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PAIRS OF NONTRIVIAL SOLUTIONS FOR RESONANT ROBIN PROBLEMS WITH INDEFINITE LINEAR PART

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ABSTRACT. We study a semilinear Robin problem driven by the Laplacian plus an indefinite and unbounded potential and a Carathéodory reaction term which exhibits linear growth near $\pm\infty$ and near zero. Resonance with respect to different eigenvalues can occur at both $\pm\infty$ and near zero. Using the saddle point reduction method and Morse theory (critical groups), we prove a multiplicity theorem producing two nontrivial smooth solutions.

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1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following semilinear Robin problem:

$$(1.1) \qquad \begin{cases} -\Delta u(z) + \xi(z)u(z) = f(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this problem the potential function $\xi \in L^s(\Omega)$ (with $s > N$) is in general sign-changing. So, the linear part of problem (1.1) is indefinite. The reaction term f is a Carathéodory function (that is, for all $\zeta \in \mathbb{R}$, $z \longmapsto f(z,\zeta)$ is measurable and for a.a. $z \in \Omega$, $\zeta \longmapsto f(z,\zeta)$ is continuous). We assume that $f(z,\cdot)$ exhibits linear growth near $\pm\infty$ and near zero and resonance is possible both at $\pm\infty$ and at zero, but with respect to different eigenvalues of $u \longmapsto -\Delta u + \xi(z)u$ with Robin boundary condition. In the boundary condition, the coefficient $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$, we have the usual homogeneous Neumann problem. We prove a multiplicity theorem, producing a pair of nontrivial smooth solutions.

Recently semilinear Robin problems were studied by Shi-Li [17] (indefinite potential and superlinear reaction term), Qian-Li [16] (zero potential and superlinear

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reaction term), Zhang-Li-Xue [19] (positive potential, thus a coercive differential operator and an autonomous reaction term with zeros), Papageorgiou-Rădulescu [15] (indefinite potential and a Carathéodory reaction term of arbitrary growth), D'Aguì-Marano-Papageorgiou [3] (indefinite potential and an asymmetric reaction term) and Filippakis-Papageorgiou [5] (indefinite potential and an odd reaction term of arbitrary growth). Also we mention the works of Papageorgiou-Papalini [11] (Dirichlet problems) and Papageorgiou-Rădulescu [12, 14] (Neumann problems) on equations driven by the Laplacian plus an indefinite potential.

We prove a multiplicity theorem producing two nontrivial smooth solutions. Our approach is based on a variant of the reduction method of Amann [1] and Castro-Lazer [2] and on Morse theory (critical groups). However, note that in our case the reduction is done on an infinite dimensional component space and this makes the situation more complicated.

2. MATHEMATICAL BACKGROUND

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given a function $\varphi \in C^1(X;\mathbb{R})$, we say that φ satisfies the "Cerami condition", if the following is true:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\varphi'(u_n) \longrightarrow 0 \quad \text{in } X^* \quad \text{as } n \to +\infty,$$

admits a strongly convergent subsequence."

Our analysis of problem (1.1), will make use of the following three spaces:

$$H^1(\Omega), \quad C^1(\overline{\Omega}), \quad L^q(\partial\Omega)\ (1 \leqslant q \leqslant +\infty).$$

We know that $H^1(\Omega)$ is a Hilbert space with inner product given by

$$(u, v)_{H^1} = \int_\Omega uv\, dz + \int_\Omega (Du, Dv)_{\mathbb{R}^N}\, dz \quad \forall u, v \in H^1(\Omega).$$

The corresponding norm is denoted by $\|\cdot\|$ and we have

$$\|u\| = \left(\|u\|_2^2 + \|Du\|_2^2\right)^{\frac{1}{2}} \quad \forall u \in H^1(\Omega).$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ. Then using this measure on $\partial\Omega$, we can define in the usual way the Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leqslant q \leqslant +\infty)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0\colon H^1(\Omega) \longrightarrow L^2(\partial\Omega)$ known as the "trace map" such that

$$\gamma_0(u) = u|_\Omega \quad \forall u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

Hence the trace map assigns "boundary values" to every Sobolev function $u \in H^1(\Omega)$. We know that

- γ_0 is compact into $L^q(\partial\Omega)$ for all $q \in [1, \frac{2(N-1)}{N-2})$ if $N \geqslant 3$ and for all $q \in [1, +\infty)$ if $N = 1, 2$;
- $R(\gamma_0) = H^{\frac{1}{2},2}(\partial\Omega)$ and $\ker \gamma_0 = H^1_0(\Omega)$.

In the sequel, for the sake of notational simplicity, we drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

We will use the spectrum of $u \longmapsto -\Delta u + \xi(z)u$ with Robin boundary condition. So, we consider the following linear eigenvalue problem

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = \widehat{\lambda} u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

This problem was studied by D'Aguì-Marano-Papageorgiou [3]. Suppose that $\xi \in L^s(\Omega)$ with $s > N$ and let $\gamma \colon H^1(\Omega) \longrightarrow \mathbb{R}$ be the C^2-functional defined by

$$\gamma(u) = \|Du\|_2^2 + \int_\Omega \xi(z)u^2\,dz + \int_{\partial\Omega} \beta(z)u^2\,d\sigma \quad \forall u \in H^1(\Omega).$$

Problem (2.1) has a smallest eigenvalue $\widehat{\lambda}_1 \in \mathbb{R}$ given by

$$\widehat{\lambda}_1 = \inf\left\{\frac{\gamma(u)}{\|u\|_2^2} :\ u \in H^1(\Omega),\ u \neq 0\right\}. \tag{2.2}$$

Then we can find $\mu > 0$ such that

$$\gamma(u) + \mu\|u\|_2^2 \geqslant c_0\|u\|^2 \quad \forall u \in H^1(\Omega), \tag{2.3}$$

for some $c_0 > 0$. Using (2.3) and the spectral theory for compact self-adjoint operators on a Hilbert space, we generate the spectrum of (2.1). This consists of a sequence $\{\widehat{\lambda}_k\}_{k\geqslant 1}$ of distinct eigenvalues such that $\widehat{\lambda}_k \longrightarrow +\infty$ as $k \to +\infty$. Let $E(\widehat{\lambda}_k)$ denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k$. Using the regularity theory of Wang [18], we know that

$$E(\widehat{\lambda}_k) \ \subseteq\ C^1(\overline{\Omega}) \quad \forall k \geqslant 1.$$

