}\right\} \\ &\le \frac{2\overline{a}\,\mathrm{meas}(\Omega)\varepsilon \max\{c_1^{a_1(p_1)_0}, \dots, c_n^{a_n(p_n)_0}\}}{\underline{m}}. \end{aligned} \tag{3.11}$$

Assumption $(\mathcal{A}_2)$ in conjunction with (3.11) yields

$$\begin{aligned}\sigma &= \sup_{u\in J^{-1}(0,\infty)} \frac{I(u)}{J(u)} = \sup_{X\setminus\{0\}} \frac{I(u)}{J(u)} \\ &\geq \frac{\int_\Omega F(x,w(x))\mathrm{d}x}{J(w(x))} = \frac{\int_\Omega F(x,w(x))\mathrm{d}x}{\sum_{i=1}^n \widehat{M}_i\left(\omega_i(w(x))\right)} \\ &> \frac{2\overline{a}\,\mathrm{meas}(\Omega)\varepsilon \max\{c_1^{a_1(p_1)_0},\ldots,c_n^{a_n(p_n)_0}\}}{\underline{m}} \geq \rho.\end{aligned}$$

Thus, all the hypotheses of Theorem 2.6 are satisfied. Clearly, $\lambda_1 = \frac{1}{\beta}$ and $\lambda_2 = \frac{1}{\alpha}$. Therefore, by using Theorem 2.6, for each compact interval $[c,d] \subset (\lambda_1,\lambda_2)$, there exists $\Lambda > 0$ with the following property: For every $\lambda \in [c,d]$ and every $G \in \mathcal{F}$, there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the system $(N_{\lambda,\mu})$ has at least three weak solutions whose norms in E are less than Λ. □

Another announced application of Theorem 2.6 is given next.

Theorem 3.2. *Suppose that $F \in \mathcal{F}$. Assume that*

$$\max\left\{\limsup_{\xi\to(0,\ldots,0)} \frac{\sup_{x\in\Omega} F(x,\xi)}{\sum_{i=1}^n |\xi_i|^{a_i\,(p_i)_0}}, \limsup_{|\xi|\to\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{\sum_{i=1}^n |\xi_i|^{a_i\,(p_i)_0}}\right\} \leq 0, \tag{3.12}$$

where $\xi = (\xi_1,\ldots,\xi_n)$ with $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$, and

$$\sup_{u\in E} \frac{\int_\Omega F(x,u(x))\mathrm{d}x}{\sum_{i=1}^n \widehat{M}_i\left(\omega_i(u_i)\right)} > 0. \tag{3.13}$$

Then, for each compact interval $[c,d] \subset (\lambda_1,\infty)$, there exists $\Lambda > 0$ with the following property: For every $\lambda \in [c,d]$ and every $G \in \mathcal{F}$, there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the system $(N_{\lambda,\mu})$ has at least three weak solutions whose norms in E are less than Λ.

Proof. In view of (3.12), there exist two constants τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(x,\xi) \leq \varepsilon \sum_{i=1}^n |\xi_i|^{a_i\,(p_i)_0}$$

for every $x \in \Omega$ and every $\xi = (\xi_1,\ldots,\xi_n)$ with $|\xi| \in [0,\tau_1) \cup (\tau_2,\infty)$. Since $F(x,\xi)$ is bounded on $x \in \Omega$ and $|\xi| \in [\tau_1,\tau_2]$, we can choose $\delta > 0$ and $\upsilon_i > a_i\,(p_i)_0$ for $i = 1,\ldots,n$ such that

$$F(x,\xi) \leq \varepsilon \sum_{i=1}^n |\xi_i|^{a_i\,(p_i)_0} + \delta \sum_{i=1}^n |\xi_i|^{\upsilon_i}$$

for all $(x,\xi) \in \Omega \times \mathbb{R}^n$. So, by the same process as in the proof of Theorem 3.1, we have the relations (3.9) and (3.10). Since ε is arbitrary, (3.9) and (3.10) give

$$\max\left\{0, \limsup_{\|u\|\to\infty} \frac{I(u)}{J(u)}, \limsup_{u\to(0,\ldots,0)} \frac{I(u)}{J(u)}\right\} \leq 0.$$

