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ATTRACTIVE AND MEAN CONVERGENCE THEOREMS FOR TWO COMMUTATIVE NONLINEAR MAPPINGS IN BANACH SPACES

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Dedicated to Professor Ravi Agarwal on the occasion of his 70th birthday

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ABSTRACT. In this paper, using the class of 2-generalized nonspreading mappings which was defined by [29] in a Banach space and covers 2-generalized hybrid mappings in a Hilbert space, we prove an attractive point theorem in a Banach space. Then we prove a mean convergence theorem of Baillon's type [2] without convexity for commutative 2-generalized nonspreading mappings in a Banach space.

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1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. Then we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [27] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

We know from [27] that $A(T)$ is closed and convex. This property is important for proving mean convergence theorems. Such a concept of attractive points was defined in a Banach space; see [20]. A mapping $T : C \to H$ is said to be *nonexpansive* [4] if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Baillon [2] proved the first mean convergence theorem in a Hilbert space.

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Theorem 1.1 ([2]). *Let C be a bounded, closed and convex subset of H and let $T : C \to C$ be nonexpansive. Then for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

This theorem for nonexpansive mappings has been extended to Banach spaces by many authors; see, for example, [3, 5]. On the other hand, in 2010, Kocourek, Takahashi and Yao [13] defined a broad class of nonlinear mappings in a Hilbert space: Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* [13] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is called $(\alpha,\ \beta)$-*generalized hybrid*. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is *nonspreading* [17, 18] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [25] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [10]. The mean convergence theorem by Baillon [2] for nonexpansive mappings has been extended to generalized hybrid mappings in a Hilbert space by Kocourek, Takahashi and Yao [13]. Furthermore, Takahashi and Takeuchi [27] proved the following mean convergence theorem without convexity in a Hilbert space.

Theorem 1.2 ([27]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let T be a generalized hybrid mapping from C into itself. Assume that $\{T^n z\}$ for some $z \in C$ is bounded and define*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

for all $x \in C$ and $n \in \mathbb{N}$. Then $\{S_n x\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n\to\infty} P_{A(T)} T^n x$ and $P_{A(T)}$ is the metric projection of H onto $A(T)$.

Maruyama, Takahashi and Yao [21] also defined a more broad class of nonlinear mappings called 2-generalized hybrid which covers generalized hybrid mappings in a Hilbert space. Let C be a nonempty subset of H and let T be a mapping of C into H. A mapping $T : C \to H$ is 2-*generalized hybrid* [21] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}(1.2) \qquad \alpha_1\|T^2x-Ty\|^2 + \alpha_2\|Tx-Ty\|^2 + (1-\alpha_1-\alpha_2)\|x-Ty\|^2 \\ \leq \beta_1\|T^2x-y\|^2 + \beta_2\|Tx-y\|^2 + (1-\beta_1-\beta_2)\|x-y\|^2\end{aligned}$$

for all $x, y \in C$.

Recently, Hojo, Takahashi and Takahashi [6] proved attractive and mean convergence theorems without convexity for commutative 2-generalized hybrid mappings in a Hilbert space. These results generalize Takahashi and Takeuchi's theorem (Theorem 1.2) and Kohsaka's theorem [15] which is a mean convergence theorem for commutative λ-hybrid mappings in a Hilbert space.

In this paper, using the class of 2-generalized nonspreading mappings which was defined by [29] in a Banach space and covers 2-generalized hybrid mappings in a Hilbert space, we prove an attractive point theorem in a Banach space. This theorem generalizes Hojo, Takahashi and Takahashi's attractive point theorem [6] in a Hilbert space. Then we prove a mean convergence theorem of Baillon's type [2] without convexity for commutative 2-generalized nonspreading mappings in a Banach space. This result is a general mean convergence theorem which extends Baillon's theorem (Theorem 1.1) to a Banach space.

2. PRELIMINARIES

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to E$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty, closed and convex subset of a strictly convex Banach space E and $T : C \to E$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [11].