In addition, from de Figueiredo-Gossez [4], we have that each eigenspace $E(\widehat{\lambda}_k)$, $k \geqslant 1$, exhibits the "unique continuation property", that is, if $u \in E(\widehat{\lambda}_k)$ and u vanishes on a set of positive Lebesgue measure, then $u \equiv 0$. We set

$$\overline{H}_m = \bigoplus_{k=1}^{m} E(\widehat{\lambda}_k) \quad \text{and} \quad \widehat{H}_m = \overline{H}_m^{\perp} = \overline{\bigoplus_{k\geqslant m+1} E(\widehat{\lambda}_k)}.$$

The space $\overline{H}_m$ is finite dimensional and we have the following orthogonal direct sum decomposition

$$H^1(\Omega) = \overline{H}_m \oplus \widehat{H}_m.$$

For the higher eigenvalues $\{\widehat{\lambda}_m\}_{m\geqslant 2}$ we have the following variational description:

$$\widehat{\lambda}_m \ = \ \inf\left\{\frac{\gamma(u)}{\|u\|_2^2} :\ u \in \widehat{H}_{m-1},\ u \neq 0\right\}$$

$$= \sup\left\{\frac{\gamma(u)}{\|u\|_2^2} : u \in \overline{H}_m, \ u \neq 0\right\}, \ m \geqslant 2. \tag{2.4}$$

From (2.2) and (2.4) we see that we have variational characterizations for all the eigenvalues. In (2.2) the infimum is realized on $E(\widehat{\lambda}_1)$, while in (2.4) both the infimum and supremum are realized on $E(\widehat{\lambda}_m)$. We know that $\widehat{\lambda}_1$ is simple (that is, $\dim E(\widehat{\lambda}_1) = 1$) and is the only eigenvalue with eigenfunctions of constant sign. For every $m \geqslant 2$, the elements of $E(\widehat{\lambda}_m)$ are nodal (sign changing). By $\widehat{u}_1$ we denote the L^2-normalized (that is, $\|u\|_2 = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1$. The strong maximum principle implies that $\widehat{u}_1(z) > 0$ for all $z \in \Omega$. As a consequence of the variational characterizations of the eigenvalues (see (2.2) and (2.4)) and of the unique continuation principle, we have the following useful inequalities.

Proposition 2.1. ***(a)*** *If $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leqslant \widehat{\lambda}_m$ for a.a. $z \in \Omega$, for some $m \geqslant 1$ and the inequality is strict on a set of positive measure, then we can find $c_1 > 0$ such that*

$$\gamma(u) - \int_\Omega \vartheta(z) u^2 \, dz \geqslant c_1 \|u\|^2 \quad \forall u \in \widehat{H}_{m-1}.$$

(b) *If $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \geqslant \widehat{\lambda}_m$ for a.a. $z \in \Omega$, for some $m \geqslant 1$ and the inequality is strict on a set of positive measure, then we can find $c_2 > 0$ such that*

$$\gamma(u) - \int_\Omega \vartheta(z) u^2 \, dz \leqslant -c_2 \|u\|^2 \quad \forall u \in \overline{H}_m.$$

Next we recall some definitions and facts from Morse theory (critical groups) which will be used in the sequel.

Let X be a Banach space, $\varphi \in C^1(X;\mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\begin{aligned} \varphi^c &= \{u \in X : \varphi(u) \leqslant c\}, \\ K_\varphi &= \{u \in X : \varphi'(u) = 0\}, \\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\}. \end{aligned}$$

Let (Y_1, Y_2) be a pair of spaces such that $Y_2 \subseteq Y_1 \subseteq X$. For every $k \geqslant 0$, by $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Suppose that $u \in K_\varphi^c$ is isolated. The critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \forall k \geqslant 0.$$

Here U is a neighbourhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that the above definition of critical groups is independent of the choice of the isolating neighbourhood U.

Suppose that $\varphi \in C^1(X;\mathbb{R})$ satisfies the Cerami condition at $\inf \varphi(K_\varphi) > -\infty$. The critical groups of φ at infinity are defined by

$$C_k(\varphi,\infty) = H_k(X,\varphi^c) \quad \forall k \geqslant 0,$$

with $c < \inf \varphi(K_\varphi)$. This definition is independent on the level $c < \inf \varphi(K_\varphi)$. Indeed, if $c' < c < \inf \varphi(K_\varphi)$, then $\varphi^{c'}$ is strong deformation retract of φ^c (see for example Motreanu-Motreanu-Papageorgiou [10, Theorem 5.34, p. 110]). Therefore

$$H_k(X,\varphi^c) = H_k(X,\varphi^{c'}) \quad \forall k \geqslant 0$$

(see Motreanu-Motreanu-Papageorgiou [10, Theorem 6.15, p. 145]).

Assume that K_φ is finite. We introduce the following quantities:

$$M(t,u) = \sum_{k\geqslant 0} \operatorname{rank} C_k(\varphi,u)t^k \quad \forall t \in \mathbb{R},\ u \in K_\varphi,$$
$$P(t,\infty) = \sum_{k\geqslant 0} \operatorname{rank} C_k(\varphi,\infty)t^k \quad \forall t \in \mathbb{R}.$$

Then the "Morse relation" says that

$$\sum_{u\in K_\varphi} M(t,u) = P(t,\infty) + (1+t)Q(t) \quad \forall t \in \mathbb{R}, \tag{2.5}$$

where $Q(t) = \sum_{k\geqslant 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k.