Then, with the notation of Theorem 2.6, we have $\rho = 0$. By (3.13), we also have $\sigma > 0$. In this case, clearly $\lambda_1 = \frac{1}{\sigma}$ and $\lambda_2 = \infty$. Thus, by using Theorem 2.6, the result is achieved. $\square$

Remark 3.3. In Assumption $(\mathcal{A}_2)$, if we choose

$$w(x) = w^\star(x) = (\delta_1, \ldots, \delta_n),$$

where $\delta_1, \ldots, \delta_n$ are positive constants, then a direct calculation shows that

$$\begin{aligned} J(w^\star) &= \sum_{i=1}^n \widehat{M}_i \left(\int_\Omega [\Phi_i(|w_i^\star(x)|) + \Phi_i(|\nabla w_i^\star(x)|)] \mathrm{d}x \right) \\ &= \sum_{i=1}^n \widehat{M}_i \left(\int_\Omega \Phi_i(\delta_i) \mathrm{d}x \right) \\ &= \operatorname{meas}(\Omega) \sum_{i=1}^n \widehat{M}_i(\Phi_i(\delta_i)). \end{aligned}$$

Then, Assumption $(\mathcal{A}_2)$ can be restated as follows:

$(\tilde{\mathcal{A}}_2)$ There exist positive constants $\delta_1, \ldots, \delta_n$ such that $\sum_{i=1}^n \widehat{M}_i(\Phi_i(\delta_i)) \neq 0$ and

$$\max\{c_1^{a_1(p_1)_0}, \ldots, c_n^{a_n(p_n)_0}\} < \frac{\underline{m}\|\theta\|_{\mathrm{L}^1(\Omega)} F(x, \delta_1, \ldots, \delta_n)}{2\overline{a}\varepsilon \operatorname{meas}(\Omega)^2 \sum_{i=1}^n \widehat{M}_i(\Phi_i(\delta_i))}.$$

4. APPLICATIONS AND EXAMPLES

Now, we point out some results in which the function F has separated variables. To be precise, consider the system

$$(N^\theta_{\lambda,\mu}) \qquad \begin{cases} M_i\left(\int_\Omega \Phi_i(|\nabla u_i|) + \Phi_i(|u_i|)\mathrm{d}x\right) \\ \quad \times \left(-\operatorname{div}(\alpha_i(|\nabla u_i|)\nabla u_i) + \alpha_i(|u_i|)u_i\right) \\ \quad = \lambda\theta(x) F_{u_i}(u_1, \ldots, u_n) + \mu G_{u_i}(x, u_1, \ldots, u_n) & \text{in } \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

for $i = 1, \ldots, n$, where $\theta : \overline{\Omega} \to \mathbb{R}$ is a nonzero function such that $\theta \in \mathrm{L}^1(\overline{\Omega})$, $F : \mathbb{R}^n \to \mathbb{R}$ is a C^1-function with $F(0, \ldots, 0) = 0$, and G is as given in $(N_{\lambda,\mu})$.

Setting $F(x, t_1, \ldots, t_n) = \theta(x) F(t_1, \ldots, t_n)$ for every $(x, t_1, \ldots, t_n) \in \overline{\Omega} \times \mathbb{R}^n$, the following existence results are consequences of Theorem 3.1.

Theorem 4.1. *Assume that the following conditions hold:*

$(\mathcal{A}_1')$ *There exists a constant* $\varepsilon > 0$ *such that*

$$\left(\sup_{x\in\Omega} \theta(x)\right) \cdot \max\left\{ \limsup_{\xi \to (0,\ldots,0)} \frac{F(\xi)}{\sum_{i=1}^n |\xi_i|^{a_i\,(p_i)_0}}, \limsup_{|\xi|\to\infty} \frac{F(\xi)}{\sum_{i=1}^n |\xi_i|^{a_i\,(p_i)_0}} \right\} < \varepsilon,$$

where $\xi = (\xi_1, \ldots, \xi_n)$ *with* $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$.