Let E be a Banach space. The *duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called *smooth*. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm to norm continuous. If E is uniformly smooth, J is uniformly norm to norm continuous on each bounded subset of E. For more details, see [23, 24].

Lemma 2.1 ([23, 24])**.** *Let E be a smooth Banach space and let J be the duality mapping on E. Then $\langle x - y, Jx - Jy \rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi\colon E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \tag{2.2}$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [12]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.3}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.4}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then from Lemma 2.1 we have

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.5}$$

The following lemma is in Xu [33].

Lemma 2.2 ([33]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Using Lemma 2.2, we have the following lemma by Kamimura and Takahashi [12].

Lemma 2.3 ([12]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $A(T)$ the set of *attractive points* of T, i.e., $A(T) = \{z \in E : \phi(z, Tx) \leq \phi(z, x),\ \forall x \in C\}$; see [20].

Lemma 2.4 ([20]). *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $A(T)$ is a closed and convex subset of E.*

Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called *generalized nonexpansive* [8] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$; see also [32]. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [8] for more details. The following results are in Ibaraki and Takahashi [8].

Lemma 2.5 ([8]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.6 ([8]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) *$z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;*
(ii) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

In 2007, Kohsaka and Takahashi [16] proved the following results:

Lemma 2.7 ([16]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) *C is a sunny generalized nonexpansive retract of E;*
(b) *C is a generalized nonexpansive retract of E;*
(c) *JC is closed and convex.*

Lemma 2.8 ([16]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) *$z = Rx$;*
(ii) *$\phi(x, z) = \min_{y \in C} \phi(x, y)$.*

Ibaraki and Takahashi [9] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Lemma 2.9 ([9]). *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following theorem is proved by using Lemmas 2.7 and 2.9.

Lemma 2.10 ([9]). *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Using Lemma 2.7, we also have the following result.

Lemma 2.11 ([26]). *Let E be a smooth, strictly convex and reflexive Banach space and let $\{C_i : i \in I\}$ be a family of sunny generalized nonexpansive retracts of E such that $\cap_{i \in I} C_i$ is nonempty. Then $\cap_{i \in I} C_i$ is a sunny generalized nonexpansive retract of E.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at

$f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [23].

3. FIXED POINT THEOREMS

Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called 2-*generalized nonspreading* [29] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha_1\phi(T^2x, Ty) + \alpha_2\phi(Tx, Ty) + (1 - \alpha_1 - \alpha_2)\phi(x, Ty) \\
&\quad + \gamma_1\{\phi(Ty, T^2x) - \phi(Ty, x)\} + \gamma_2\{\phi(Ty, Tx) - \phi(Ty, x)\} \\
&\le \beta_1\phi(T^2x, y) + \beta_2\phi(Tx, y) + (1 - \beta_1 - \beta_2)\phi(x, y) \\
&\quad + \delta_1\{\phi(y, T^2x) - \phi(y, x)\} + \delta_2\{\phi(y, Tx) - \phi(y, x)\}
\end{aligned} \tag{3.1}$$

for all $x, y \in C$; see also [30]. Such a mapping is called $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$-generalized nonspreading. We know that a $(0, \alpha_2, 0, \beta_2, 0, \gamma_2, 0, \delta_2)$-generalized nonspreading mapping is generalized nonspreading in the sense of [14]. We also know that a $(0, 1, 0, 1, 0, 1, 0, 0)$-generalized nonspreading mapping is nonspreading in the sense of [18].

Now we prove an attractive point theorem for commutative 2-generalized nonspreading mappings in a Banach space. Before proving it, we prove the following result.

Lemma 3.1. *Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J and let C be a nonempty subset of E. Let S and T be mappings of C into itself. Let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on l^∞. Suppose that*

$$\mu_n\phi(x_n, Sy) \le \mu_n\phi(x_n, y) \text{ and } \mu_n\phi(x_n, Ty) \le \mu_n\phi(x_n, y)$$

for all $y \in C$. Then $A(S) \cap A(T)$ is nonemppty. Additionally, if C is closed and convex and $\{x_n\} \subset C$, then $F(S) \cap F(T)$ is nonempty.