Suppose that $X = H$ is a Hilbert space, $\varphi \in C^2(H;\mathbb{R})$ and $u \in K_\varphi$. We say that u is "nondegenerate", if $\varphi''(u) \in \mathcal{L}(H)$ is invertible. The "Morse index" m of u is defined to be the supremum of the dimensions of the vector subspaces of H on which $\varphi''(u)$ is negative definite. If $u \in K_\varphi$ is isolated, nondegenerate, then

$$C_k(\varphi,u) = \delta_{k,m}\mathbb{Z} \quad \forall k \geqslant 0,$$

where m is the Morse index of u and $\delta_{k,m}$ is the Kronecker symbol, that is,

$$\delta_{k,m} = \begin{cases} 1 & \text{if} \quad k = m, \\ 0 & \text{if} \quad k \neq m. \end{cases}$$

3. **PAIRS OF NONTRIVIAL SOLUTIONS**

In this section we prove the existence of two nontrivial smooth solutions for problem (1.1) under conditions which permit resonance at $\pm\infty$ and at zero.

More precisely the conditions on the reaction term f are the following:

<u>$H(f)$:</u> $f\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i): for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)_+$ such that

$$|f(z,\zeta)| \leqslant a_\varrho(z) \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \leqslant \varrho;$$

(ii): there exist $m \geqslant 1$ and a function $\vartheta \in L^\infty(\Omega)$ such that

$$\vartheta(z) \geqslant \widehat{\lambda}_m \quad \text{for a.a. } z \in \Omega, \ \vartheta \not\equiv \widehat{\lambda}_m,$$

and

$$(f(z,\zeta) - f(z,y))(\zeta - y) \geqslant \vartheta(z)(\zeta - y)^2 \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta, y \in \mathbb{R};$$

(iii): $\limsup\limits_{\zeta \to \pm\infty} \frac{f(z,\zeta)}{\zeta} \leqslant \widehat{\lambda}_{m+1}$ uniformly for a.a. $z \in \Omega$;

(iv): if $F(z,\zeta) = \int_0^\zeta f(z,s)\,ds$, then

$$\lim_{\zeta \to \pm\infty} (f(z,\zeta)\zeta - 2F(z,\zeta)) = +\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(v): there exist $l \in \mathbb{N}$, $l > m$ and $\delta > 0$ such that

$$\widehat{\lambda}_l \zeta^2 \leqslant f(z,\zeta)\zeta \leqslant \widehat{\lambda}_{l+1}\zeta^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \leqslant \delta.$$

Remark 3.1. Hypothesis $H(f)(iii)$ implies that at $\pm\infty$ we can have resonance with respect to any nonprincipal eigenvalue. Similarly, hypothesis $H(f)(v)$ says that at zero we can have double resonance at any spectral interval higher than the one corresponding to the asymptotic behaviour of $\frac{f(z,\zeta)}{\zeta}$ and $\zeta \to \pm\infty$.

The hypotheses on the potential function ξ and the boundary coefficient β are the following:

$\underline{H(\xi)}$: $\xi \in L^s(\Omega)$ with $s > N$.

$\underline{H(\beta)}$: $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Let $\varphi\colon H^1(\Omega) \longrightarrow \mathbb{R}$ be the energy (Euler) functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{2}\gamma(u) - \int_\Omega F(z,u)\,dz \quad \forall u \in H^1(\Omega).$$

Evidently $\varphi \in C^1(H^1(\Omega))$. We set

$$Y = \bigoplus_{k=1}^{m} E(\widehat{\lambda}_k) \quad \text{and} \quad V = Y^\perp = \overline{\bigoplus_{k \geqslant m+1} E(\widehat{\lambda}_k)}.$$

We have the following orthogonal direct sum decomposition:

$$H^1(\Omega) = Y \oplus V.$$

Proposition 3.2. *If hypotheses $H(f)$, $H(\xi)$ and $H(\beta)$ hold, then there exists a continuous map $\tau\colon V \longrightarrow Y$ such that*

$$\varphi(v + \tau(v)) = \max_{y \in Y} \varphi(v + y) \quad \forall v \in V.$$

Proof. Fix $v \in V$ and consider the C^1-functional $\varphi_v \colon H^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_v(u) = \varphi(v+u) \quad \forall u \in H^1(\Omega).$$

Let $i_Y \colon Y \longrightarrow H^1(\Omega)$ be the inclusion map. We set

$$\widetilde{\varphi}_v = \varphi_v \circ i.$$

Using the chain rule, we have

$$\widetilde{\varphi}_v' = p_{Y^*} \circ \varphi_v', \tag{3.1}$$

with p_{Y^*} being the orthogonal projection of $H^1(\Omega)^*$ onto Y^*. In the sequel by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (Y^*, Y). For $y_1, y_2 \in Y$ we have

$$\begin{aligned}
&\langle \widetilde{\varphi}_v'(y_1) - \widetilde{\varphi}_v'(y_2), y_1 - y_2 \rangle_Y \\
&\quad = \langle \varphi_v'(y_1) - \varphi_v'(y_2), y_1 - y_2 \rangle \\
&\quad = \gamma(y_1 - y_2) - \int_\Omega (f(z, v+y_1) - f(z, v+y_2))(y_1 - y_2)\, dz \\
&\quad \leqslant \gamma(y_1 - y_2) - \int_\Omega \vartheta(z)(y_1 - y_2)^2\, dz \\
&\quad \leqslant -c_2 \|y_1 - y_2\|^2
\end{aligned} \tag{3.2}$$

(see (3.1), hypothesis $H(f)(ii)$ and Proposition 2.1(b)).

Therefore $-\widetilde{\varphi}_v'$ is strongly monotone and $-\widetilde{\varphi}_v$ is strictly convex. Note that

$$\langle \widetilde{\varphi}_v'(y), y \rangle_Y = \langle \widetilde{\varphi}_v'(y) - \widetilde{\varphi}_v'(0), y \rangle_Y + \langle \widetilde{\varphi}_v'(0), y \rangle_Y \leqslant -c_2 \|y\|^2 + c_3 \|y\|, \tag{3.3}$$

for some $c_3 > 0$ (see (3.2)), so

$$-\widetilde{\varphi}_v' \text{ is coercive.} \tag{3.4}$$

Since $-\widetilde{\varphi}_v'$ is continuous, monotone, it follows that

$$-\widetilde{\varphi}_v' \text{ is maximal monotone.} \tag{3.5}$$

From (3.4) and (3.5) we infer that $-\widetilde{\varphi}_v'$ is surjective (see Gasiński-Papageorgiou [6, Corollary 3.2.31, p. 319]). Therefore we can find $y_0 \in Y$ such that

$$\widetilde{\varphi}_v'(y_0) = 0. \tag{3.6}$$

Since $-\widetilde{\varphi}_v'$ is strongly monotone, the solution $y_0 \in Y$ of (3.6) is unique and is the unique minimizer of the strictly convex functional $-\widetilde{\varphi}_v = -\varphi_v|_Y$. Now, let $\tau \colon V \longrightarrow Y$ be the map which to each $v \in V$ assigns the unique solution $y_0 \in Y$ of (3.6). From (3.1) and (3.6), we have

$$p_{Y^*} \varphi'(v + \tau(v)) = 0 \quad \text{and} \quad \varphi(v + \tau(v)) = \max_{y \in Y} \varphi(v + y). \tag{3.7}$$