($\mathcal{A}_2'$) *There exist positive constants* $\delta_1, \dots, \delta_n$ *such that*

$$2\overline{a}\varepsilon \max\{c_1^{a_1(p_1)_0}, \dots, c_n^{a_n(p_n)_0}\} < \frac{\underline{m}\|\theta\|_{L^1(\Omega)} F(\delta_1, \dots, \delta_n)}{\operatorname{meas}(\Omega)^2 \sum_{i=1}^n \widehat{M}_i(\Phi_i(\delta_i))}.$$

Then, for each compact interval $[c,d] \subset (\lambda_3, \lambda_4)$, *where* λ_3 *and* λ_4 *are* λ_1 *and* λ_2 *with* $\int_\Omega F(x,u(x))\mathrm{d}x$ *replaced by* $\int_\Omega \theta(x)F(u(x))\mathrm{d}x$, *there exists* $\Lambda > 0$ *with the following property: For every* $\lambda \in [c,d]$ *and every* $G \in \mathcal{F}$, *there exists* $\gamma > 0$ *such that, for each* $\mu \in [0,\gamma]$, *the system* $(N^\theta_{\lambda,\mu})$ *has at least three weak solutions whose norms in* E *are less than* Λ.

The next result immediately follows from Theorem 4.1 by setting $n = 2$, $\alpha_1(|t|) = |t|^{p_1-2}$, $\alpha_2(|t|) = |t|^{p_2-2}$ for all $t > 0$ and $M_i(t) = 1$, $i = 1,2$ for all $t \in \mathbb{R}$.

Corollary 4.2. *Let* $p_1, p_2 > N$ *and* $F : \mathbb{R}^2 \to \mathbb{R}$ *be a* C^1*-function with* $F(0,0) = 0$. *Assume that there exists a positive constant* ε *such that*

$$\max\left\{ \limsup_{(\xi_1,\xi_2)\to(0,0)} \frac{F(\xi_1,\xi_2)}{|\xi_1|^{p_1} + |\xi_2|^{p_2}}, \limsup_{|(\xi_1,\xi_2)|\to\infty} \frac{F(\xi_1,\xi_2)}{|\xi_1|^{p_1} + |\xi_2|^{p_2}} \right\} < \varepsilon,$$

where $|(\xi_1,\xi_2)| = \sqrt{\xi_1^2 + \xi_2^2}$ *and there exist two positive constants* δ_1, δ_2 *such that*

$$2\varepsilon \max\{\kappa_1^{p_1}, \kappa_2^{p_2}\} \le \frac{F(\delta_1,\delta_2)}{\operatorname{meas}(\Omega)(|\delta_1|^{p_1} + |\delta_2|^{p_2})},$$

where κ_i, $i = 1,2$ *are two constants such that*

$$\|u\|_\infty \le \kappa_i \|u\|_{W^{1,p_i}(\Omega)}, \quad i = 1,2$$

for every $u \in W^{1,p_i}(\Omega)$ *and*

$$\|u\|_{W^{1,p_i}(\Omega)} := \left(\int_\Omega |\nabla u(x)|^{p_i} \mathrm{d}x + \int_\Omega |u(x)|^{p_i} \mathrm{d}x \right)^{1/p_i}, \quad i = 1,2.$$