Proof. Using a mean μ and a bounded sequence $\{x_n\}$, we define a function $g : E^* \to \mathbb{R}$ as follows:

$$g(x^*) = \mu_n\langle x_n, x^*\rangle$$

for all $x^* \in E^*$. Since μ is linear, g is also linear. Furthermore, we have

$$\begin{aligned} |g(x^*)| &= |\mu_n \langle x_n, x^* \rangle| \\ &\leq \|\mu\| \sup_{n \in \mathbb{N}} |\langle x_n, x^* \rangle| \\ &\leq \|\mu\| \sup_{n \in \mathbb{N}} \|x_n\| \|x^*\| \\ &= \sup_{n \in \mathbb{N}} \|x_n\| \|x^*\| \end{aligned}$$

for all $x^* \in E^*$. Then g is a linear and bounded real-valued function on E^*. Since E is reflexive, there exists a unique element z of E such that

$$g(x^*) = \mu_n \langle x_n, x^* \rangle = \langle z, x^* \rangle$$

for all $x^* \in E^*$. From (2.3) we have that for $y \in C$ and $n \in \mathbb{N}$,

$$\phi(x_n, y) = \phi(x_n, Sy) + \phi(Sy, y) + 2\langle x_n - Sy, JSy - Jy \rangle.$$

So, we have that for $y \in C$,

$$\begin{aligned} \mu_n \phi(x_n, y) &= \mu_n \phi(x_n, Sy) + \mu_n \phi(Sy, y) + 2\mu_n \langle x_n - Sy, JSy - Jy \rangle \\ &= \mu_n \phi(x_n, Sy) + \phi(Sy, y) + 2\langle z - Sy, JSy - Jy \rangle. \end{aligned}$$

Since, by assumption, $\mu_n \phi(x_n, Sy) \leq \mu_n \phi(x_n, y)$ for all $y \in C$, we have

$$\mu_n \phi(x_n, y) \leq \mu_n \phi(x_n, y) + \phi(Sy, y) + 2\langle z - Sy, JSy - Jy \rangle.$$

This implies that

$$0 \leq \phi(Sy, y) + 2\langle z - Sy, JSy - Jy \rangle.$$

Using (2.4), we have that

$$0 \leq \phi(Sy, y) + \phi(z, y) + \phi(Sy, Sy) - \phi(z, Sy) - \phi(Sy, y)$$

and hence $\phi(z, Sy) \leq \phi(z, y)$. This implies that z is an element of $A(S)$. Similarly, we have that $\phi(z, Ty) \leq \phi(z, y)$ and hence $z \in A(T)$. Therefore we have $z \in A(S) \cap A(T)$. Additionally, if C is closed and convex and $\{x_n\} \subset C$, we have that $z \in \overline{co}\{x_n : n \in \mathbb{N}\} \subset C$. In fact, if $z \notin C$, then there exists $y^* \in E^*$ by the separation theorem [23] such that $\langle z, y^* \rangle < \inf_{y \in C} \langle y, y^* \rangle$. So, from $\{x_n\} \subset C$ we have

$$\langle z, y^* \rangle < \inf_{y \in C} \langle y, y^* \rangle \leq \inf_{n \in \mathbb{N}} \langle x_n, y^* \rangle \leq \mu_n \langle x_n, y^* \rangle = \langle z, y^* \rangle.$$

This is a contradiction. Then we have $z \in C$. Since $z \in A(S) \cap A(T)$ and $z \in C$, we have that

$$\phi(z, Sz) \leq \phi(z, z) = 0 \quad \text{and} \quad \phi(z, Tz) \leq \phi(z, z) = 0$$

and hence $\phi(z, Sz) = 0$ and $\phi(z, Tz) = 0$. Since E is strictly convex, we have $z \in F(S) \cap F(T)$. This completes the proof. $\square$

Using Lemma 3.1, we prove an attractive point theorem for commutative 2-generalized nonspreading mappings in a Banach space.