We examine the continuity of the map $\tau \colon V \longrightarrow Y$. So, let $v_n \longrightarrow v$ in V. We have

$$0 = \langle \widetilde{\varphi}_{v_n}'(\tau(v_n), \tau(v_n)) \leqslant -c_2 \|\tau(v_n)\|^2 + c_3 \|\tau(v_n)\|$$

(see (3.6) and (3.3)), so the sequence $\{\tau(v_n)\}_{n\geqslant 1} \subseteq Y$ is bounded. Since Y is finite dimensional, by passing to a subsequence if necessary, we may assume that

$$\tau(v_n) \longrightarrow \widetilde{y} \in Y \quad \text{in } Y. \tag{3.8}$$

From the continuity of φ, we have

$$\varphi(v + \widetilde{y}) = \lim_{n\to+\infty} \varphi(v_n + \tau(v_n)) \tag{3.9}$$

(see (3.8)). From (3.7), we have

$$\varphi(v_n + y) \leqslant \varphi(v_n + \tau(v_n)) \quad \forall y \in Y,\ n \geqslant 1,$$

so

$$\varphi(v + y) \leqslant \varphi(v + \widetilde{y}) \quad \forall y \in Y,$$

(see (3.9)), thus $\widetilde{y} = \tau(v)$ (see (3.7)).

By the Urysohn criterion for the convergence of subsequences (see, for example, Gasiński-Papageorgiou [7, Problem 1.3, p. 33]), for the original sequence we have

$$\tau(v_n) \longrightarrow \tau(v) \quad \text{in } Y,$$

so $\tau\colon V \longrightarrow Y$ is continuous. □

We set

$$\psi(v) = \varphi(v + \tau(v)) \quad \forall v \in V.$$

From Proposition 3.2 it follows that $\psi \in C(V;\mathbb{R})$. In fact we can say more, namely that ψ is continuously differentiable on V.

Proposition 3.3. *If hypotheses $H(f)$, $H(\xi)$ and $H(\beta)$ hold, then $\psi \in C^1(V;\mathbb{R})$ and*

$$\psi'(v) = p_{V^*}\varphi(v + \tau(v)) \quad \forall v \in V,$$

here p_{V^*} is the orthogonal projection of $H^1(\Omega)^*$ onto V^*.

Proof. Let $v, w \in V$ and $t > 0$. We have

$$\frac{1}{t}(\psi(v + tw) - \psi(v)) \geqslant \frac{1}{t}(\varphi(v + tw + \tau(v)) - \varphi(v + \tau(v)))$$

(see (3.7)), so

$$\liminf_{t\to 0^+} \frac{1}{t}(\psi(v + tw) - \psi(v)) \geqslant \langle \varphi'(v + \tau(v)), w\rangle. \tag{3.10}$$

Also, we have

$$\frac{1}{t}(\psi(v + tw) - \psi(v)) \leqslant \frac{1}{t}(\varphi(v + tw + \tau(v + tw)) - \varphi(v + \tau(v + w)))$$

(see (3.7)), so

$$\limsup_{t\to 0^+} \frac{1}{t}(\psi(v + tw) - \psi(v)) \leqslant \langle \varphi'(v + \tau(v)), w\rangle \tag{3.11}$$

(since τ is continuous by Proposition 3.2 and $\varphi \in C^1(H^1(\Omega);\mathbb{R})$). Then from (3.10) and (3.11) it follows that

$$\lim_{t\to 0^+} \frac{1}{t}(\psi(v+tw)-\psi(v)) = \langle \varphi'(v+\tau(v)), w\rangle. \tag{3.12}$$

In a similar fashion we show that

$$\lim_{t\to 0^-} \frac{1}{t}(\psi(v+tw)-\psi(v)) = \langle \varphi'(v+\tau(v)), w\rangle. \tag{3.13}$$

By $\langle\cdot,\cdot\rangle_V$ we denote the duality brackets for the pair (V^*, V). From (3.12) and (3.13), we conclude that

$$\langle \psi'(v), w\rangle_V = \langle \varphi'(v+\tau(v)), w\rangle \quad \forall v, w \in V,$$

so

$$\psi'(v) = p_{V^*}\varphi'(v+\tau(v)),$$

thus $\psi \in C^1(V;\mathbb{R})$ (recall that τ is continuous; see Proposition 3.2). $\square$

Note that in contrast to the usual reduction method (see Amann [1] and Castro-Lazer [2]), in our case the reduction is done on infinite dimensional space (the space V). This is a consequence of hypothesis $H(f)(ii)$ (in the spectral interval $[\widehat{\lambda}_m, \widehat{\lambda}_{m+1}]$ at $\pm\infty$ we can have resonance with $\widehat{\lambda}_{m+1}$, but only nonuniform nonresonance with respect to $\widehat{\lambda}_m$). In addition, here the reaction term f is only a Carathéodory function (no differentiability condition on $f(z,\cdot)$ is required) and so the energy functional is only C^1 and not C^2 on $H^1(\Omega)$. These special features, lead to some technical difficulties. Nevertheless, with no extra conditions, we are able to overcome these difficulties and prove the following result.

Proposition 3.4. *If hypotheses $H(f)$, $H(\xi)$ and $H(\beta)$ hold, then the functional ψ is coercive.*

Proof. Let $\widehat{\varphi} = \varphi|_V$. Clearly $\widehat{\varphi} \in C^1(V;\mathbb{R})$ and so as before via the chain rule, we have

$$\widehat{\varphi}' = p_{V^*} \circ \varphi' \tag{3.14}$$

(recall that p_{V^*} denotes the orthogonal projection of $H^1(\Omega)^*$ onto V^*).