Then, for each compact interval $[c,d] \subset (\lambda_1', \lambda_2')$, *where*

$$\lambda_1' = \inf\left\{ \frac{\sum_{i=1}^2 \int_\Omega (|\nabla u_i(x)|^{p_i} + |u_i(x)|^{p_i})\,\mathrm{d}x}{2\int_\Omega F(u_1(x),u_2(x))\mathrm{d}x} : u \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega), \int_\Omega F(u_1(x),u_2(x))\mathrm{d}x > 0 \right\}$$

and $\lambda_2' = (\max\{0, \lambda_0', \lambda_\infty'\})^{-1}$ *with*

$$\lambda_0' = \limsup_{|(u_1,u_2)|\to(0,0)} \frac{2\int_\Omega F(u_1(x),u_2(x))\mathrm{d}x}{\sum_{i=1}^2 \int_\Omega (|\nabla u_i(x)|^{p_i} + |u_i(x)|^{p_i})\,\mathrm{d}x}$$

and

$$\lambda_\infty' = \limsup_{\sum_{i=1}^2 \|u_i\|_{W^{1,p_i}(\Omega)}\to\infty} \frac{2\int_\Omega F(u_1(x),u_2(x))\mathrm{d}x}{\sum_{i=1}^2 \int_\Omega (|\nabla u_i(x)|^{p_i} + |u_i(x)|^{p_i})\,\mathrm{d}x},$$

there exists $\Lambda > 0$ *with the following property: For every* $\lambda \in [c,d]$ *and for every* $G \in \mathrm{C}^1(\mathbb{R}^2,\mathbb{R})$ *satisfying* $G(0,0)=0$*, there exists* $\gamma > 0$ *such that, for each* $\mu \in [0,\gamma]$*, the system*

$$\begin{cases} -\Delta_{p_1}u_1 + |u_1|^{p_1-2}u_1 = \lambda F_{u_1}(u_1,u_2) + \lambda G_{u_1}(u_1,u_2) & \text{in}\, \Omega, \\ -\Delta_{p_2}u_2 + |u_2|^{p_2-2}u_2 = \lambda F_{u_2}(u_1,u_2) + \lambda G_{u_2}(u_1,u_2) & \text{in}\, \Omega, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0 & \text{on}\, \partial\Omega \end{cases}$$

has at least three weak solutions whose norms in $\mathrm{W}^{1,p_1}(\Omega) \times \mathrm{W}^{1,p_2}(\Omega)$ *are less than* Λ*.*

Theorem 4.3. *Assume that there exist positive constants* $\delta_1, \ldots, \delta_n$ *such that*

$$\sum_{i=1}^n \widehat{M_i}(\Phi_i(\delta_i)) > 0 \quad \text{and} \quad F(\delta_1,\ldots,\delta_n) > 0. \tag{4.1}$$

Moreover, suppose that

$$\limsup_{\xi \to (0,\ldots,0)} \frac{F(\xi)}{\sum_{i=1}^n |\xi_i|^{a_i\,(p_i)_0}} = \limsup_{|\xi| \to \infty} \frac{F(\xi)}{\sum_{i=1}^n |\xi_i|^{a_i\,(p_i)_0}} = 0, \tag{4.2}$$

where $\xi = (\xi_1,\ldots,\xi_n)$ *with* $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$*. Then, for each compact interval* $[c,d] \subset (\lambda_3,\infty)$*, where* λ_3 *is* λ_1 *with* $\int_\Omega F(x,u(x))\mathrm{d}x$ *replaced by* $\int_\Omega \theta(x)F(u(x))\mathrm{d}x$*, there exists* $\Lambda > 0$ *with the following property: For every* $\lambda \in [c,d]$ *and every* $G \in \mathcal{F}$*, there exists* $\gamma > 0$ *such that, for each* $\mu \in [0,\gamma]$*, the system* $(N^\theta_{\lambda,\mu})$ *has at least three weak solutions whose norms in* E *are less than* Λ*.*

Proof. From (4.2), we easily observe that $(\mathcal{A}_1')$ is satisfied for every $\varepsilon > 0$. Moreover, using (4.1), by choosing $\varepsilon > 0$ small enough, one can derive $(\mathcal{A}_2')$. Hence, the conclusion follows from Theorem 4.1. □

Now, we exhibit an example in which the hypotheses of Theorem 4.3 are satisfied.