Theorem 3.2. *Let C be a nonempty subset of a smooth, strictly convex and reflexive Banach space E and let S and T be commutative 2-generalized nonspreading mappings of C into itself. Suppose that there exists an element $z \in C$ such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

Proof. Since S is a 2-generalized nonspreading mapping of C into itself, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that for all $x, y \in C$,

$$\begin{aligned}
&\alpha_1\phi(S^2x, Sy) + \alpha_2\phi(Sx, Sy) + (1 - \alpha_1 - \alpha_2)\phi(x, Sy) \\
&\qquad + \gamma_1\{\phi(Sy, S^2x) - \phi(Sy, x)\} + \gamma_2\{\phi(Sy, Sx) - \phi(Sy, x)\} \\
&\quad \le \beta_1\phi(S^2x, y) + \beta_2\phi(Sx, y) + (1 - \beta_1 - \beta_2)\phi(x, y) \\
&\qquad + \delta_1\{\phi(y, S^2x) - \phi(y, x)\} + \delta_2\{\phi(y, Sx) - \phi(y, x)\}.
\end{aligned} \tag{3.2}$$

By assumption, we can take $z \in C$ such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Replacing x by S^kT^lz in (3.2), we have that for any $y \in C$ and $k, l \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\alpha_1\phi(S^{k+2}T^lz, Sy) + \alpha_2\phi(S^{k+1}T^lz, Sy) + (1 - \alpha_1 - \alpha_2)\phi(S^kT^lz, Sy) \\
&\quad + \gamma_1\{\phi(Sy, S^{k+2}T^lz) - \phi(Sy, S^kT^lz)\} + \gamma_2\{\phi(Sy, S^{k+1}T^lz) - \phi(Sy, S^kT^lz)\} \\
&\le \beta_1\phi(S^{k+2}T^lz, y) + \beta_2\phi(S^{k+1}T^lz, y) + (1 - \beta_1 - \beta_2)\phi(S^kT^lz, y) \\
&\quad + \delta_1\{\phi(y, S^{k+2}T^lz) - \phi(y, S^kT^lz)\} + \delta_2\{\phi(y, S^{k+1}T^lz) - \phi(y, S^kT^lz)\} \\
&= \beta_1\{\phi(S^{k+2}T^lz, Sy) + \phi(Sy, y) + 2\langle S^{k+2}T^lz - Sy, JSy - Jy\rangle\} \\
&\quad + \beta_2\{\phi(S^{k+1}T^lz, Sy) + \phi(Sy, y) + 2\langle S^{k+1}T^lz - Sy, JSy - Jy\rangle\} \\
&\quad + (1 - \beta_1 - \beta_2)\{\phi(S^kT^lz, Sy) + \phi(Sy, y) + 2\langle S^kT^lz - Sy, JSy - Jy\rangle\} \\
&\quad + \delta_1\{\phi(y, S^{k+2}T^lz) - \phi(y, S^kT^lz)\} + \delta_2\{\phi(y, S^{k+1}T^lz) - \phi(y, S^kT^lz)\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
0 &\le (\beta_1 - \alpha_1)\{\phi(S^{k+2}T^lz, Sy) - \phi(S^kT^lz, Sy)\} \\
&\quad + (\beta_2 - \alpha_2)\{\phi(S^{k+1}T^lz, Sy) - \phi(S^kT^lz, Sy)\} + \phi(Sy, y) \\
&\quad + 2\langle S^kT^lz - Sy + \beta_1(S^{k+2}T^lz - S^kT^lz) + \beta_2(S^{k+1}T^lz - S^kT^lz), JSy - Jy\rangle \\
&\quad - \gamma_1\{\phi(Sy, S^{k+2}T^lz) - \phi(Sy, S^kT^lz)\} - \gamma_2\{\phi(Sy, S^{k+1}T^lz) - \phi(Sy, S^kT^lz)\} \\
&\quad + \delta_1\{\phi(y, S^{k+2}T^lz) - \phi(y, S^kT^lz)\} + \delta_2\{\phi(y, S^{k+1}T^lz) - \phi(y, S^kT^lz)\}.
\end{aligned}$$