Claim 1. The functional $\widehat{\varphi}$ satisfies the Cerami condition.

We consider a sequence $\{v_n\} \subseteq V$ such that

$$|\widehat{\varphi}(v_n)| \leqslant M_1 \quad \forall n \in \mathbb{N} \text{ (for some } M_1 > 0), \tag{3.15}$$

$$(1+\|v_n\|)\widehat{\varphi}'(v_n) \longrightarrow 0 \quad \text{in } V^* \quad \text{as } n \to +\infty. \tag{3.16}$$

From (3.16), we have

$$|\langle \widehat{\varphi}(v_n), h\rangle_V| \leqslant \frac{\varepsilon_n \|h\|}{1+\|v_n\|} \quad \forall n \geqslant 1,\ h \in V,$$

with $\varepsilon_n \to 0^+$, so

$$\langle \varphi(v_n), h \rangle \leqslant \frac{\varepsilon_n \|h\|}{1 + \|v_n\|} \quad \forall n \geqslant 1,\ h \in V \tag{3.17}$$

(see (3.14)).

In (3.17) we choose $h = v_n \in V$. Then

$$\gamma(v_n) - \int_\Omega f(z, v_n) v_n \, dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{3.18}$$

We will show that the sequence $\{v_n\} \subseteq V$ is bounded. Arguing by contradiction, suppose that at least for a subsequence, we have

$$\|v_n\| \longrightarrow +\infty. \tag{3.19}$$

Let $\widehat{v}_n = \frac{v_n}{\|v_n\|}$, $n \geqslant 1$. Then $\|\widehat{v}_n\| = 1$ for all $n \geqslant 1$ and so passing to a next subsequence if necessary, we may assume that

$$\widehat{v}_n \xrightarrow{w} \widehat{v} \quad \text{in } H^1(\Omega) \quad \text{and} \quad \widehat{v}_n \longrightarrow \widehat{v} \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.20}$$

Hypotheses $H(f)$ imply that

$$|f(z, \zeta)| \leqslant c_4 |\zeta| \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R}, \tag{3.21}$$

for some $c_4 > 0$. We return to (3.18) and use (3.21). We obtain

$$\gamma(v_n) - c_4 \|v_n\|_2^2 \leqslant \varepsilon_n \quad \forall n \geqslant 1,$$

so

$$\gamma(v_n) + \mu \|v_n\|_2^2 - (c_4 + \mu) \|v_n\|_2^2 \leqslant \varepsilon_n,$$

with $\mu > 0$ as in (2.3). Using also (2.3), we get

$$c_0 \|v_n\|^2 - (c_4 + \mu) \|v_n\|_2^2 \leqslant \varepsilon_n,$$

so

$$c_0 - (c_4 + \mu) \|\widehat{v}_n\|_2^2 \leqslant \frac{\varepsilon_n}{\|v_n\|^2} \quad \forall n \geqslant 1,$$

thus by (3.19) and (3.20), we get

$$c_0 \leqslant (c_4 + \mu) \|\widehat{v}\|_2^2,$$

hence

$$\widehat{v} \neq 0. \tag{3.22}$$

Let $\Omega^* = \{z \in \Omega : \ \widehat{v}(z) \neq 0\}$. We have $|\Omega^*|_N > 0$, with $|\cdot|_N$ being the Lebesgue measure on $\mathbb{R}^N$ and

$$|v_n(z)| \longrightarrow +\infty \quad \text{for a.a. } z \in \Omega^*,$$

so

$$f(z, v_n(z)) v_n(z) - 2F(z, v_n(z)) \longrightarrow +\infty \quad \text{for a.a. } z \in \Omega^*$$

(see hypothesis $H(f)(iv)$), thus

$$\int_{\Omega^*}(f(z,v_n)v_n - 2F(z,v_n))\,dz \longrightarrow +\infty \tag{3.23}$$

(see hypothesis $H(f)(iv)$ and use Fatou's lemma).

Hypothesis $H(f)(iv)$ implies that we can find $M_2 > 0$ such that

$$f(z,\zeta)\zeta - 2F(z,\zeta) \geqslant 0 \quad \text{for a.a. } z \in \Omega,\ \text{all } |\zeta| \geqslant M_2. \tag{3.24}$$

Then we have

$$\begin{aligned}
&\int_\Omega (f(z,v_n)v_n - 2F(z,v_n))\,dz \\
&\quad = \int_{(\Omega\setminus\Omega^*)\cap\{|v_n|\geqslant M_2\}} (f(z,v_n)v_n - 2F(z,v_n))\,dz \\
&\qquad + \int_{(\Omega\setminus\Omega^*)\cap\{|v_n|< M_2\}} (f(z,v_n)v_n - 2F(z,v_n))\,dz \\
&\qquad + \int_{\Omega^*} (f(z,v_n)v_n - 2F(z,v_n))\,dz \\
&\quad \geqslant -c_5|\Omega|_N + \int_{\Omega^*} (f(z,v_n)v_n - 2F(z,v_n))\,dz \quad \forall n \geqslant 1,
\end{aligned}$$

for some $c_5 > 0$ (see (3.24) and hypothesis $H(f)(i)$), so

$$\int_\Omega (f(z,v_n)v_n - 2F(z,v_n))\,dz \longrightarrow +\infty \quad \text{as } n \to +\infty \tag{3.25}$$

(see (3.23)).

From (3.15), we have

$$\gamma(v_n) - \int_\Omega 2F(z,v_n)\,dz \leqslant 2M_1 \quad \forall n \geqslant 1. \tag{3.26}$$

Also from (3.17) with $h = v_n \in V$, we have

$$-\gamma(v_n) + \int_\Omega f(z,v_n)v_n\,dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{3.27}$$

We add (3.26) and (3.27) and obtain

$$\int_\Omega (f(z,v_n)v_n - 2F(z,v_n))\,dz \leqslant M_3 \quad \forall n \geqslant 1, \tag{3.28}$$

for some $M_3 > 0$. Comparing (3.25) and (3.28) we get a contradiction. This proves that the sequence $\{v_n\}_{n\geqslant 1} \subseteq V$ is bounded.