Example 4.4. Let $\Omega \subset \mathbb{R}^N$, $M_1(t) = 1+t^2$ and $M_2(t) = e^t$ for all $t > 0$. Thus, the assumption (M_0) holds by choosing $m_1 = m_2 = 1$ and $a_1 = a_2 = 2$. Now let

$$\varphi_1(t) = |t|^{p_1-2}t\log(1+\eta+|t|), \quad t \in \mathbb{R}$$

and

$$\varphi_2(t) = |t|^{p_2-2}t, \quad t \in \mathbb{R}$$

with $3 \le N < p_1$ and $3 \le N < p_2$. We observe that

$$\Phi_1(t) = \frac{|t|^{p_1}}{p_1}\log(1+\eta+|t|) - \frac{1}{p_1}\int_0^{|t|} \frac{s^{p_1}}{1+\eta+s}\mathrm{d}s$$

and

$$\Phi_2(t) = \frac{1}{p_2}|t|^{p_2}$$

for all $t \in \mathbb{R}$. It is easy to see that $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ are odd and strictly increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ such that the relations (Φ_0), (Φ_1) and (Φ_2) are satisfied, and we have

$$(p_1)_0 = p_1 \quad \text{and} \quad (p_1)^0 = \sup_{t>0} \frac{t\varphi_1(t)}{\Phi_1(t)} < \infty$$

(see [23, Example III] for more details) and $(p_2)^0 = (p_2)_0 = p_2$ (see [23, Example I] for more details). Let

$$F(\xi_1, \xi_2) = \begin{cases} (|\xi_1|^{2p_1} + |\xi_2|^{2p_2})^2 & \text{if } |\xi_1|^{2p_1} + |\xi_2|^{2p_2} < 1, \\ 1 & \text{if } |\xi_1|^{2p_1} + |\xi_2|^{2p_2} \geq 1. \end{cases}$$

Thus, F is a C^1-function. By choosing $\delta_1 = \delta_2 = 1$, we have

$$F(\delta_1, \delta_2) = 1 > 0 \quad \text{and} \quad \widehat{M_1}(\Phi_1(\delta_1)) + \widehat{M_2}(\Phi_2(\delta_2)) > 0.$$

Moreover, we have

$$\begin{aligned} \lim_{(\xi_1,\xi_2)\to(0,0)} \frac{F(\xi_1,\xi_2)}{\sum_{i=1}^2 |\xi_i|^{a_i\,(p_i)_0}} &= \lim_{(\xi_1,\xi_2)\to(0,0)} \frac{(|\xi_1|^{2p_1} + |\xi_2|^{2p_2})^2}{|\xi_1|^{2p_1} + |\xi_2|^{2p_2}} \\ &= \lim_{(\xi_1,\xi_2)\to(0,0)} (|\xi_1|^{2p_1} + |\xi_2|^{2p_2}) = 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{|(\xi_1,\xi_2)|\to\infty} \frac{F(\xi_1,\xi_2)}{\sum_{i=1}^2 |\xi_i|^{a_i\,(p_i)_0}} &= \lim_{|(\xi_1,\xi_2)|\to\infty} \frac{F(\xi_1,\xi_2)}{|\xi_1|^{2p_1} + |\xi_2|^{2p_2}} \\ &= \lim_{|(\xi_1,\xi_2)|\to\infty} \frac{1}{|\xi_1|^{2p_1} + |\xi_2|^{2p_2}} = 0, \end{aligned}$$

where $|(\xi_1,\xi_2)| = \sqrt{\xi_1^2 + \xi_2^2}$. Hence, since all assumptions of Theorem 4.3 are satisfied, it follows that for each compact interval $[c,d] \subset (0,\infty)$, there exists $\Lambda > 0$ with the following property: For every $\lambda \in [c,d]$ and every $G \in \mathrm{C}^1(\mathbb{R}^2,\mathbb{R})$ with $G(0,0) = 0$, there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the system