Summing up these inequalities with respect to $k = 0, 1, \dots, n$, we have

$$\begin{aligned}
0 \le (\beta_1 - \alpha_1)\{&\phi(S^{n+2}T^lz, Sy) + \phi(S^{n+1}T^lz, Sy) \\
&- \phi(ST^lz, Sy) - \phi(T^lz, Sy)\}
\end{aligned}$$

$$
\begin{aligned}
&+ (\beta_2 - \alpha_2)\{\phi(S^{n+1}T^l z, Sy) - \phi(T^l z, Sy)\} + (n+1)\phi(Sy, y) \\
&+ 2\Big\langle \sum_{k=0}^{n} S^k T^l z + \beta_1(S^{n+2}T^l z + S^{n+1}T^l z - ST^l z - T^l z) \\
&\quad + \beta_2(S^{n+1}T^l z - T^l z) - (n+1)Sy, JSy - Jy\Big\rangle \\
&- \gamma_1\{\phi(Sy, S^{n+2}T^l z) + \phi(Sy, S^{n+1}T^l z) - \phi(Sy, ST^l z) - \phi(Sy, T^l z)\} \\
&- \gamma_2\{\phi(Sy, S^{n+1}T^l z) - \phi(Sy, T^l z)\} \\
&+ \delta_1\{\phi(y, S^{n+2}T^l z) + \phi(y, S^{n+1}T^l z) - \phi(y, ST^l z) - \phi(y, T^l z)\} \\
&+ \delta_2\{\phi(y, S^{n+1}T^l z) - \phi(y, T^l z)\}.
\end{aligned}
$$

Furthermore, summing up these inequalities with respect to $l = 0, 1, \ldots, n$, we have

$$
\begin{aligned}
0 \le\ & (\beta_1 - \alpha_1)\sum_{l=0}^{n}\{\phi(S^{n+2}T^l z, Sy) + \phi(S^{n+1}T^l z, Sy) \\
&\quad - \phi(ST^l z, Sy) - \phi(T^l z, Sy)\} \\
&+ (\beta_2 - \alpha_2)\sum_{l=0}^{n}\{\phi(S^{n+1}T^l z, Sy) - \phi(T^l z, Sy)\} + (n+1)^2\phi(Sy, y) \\
&+ 2\Big\langle \sum_{l=0}^{n}\sum_{k=0}^{n} S^k T^l z + \beta_1\sum_{l=0}^{n}(S^{n+2}T^l z + S^{n+1}T^l z - ST^l z - T^l z) \\
&\quad + \beta_2\sum_{l=0}^{n}(S^{n+1}T^l z - T^l z) - (n+1)^2 Sy, JSy - Jy\Big\rangle \\
&- \gamma_1\sum_{l=0}^{n}\{\phi(Sy, S^{n+2}T^l z) + \phi(Sy, S^{n+1}T^l z) - \phi(Sy, ST^l z) - \phi(Sy, T^l z)\} \\
&- \gamma_2\sum_{l=0}^{n}\{\phi(Sy, S^{n+1}T^l z) - \phi(Sy, T^l z)\} \\
&+ \delta_1\sum_{l=0}^{n}\{\phi(y, S^{n+2}T^l z) + \phi(y, S^{n+1}T^l z) - \phi(y, ST^l z) - \phi(y, T^l z)\} \\
&+ \delta_2\sum_{l=0}^{n}\{\phi(y, S^{n+1}T^l z) - \phi(y, T^l z)\}.
\end{aligned}
$$