Therefore, passing to a subsequence if necessary, we may assume that

$$v_n \xrightarrow{w} v \quad \text{in } H^1(\Omega) \quad \text{and} \quad v_n \longrightarrow v \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.29}$$

From (3.17), we have

$$\langle A(v_n),h\rangle + \int_\Omega \xi(z)v_n h\,dz + \int_{\partial\Omega} \beta(z)v_n h\,d\sigma - \int_\Omega f(z,u_n)\,dz$$

$$\leqslant \frac{\varepsilon_n \|h\|}{1+\|u_n\|} \quad \forall h \in H^1(\Omega), \tag{3.30}$$

with $\varepsilon_n \to 0^+$. Choosing $h = v_n - v \in H^1(\Omega)$ in (3.30), passing to the limit as $n \to +\infty$ and using (3.29), we obtain

$$\lim_{n\to+\infty} \langle A(v_n), v_n - v\rangle = 0,$$

so

$$\|Dv_n\|_2 \longrightarrow \|Dv\|_2,$$

thus, by the Kadec-Klee property for Hilbert spaces (see (3.29)), we get

$$v_n \longrightarrow v \quad \text{in } H^1(\Omega),$$

hence $\widehat{\varphi}$ satisfies the Cerami condition. This proves Claim 1.

Claim 2. $\widehat{\lambda}_{m+1}\xi^2 - 2F(z,\xi) \longrightarrow +\infty$ uniformly for a.a. $z \in \Omega$ as $\xi \to +\infty$.

Hypothesis $H(f)(iv)$ implies that given $\eta > 0$, we can find $M_4 = M_4(\eta) > 0$ such that

$$f(z,\zeta)\zeta - 2F(z,\zeta) \geqslant \eta \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \geqslant M_4. \tag{3.31}$$

We have

$$\begin{aligned} \frac{d}{d\zeta}\Big(\frac{F(z,\zeta)}{|\zeta|^2}\Big) &= \frac{f(z,\zeta)|\zeta|^2 - 2\zeta F(z,\zeta)}{|\zeta|^4} \\ &= \frac{f(z,\zeta)\zeta - 2F(z,\zeta)}{|\zeta|^2\zeta} \\ &\begin{cases} \geqslant \frac{\eta}{\zeta^3} & \text{for a.a. } z \in \Omega, \text{ all } \zeta \geqslant M_4, \\ \leqslant \frac{\eta}{|\zeta|^2\zeta} & \text{for a.a. } z \in \Omega, \text{ all } \zeta \leqslant -M_4 \end{cases} \end{aligned}$$

(see (3.31)), so

$$\begin{gathered} \frac{F(z,y)}{|y|^2} - \frac{F(z,u)}{|u|^2} \geqslant \frac{\eta}{2}\Big(\frac{1}{|u|^2} - \frac{1}{|y|^2}\Big) \\ \text{for a.a. } z \in \Omega, \text{ all } |y| \geqslant |u| \geqslant M_4. \end{gathered} \tag{3.32}$$

Hypotheses $H(f)(ii)$ and (iii) imply that

$$\begin{gathered} \vartheta(z) \leqslant \liminf_{\zeta\to\pm\infty} \frac{f(z,\zeta)}{\zeta} \leqslant \limsup_{\zeta\to\pm\infty} \frac{f(z,\zeta)}{\zeta} \leqslant \widehat{\lambda}_{m+1} \\ \text{uniformly for a.a. } z \in \Omega, \end{gathered}$$

so

$$\begin{gathered} \vartheta(z) \leqslant \liminf_{\zeta\to\pm\infty} \frac{2F(z,\zeta)}{\zeta^2} \leqslant \limsup_{\zeta\to\pm\infty} \frac{2F(z,\zeta)}{\zeta^2} \leqslant \widehat{\lambda}_{m+1} \\ \text{uniformly for a.a. } z \in \Omega. \end{gathered} \tag{3.33}$$

In (3.32) we let $|y| \to +\infty$. Using (3.33) we obtain

$$\widehat{\lambda}_{m+1}|u|^2 - 2F(z,u) \geqslant \eta \quad \text{for a.a. } z \in \Omega, \text{ all } |u| \geqslant M_4.$$

Since $\eta > 0$ is arbitrary, it follows that

$$\widehat{\lambda}_{m+1}|u|^2 - 2F(z,u) \longrightarrow +\infty \quad \text{uniformly for a.a. } z \in \Omega, \text{ as } u \to \pm\infty.$$

This proves Claim 2.

For every $v \in V$, we have

$$2\widehat{\varphi}(v) = \gamma(v) - 2\int_\Omega F(z,v)\,dz \geqslant \int_\Omega (\widehat{\lambda}_{m+1}v^2 - 2F(z,v))\,dz \geqslant -c_5,$$

for some $c_5 > 0$ (recall that $\widehat{\varphi} = \varphi|_V$, see (2.4), Claim 2 and hypothesis $H(f)(i)$), so

$$\widehat{\varphi} \text{ is bounded below.} \tag{3.34}$$

Then (3.34), Claim 1 and Proposition 5.22 of Motreanu-Motreanu-Papageorgiou [10, p. 103] imply that $\widehat{\varphi}$ is coercive. From Proposition 2.1 we have

$$\psi(v) = \varphi(v + \tau(v)) = \max_{y\in Y} \varphi(v+y) \geqslant \varphi(v) = \widehat{\varphi}(v) \quad \forall v \in V,$$

so ψ is coercive (since $\widehat{\varphi}$ is coercive). $\square$

Corollary 3.5. *If hypotheses $H(f)$, $H(\xi)$ and $H(\beta)$ hold, then ψ is bounded below, satisfies the Cerami condition and*

$$C_k(\psi,\infty) = \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0.$$

We assume that K_φ is finite. Otherwise we already have an infinity of solutions and so we are done.

Proposition 3.6. *If hypotheses $H(f)$, $H(\xi)$ and $H(\beta)$ hold, then*

$$C_k(\varphi,0) = \delta_{k,d_l}\mathbb{Z} \quad \forall k \geqslant 0,$$

with $d_l = \dim \bigoplus_{k=1}^{l} E(\widehat{\lambda}_k)$.