$$\begin{cases} \left(1 + \left(\int_\Omega (\Phi_1(|\nabla u_1|) + \Phi_1(|u_1|))\mathrm{d}x\right)^2\right)(-\operatorname{div}(\varphi_1(|\nabla u_1|)) + \varphi_1(|u_1|)) & \\ \quad = \lambda F_{u_1}(u_1,u_2) + \mu G_{u_1}(u_1,u_2) & \text{in } \Omega, \\ e^{\left(\int_\Omega (\Phi_2(|\nabla u_2|) + \Phi_2(|u_2|))\mathrm{d}x\right)}(-\operatorname{div}(\varphi_2(|\nabla u_2|)) + \varphi_2(|u_2|)) & \\ \quad = \lambda F_{u_2}(u_1,u_2) + \mu G_{u_2}(u_1,u_2) & \text{in } \Omega, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions whose norms in $\mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega) \times \mathrm{W}^1\mathrm{L}_{\Phi_2}(\Omega)$ are less than Λ.

Let $n = 1$. As an application of the results, we consider the problem

$$(N_{\lambda,\mu}^{f,g}) \quad \begin{cases} M_1\left(\int_\Omega \Phi_1(|\nabla u|) + \Phi_1(|u|)\mathrm{d}x\right)(-\operatorname{div}(\alpha_1(|\nabla u|)\nabla u) + \alpha_1(|u|)u) & \\ \quad = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $M_1 : \mathbb{R}^+ \to \mathbb{R}$ is a nondecreasing continuous function which satisfies the assumption (M_0) and $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions.

Let $\widehat{M_1} : \mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega) \to \mathbb{R}$ and $\omega_1 : \mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega) \to \mathbb{R}$ be as in (2.4) and (2.5), respectively. Put

$$F(x,\xi) = \int_0^\xi f(x,t)\mathrm{d}t \quad \text{for all} \quad (x,\xi) \in \Omega \times \mathbb{R}.$$

The following two corollaries are consequences of Theorems 3.1 and 3.2, respectively.

Corollary 4.5. *Assume that the following conditions hold:*

$(\mathcal{B}_1)$ *There exists a constant* $\varepsilon > 0$ *such that*

$$\max\left\{\limsup_{\xi\to 0} \frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^{a_1p_0}}, \limsup_{|\xi|\to\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^{a_1p_0}}\right\} < \varepsilon.$$

$(\mathcal{B}_2)$ *There exists a function* $w \in \mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega)$ *with* $\widehat{M_1}(\omega_1(w)) \neq 0$ *such that*

$$2a\varepsilon c_1^{a_1p_0} < \frac{m_1 \int_\Omega F(x,w(x))\mathrm{d}x}{\mathrm{meas}(\Omega)\widehat{M_1}(\omega_1(w))}.$$

Then, for each compact interval $[c,d] \subset (\bar{\lambda}_1, \bar{\lambda}_2)$, *where*

$$\bar{\lambda}_1 = \inf\left\{\frac{\widehat{M_1}(\omega_1(u))}{2\int_\Omega F(x,u(x))\mathrm{d}x} : \ u \in \mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega), \ \int_\Omega F(x,u(x))\mathrm{d}x > 0\right\}$$

and

$$\bar{\lambda}_2 = \left(\max\left\{0, \limsup_{u\to 0} \frac{2\int_\Omega F(x,u(x))\mathrm{d}x}{\widehat{M_1}(\omega_1(u))}, \limsup_{\|u\|_\Phi\to\infty} \frac{2\int_\Omega F(x,u(x))\mathrm{d}x}{\widehat{M_1}(\omega_1(u))}\right\}\right)^{-1},$$

there exists $\Lambda > 0$ *with the following property: For every* $\lambda \in [c,d]$ *and for every* L^1-*Carathéodory function* $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, *there exists* $\gamma > 0$ *such that, for each* $\mu \in [0,\gamma]$, *the problem* $(N_{\lambda,\mu}^{f,g})$ *has at least three weak solutions whose norms in* $\mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega)$ *are less than* Λ.

Corollary 4.6. *Assume that*

$$\max\left\{\limsup_{\xi\to 0} \frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^{a_1p_0}}, \limsup_{|\xi|\to\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^{a_1p_0}}\right\} \leq 0$$

and

$$\sup_{u\in \mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega)} \frac{\int_\Omega F(x,u(x))\mathrm{d}x}{\widehat{M_1}(\omega_1(u))} > 0.$$

Then, for each compact interval $[c,d] \subset (\bar{\lambda}_1, \infty)$, *where* $\bar{\lambda}_1$ *is given in Corollary* 4.5, *there exists* $\Lambda > 0$ *with the following property: For every* $\lambda \in [c,d]$ *and for every* L^1-*Carathéodory function* $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, *there exists* $\gamma > 0$ *such that, for each* $\mu \in [0,\gamma]$, *the problem* $(N_{\lambda,\mu}^{f,g})$ *has at least three weak solutions whose norms in* $\mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega)$ *are less than* Λ.

Remark 4.7. If f, g are nonnegative, as proved in [15], the weak solutions ensured by Corollaries 4.5, 4.6, 4.8 and 4.9 are nonnegative. In addition, if either $f(x,0) \neq 0$ for all $x \in \overline{\Omega}$ or $g(x,0) \neq 0$ for all $x \in \overline{\Omega}$, or both are true, then the solutions are positive.

Now we present the following corollaries as immediate consequences of Theorems 4.1 and 4.3, respectively, in which f has separated variables, i.e., $f(x,t) = \theta(x)h(t)$ for each $(x,t) \in \overline{\Omega} \times \mathbb{R}$, where $\theta : \overline{\Omega} \to \mathbb{R}$ is a nonzero function such that $\theta \in \mathrm{L}^1(\overline{\Omega})$ and $h : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Put $H(\xi) = \int_0^\xi h(t)\mathrm{d}t$ for all $\xi \in \mathbb{R}$.

Corollary 4.8. *Assume that the following conditions hold:*

$(\mathcal{B}_1')$ *There exists a constant* $\varepsilon > 0$ *such that*

$$\left(\sup_{x\in\Omega} \theta(x)\right) \cdot \max\left\{\limsup_{\xi\to 0} \frac{H(\xi)}{|\xi|^{a_1 p_0}}, \limsup_{|\xi|\to\infty} \frac{H(\xi)}{|\xi|^{a_1 p_0}}\right\} < \varepsilon.$$

$(\mathcal{B}_2')$ *There exists a positive constant* δ *such that*

$$2ac_1^{a_1 p_0}\varepsilon < \frac{m_1\|\theta\|_{\mathrm{L}^1(\Omega)} H(\delta)}{\mathrm{meas}(\Omega)^2 \widehat{M_1}(\Phi_1(\delta))}.$$

Then, for each compact interval $[c,d] \subset (\bar\lambda_3, \bar\lambda_4)$, *where* $\bar\lambda_3$ *and* $\bar\lambda_4$ *are* $\bar\lambda_1$ *and* $\bar\lambda_2$ *in Corollary* 4.5 *with* $\int_\Omega F(x,u(x))\mathrm{d}x$ *replaced by* $\int_\Omega \theta(x)H(u(x))\mathrm{d}x$, *there exists* $\Lambda > 0$ *with the following property: For every* $\lambda \in [c,d]$ *and every* L^1*-Carathéodory function* $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, *there exists* $\gamma > 0$ *such that, for each* $\mu \in [0,\gamma]$, *the problem*

$$(N_{\lambda,\mu}^{\theta,h,g}) \qquad \begin{cases} M_1\left(\int_\Omega \Phi_1(|\nabla u|) + \Phi_1(|u|)\mathrm{d}x\right) & \\ \quad \times(-\operatorname{div}(\alpha_1(|\nabla u|)\nabla u) + \alpha_1(|u|)u) & \\ \quad = \lambda\theta(x)h(u) + \mu g(x,u) & in\,\Omega, \\ \frac{\partial u}{\partial \nu} = 0 & on\,\partial\Omega \end{cases}$$

has at least three weak solutions whose norms in $\mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega)$ *are less than* Λ.