Proof. Let

$$\widetilde{Y} = \bigoplus_{k=1}^{l} E(\widehat{\lambda}_k) \quad \text{and} \quad \widetilde{V} = \overline{\bigoplus_{k\geqslant l+1} E(\widehat{\lambda}_k)}.$$

We have the following orthogonal direct sum decomposition

$$H^1(\Omega) = \widetilde{Y} \oplus \widetilde{V}.$$

Then every $u \in H^1(\Omega)$ admits a unique sum decomposition of the form

$$u = \widetilde{y} + \widetilde{v}, \quad \text{with } \widetilde{y} \in \widetilde{Y},\ \widetilde{v} \in \widetilde{V}. \tag{3.35}$$

Let $\lambda \in (\widehat{\lambda}_l, \widehat{\lambda}_{l+1})$ and consider the C^2-functional $\varphi_0\colon H^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{2}\gamma(u) - \frac{\lambda}{2}\|u\|_2^2 \quad \forall u \in H^1(\Omega).$$

We consider the homotopy $h(t,u)$ defined by

$$h(t,u)=(1-t)\varphi(u)+t\varphi_0(u) \quad \forall (t,u)\in[0,1]\times H^1(\Omega).$$

Suppose that we can find two sequences $\{t_n\}_{n\geqslant 1}\subseteq[0,1]$ and $\{u_n\}_{n\geqslant 1}\subseteq H^1(\Omega)\setminus\{0\}$ such that

$$\begin{cases} t_n\longrightarrow t \quad \text{in } \mathbb{R}, \quad u_n\longrightarrow 0 \quad \text{in } H^1(\Omega)\\ h'_u(t_n,u_n)=0 \quad \forall n\geqslant 1. \end{cases} \tag{3.36}$$

Since K_φ is finite, we may assume that $t_n\neq 0$ for all $n\geqslant 1$. We have

$$(1-t_n)\langle\varphi'(u_n),h\rangle+t_n\langle\varphi_0'(u_n),h\rangle=0 \quad \forall n\geqslant 1,\ h\in H^1(\Omega), \tag{3.37}$$

so

$$\begin{aligned}&\langle A(u_n),h\rangle+\int_\Omega \xi(z)u_nh\,dz+\int_{\partial\Omega}\beta(z)u_nh\,d\sigma\\ &\quad=\int_\Omega((1-t_n)f(z,u_n)+t_n\lambda u_n)\,dz \quad \forall h\in H^1(\Omega),\ n\geqslant 1,\end{aligned} \tag{3.38}$$

so

$$\begin{cases} -\Delta u_n(z)+\xi(z)u_n(z)=(1-t_n)f(z,u_n(z))+t_n\lambda u_n(z) \quad \text{for a.a. } z\in\Omega,\\ \frac{\partial u_n}{\partial n}+\beta(z)u_n=0 \quad \text{on } \partial\Omega \end{cases}$$

(see Papageorgiou-Rădulescu [13]). The regularity theory of Wang [18] implies that there exists $\alpha\in(0,1)$ and $M_5>0$ such that

$$u_n\in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})}\leqslant M_5 \quad \forall n\geqslant 1. \tag{3.39}$$

Recall that $C^{1,\alpha}(\overline{\Omega})$ is embedded compactly into $C^1(\overline{\Omega})$. So, from (3.36) and (3.39), we have

$$u_n\longrightarrow 0 \quad \text{in } C^1(\overline{\Omega}) \quad \text{as } n\to+\infty. \tag{3.40}$$

From (3.40) it follows that we can find $n_0\geqslant 1$ such that

$$u_n(z)\ \in\ [-\delta,\delta] \quad \forall z\in\overline{\Omega},\ \text{all } n\geqslant n_0. \tag{3.41}$$

In (3.38) we choose $h=\widetilde{v}_n-\widetilde{y}_n\in H^1(\Omega)$ (see (3.35)). Exploiting the orthogonality of the component spaces, we have

$$\gamma(\widetilde{v}_n)-\gamma(\widetilde{y}_n)=\int_\Omega((1-t_n)f(z,u_n)+t_n\lambda u_n)(\widetilde{v}_n-\widetilde{y}_n)\,dz. \tag{3.42}$$

Note that when $u_n(z)\neq 0$, we have

$$\begin{aligned}&f(z,u_n(z))(\widetilde{v}_n-\widetilde{y}_n)(z)\\ &\quad=\frac{f(z,u_n(z))}{u_n(z)}u_n(z)(\widetilde{v}_n-\widetilde{y}_n)(z)\\ &\quad\leqslant\begin{cases}\widehat{\lambda}_{l+1}(\widetilde{v}_n(z)^2-\widetilde{y}_n(z)^2) & \text{if} \quad u_n(z)(\widetilde{v}_n-\widetilde{y}_n)(z)\geqslant 0,\\ \widehat{\lambda}_l(\widetilde{v}_n(z)^2-\widetilde{y}_n(z)^2) & \text{if} \quad u_n(z)(\widetilde{v}_n-\widetilde{y}_n)(z)\leqslant 0\end{cases}\\ &\quad\leqslant\widehat{\lambda}_{l+1}\widetilde{v}_n(z)^2-\widehat{\lambda}_l\widetilde{y}_n(z)^2 \quad \forall n\geqslant n_0\end{aligned} \tag{3.43}$$

(see (3.35) and hypothesis $H(f)(v)$).

When $u_n(z) = 0$, then $f(z, u_n(z)) = 0$ and $\widetilde{v}_n(z) = -\widetilde{y}_n(z)$ (see (3.35)). Hence (3.43) remains valid.

We return to (3.42) and use (3.43). Recalling that $t_n \neq 0$ for all $n \geqslant 1$ and that $\lambda \in (\widehat{\lambda}_l, \widehat{\lambda}_{l+1})$, we have

$$\gamma(\widetilde{v}_n) - \int_\Omega ((1-t_n)\widehat{\lambda}_{l+1} + t_n\lambda)\widetilde{v}_n^2\,dz \leqslant \gamma(\widetilde{y}_n) - \int_\Omega ((1-t_n)\widehat{\lambda}_l + t_n\lambda)\widetilde{y}_n^2\,dz,$$

so

$$c_1\|\widetilde{v}_n\|^2 \leqslant -c_2\|\widetilde{y}_n\|^2 \quad \forall n \geqslant n_0$$

(see Proposition 2.1), thus

$$\widetilde{v}_n = \widetilde{y}_n = 0 \quad \forall n \geqslant n_0,$$

hence

$$u_n = 0 \quad \forall n \geqslant n_0,$$

a contradiction. This shows that (3.36) cannot occur. Then the homotopy invariance of critical groups (see Gasiński-Papageorgiou [8, Theorem 5.125, p. 836]) implies that

$$C_k(\varphi, 0) = C_k(\varphi_0, 0) \quad \forall k \geqslant 0. \tag{3.44}$$

Recall that $\lambda \in (\widehat{\lambda}_l, \widehat{\lambda}_{l+1})$. Hence $K_{\varphi_0} = \{0\}$ and $u = 0$ is a nondegenerate critical point of φ_0 with Morse index $d_l = \dim \bigoplus_{k=1}^{l} E(\widehat{\lambda}_k)$. Since $\varphi_0 \in C^2(H^1(\Omega))$, we can apply Theorem 6.51 of Motreanu-Motreanu-Papageorgiou [10, p. 155] and have that

$$C_k(\varphi_0, 0) = \delta_{k,d_l}\mathbb{Z} \quad \forall k \geqslant 0,$$

so

$$C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \forall k \geqslant 0$$

(see (3.44)) $\square$

Using Proposition 3.6 and Theorem 1.2 of Li-Liu [9], we obtain the following result.

Corollary 3.7. *If hypotheses $H(f)$, $H(\xi)$ and $H(\beta)$ hold, then*

$$C_k(\varphi, 0) = C_{k-d_m}(\psi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \forall k \geqslant 0.$$

The next result is an easy observation which relates the critical sets K_φ and K_ψ.

Proposition 3.8. *If hypotheses $H(f)$, $H(\xi)$ and $H(\beta)$ hold, then $v \in K_\psi$ if and only if $v + \tau(v) \in K_\varphi$.*

Proof. "$\Longrightarrow$": We have

$$0 = \psi'(v) = p_{V^*}\varphi'(v + \tau(v)) \tag{3.45}$$

(see Proposition 3.3). We know that $H^1(\Omega)^* = Y^* \oplus V^*$. So, from (3.45), it follows that $\varphi'(v+\tau(v)) \in Y^*$. But from (3.7), we have $p_{Y^*}\varphi'(v+\tau(v)) = 0$, so $\varphi'(v+\tau(v)) = 0$, thus $v + \tau(v) \in K_\varphi$.

"$\Longleftarrow$": Follows from Proposition 3.3. $\square$

Now we are ready for the multiplicity theorem for problem (1.1). We produce two nontrivial smooth solution.

Theorem 3.9. *If hypotheses $H(f)$, $H(\xi)$ and $H(\beta)$ hold, then problem* (1.1) *admits at least two nontrivial solutions*

$$u_0, \widehat{u} \in C^1(\overline{\Omega}),\ u_0 \neq \widehat{u}.$$

Proof. From Proposition 3.4 we know that ψ is coercive. Also, the Sobolev embedding theorem, the compactness of the trace operator and the continuity of the map τ (see Proposition 3.2) imply that ψ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $v_0 \in V$ such that

$$\psi(v_0) = \inf_{v\in V} \psi(v),$$

so

$$C_k(\psi, v_0) = \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0. \tag{3.46}$$

From Corollary 3.7, we have

$$C_k(\psi, 0) = \delta_{k,d_l-d_m}\mathbb{Z} \quad \forall k \geqslant 0. \tag{3.47}$$

According to hypothesis $H(f)(v)$, $l > m$. Hence $d_l > d_m$. So, comparing (3.46) and (3.47), we infer that $v_0 \neq 0$, therefore $u_0 = v_0+\tau(v_0) \in K_\varphi\setminus\{0\}$ (see Proposition 3.8).

Hypotheses $H(f)$ imply that

$$|f(z,\zeta)| \leqslant c_6|\zeta| \quad \text{for a.a. } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}, \tag{3.48}$$

for some $c_6 > 0$. We have

$$\begin{cases} -\Delta u_0(z) + \xi(z)u_0(z) = f(z, u_0(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial u_0}{\partial n} + \beta(z)u_0 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.49}$$

We define

$$\eta(z) = \begin{cases} \frac{f(z,u_0(z))}{u_0(z)} & \text{if } \ u_0(z) \neq 0, \\ 0 & \text{if } \ u_0(z) = 0. \end{cases}$$

Evidently $\eta \in L^\infty(\Omega)$ (see (3.48)). From (3.49), we have

$$\begin{cases} -\Delta u_0(z) = \widehat{\eta}(z)u_0(z) & \text{for a.a. } z \in \Omega, \\ \frac{\partial u_0}{\partial n} + \beta(z)u_0 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

with $\widehat{\eta} = \eta - \xi \in L^s(\Omega)$, $s > N$ (see hypothesis $H(\xi)$). From Lemma 5.1 of Wang [18], we have that $u_0 \in L^\infty(\Omega)$. Then using the Calderon-Zygmund estimates (see Lemma 5.2 of Wang [18]), we obtain

$$u_0 \in W^{2,s}(\Omega),$$

so

$$u_0 \in C^{1,\alpha}(\overline{\Omega}),$$

with $\alpha = 1 - \frac{N}{s} > 0$ (by the Sobolev embedding theorem).

Suppose that $K_\psi = \{0, v_0\}$. Then from (3.46), (3.47), Corollary 3.5 and the Morse relation with $t = -1$ (see (2.5)), we have

$$(-1)^{d_l - d_m} = 0,$$

a contradiction. So, there exists $\widehat{v} \in K_\psi \setminus \{0, v_0\}$. Then

$$\widehat{u} = \widehat{v} + \tau(\widehat{v}) \in K_\varphi \setminus \{0, u_0\}$$

and as before using the regularity theory of Wang [18], we have $\widehat{u} \in C^1(\overline{\Omega})$. $\square$

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