Corollary 4.9. *Let* $\theta : \overline{\Omega} \to \mathbb{R}$ *be a positive function such that* $\theta \in \mathrm{L}^1(\overline{\Omega})$. *Assume that there exists a positive constant* δ *such that*

$$\widehat{M_1}(\Phi_1(\delta)) > 0 \quad and \quad H(\delta) > 0.$$

Moreover, suppose that

$$\limsup_{t\to 0} \frac{h(t)}{|t|^{a_1 p_0 - 1}} = \limsup_{|t|\to\infty} \frac{h(t)}{|t|^{a_1 p_0 - 1}} = 0.$$

Then, for each compact interval $[c,d] \subset (\bar\lambda_3, \infty)$, *where* $\bar\lambda_3$ *is* $\bar\lambda_1$ *in Corollary* 4.5 *with* $\int_\Omega F(x,u(x))\mathrm{d}x$ *replaced by* $\int_\Omega \theta(x)H(u(x))\mathrm{d}x$, *there exists* $\Lambda > 0$ *with the following property: For every* $\lambda \in [c,d]$ *and every* L^1*-Carathéodory function* $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, *there exists $\gamma > 0$ such that, for each $\mu \in [0, \gamma]$, the problem $(N^{\theta,h,g}_{\lambda,\mu})$ has at least three weak solutions whose norms in $\mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega)$ are less than Λ.*

Finally, we present the following example in order to illustrate Corollary 4.9.

Example 4.10. Let $\Omega \subset \mathbb{R}^N$, $M_1(t) = t^3$ for all $t > 0$,

$$\varphi_1(t) = \begin{cases} \frac{|t|^{p-2}t}{\log(1+|t|)} & \text{if } t \neq 0, \\ 0 & \text{if } \ t = 0 \end{cases}$$

with $3 \leq N < p$, $\theta(x) = e^x$ for all $x \in \overline{\Omega}$ and

$$h(t) = t^{2p}e^{-|t|} \quad \text{for all} \quad t \in \mathbb{R}.$$

By choosing $m_1 = 1$ and $a_2 = 2$, we observe that assumption (M_0) is satisfied. It is also easy to see that $\varphi_1 : \mathbb{R} \to \mathbb{R}$ is an odd and strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and by [10, Example 3], one has

$$(p_1)_0 = p - 1 < (p_1)^0 = p = \liminf_{t\to\infty} \frac{\log(\Phi_1(t))}{\log(t)},$$

where

$$\Phi_1(t) = \int_0^t \varphi_1(s)\mathrm{d}s.$$

By choosing $\delta = 1$, $\Phi_1(\delta) = \Phi_1(1) > 0$, we have

$$H(\delta) = H(1) = \int_0^1 t^{2p}e^{-t} > 0,$$
$$\widehat{M_1}(\Phi_1(\delta)) = \int_0^{\Phi_1(1)} s^3\mathrm{d}s > 0$$

and

$$\lim_{t\to 0} \frac{h(t)}{|t|^{2p-1}} = \lim_{|t|\to\infty} \frac{h(t)}{|t|^{2p-1}} = 0.$$

Hence, since all assumptions of Corollary 4.9 are fulfilled, it follows that for each compact interval $[c, d] \subset (0, \infty)$, there exists $\Lambda > 0$ with the following property: For every $\lambda \in [c, d]$ and every L^1-Carathéodory function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that, for each $\mu \in [0, \gamma]$, the problem

$$\begin{cases} \left(\int_\Omega \Phi_1(|\nabla u_1|) + \Phi_1(|u_1|)\mathrm{d}x\right)^3 \left(-\operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{\log(1+|\nabla u|)}\right) + \frac{|\nabla u|^{p-2}\nabla u}{\log(1+|\nabla u|)}\right) & \\ \quad = \lambda e^x u^{2p}e^{-|u|} + \mu g(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions whose norms in $\mathrm{W}^1\mathrm{L}_{\Phi_1}(\Omega)$ are less than Λ.

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