Dividing by $(n+1)^2$, we have

$$
\begin{aligned}
0 \le\ & (\beta_1 - \alpha_1)\frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(S^{n+2}T^l z, Sy) + \phi(S^{n+1}T^l z, Sy) \\
&\quad - \phi(ST^l z, Sy) - \phi(T^l z, Sy)\} \\
&+ (\beta_2 - \alpha_2)\frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(S^{n+1}T^l z, Sy) - \phi(T^l z, Sy)\} + \phi(Sy, y)
\end{aligned}
$$

$$+ 2\Big\langle S_n z + \beta_1 \frac{1}{(n+1)^2} \sum_{l=0}^{n} (S^{n+2}T^l z + S^{n+1}T^l z - ST^l z - T^l z)$$

$$+ \beta_2 \frac{1}{(n+1)^2} \sum_{l=0}^{n} (S^{n+1}T^l z - T^l z) - Sy, JSy - Jy\Big\rangle$$

$$- \gamma_1 \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(Sy, S^{n+2}T^l z) + \phi(Sy, S^{n+1}T^l z)$$

$$- \phi(Sy, ST^l z) - \phi(Sy, T^l z)\}$$

$$- \gamma_2 \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(Sy, S^{n+1}T^l z) - \phi(Sy, T^l z)\}$$

$$+ \delta_1 \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(y, S^{n+2}T^l z) + \phi(y, S^{n+1}T^l z) - \phi(y, ST^l z) - \phi(y, T^l z)\}$$

$$+ \delta_2 \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(y, S^{n+1}T^l z) - \phi(y, T^l z)\},$$

where $S_n z = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l z$. Since $\{S^k T^l z\}$ is bounded by assumption, $\{S_n z\}$ is bounded. Taking a Banach limit μ to both sides of this inequality, we have that

$$0 \le \phi(Sy, y) + 2\mu_n \langle S_n z - Sy, JSy - Jy\rangle$$

and hence

$$0 \le \phi(Sy, y) + \mu_n \phi(S_n z, y) + \phi(Sy, Sy) - \mu_n \phi(S_n z, Sy) - \phi(Sy, y).$$

Thus, we have

$$\mu_n \phi(S_n z, Sy) \le \mu_n \phi(S_n z, y).$$

Similarly, replacing S and T by T and S, respectively, we have

$$\mu_n \phi(S_n z, Ty) \le \mu_n \phi(S_n z, y).$$

Using Lemma 3.1, we have that $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty. $\square$

Since commutative 2-generalized hybrid mappings in a Hilbert space are commutative 2-generalized nonspreading mappings in a Banach space, as a direct sequence of Theorem 3.2, we have the following theorem proved by Hojo, Takahashi and Takahashi [6] in a Hilbert space.

Theorem 3.3 ([6]). *Let H be a Hilbert space, let C be a nonempty subset of H and let S and T be commutative 2-generalized hybrid mappings of C into itself. Suppose that there exists an element $z \in C$ such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

4. NONLINEAR ERGODIC THEOREMS

Let E be a smooth Banach space, let C be a nonempty subset of E and let J be the duality mapping from E into E^*. Observe that if $T : C \to E$ is a 2-generalized nonspreading mapping and $F(T) \neq \emptyset$, then

$$\phi(u, Ty) \le \phi(u, y)$$

for all $u \in F(T)$ and $y \in C$. Indeed, putting $x = u \in F(T)$ in (3.1), we obtain that

$$\begin{aligned}
&\alpha_1\phi(u, Ty) + \alpha_2\phi(u, Ty) + (1 - \alpha_1 - \alpha_2)\phi(u, Ty)\\
&\qquad + \gamma_1\{\phi(Ty, u) - \phi(Ty, u)\} + \gamma_2\{\phi(Ty, u) - \phi(Ty, u)\}\\
&\quad \le \beta_1\phi(u, y) + \beta_2\phi(u, y) + (1 - \beta_1 - \beta_2)\phi(u, y)\\
&\qquad + \delta_1\{\phi(y, u) - \phi(y, u)\} + \delta_2\{\phi(y, u) - \phi(y, u)\}.
\end{aligned}$$

So, we have that

$$\phi(u, Ty) \le \phi(u, y) \tag{4.1}$$

for all $u \in F(T)$ and $y \in C$. Similarly, putting $y = u \in F(T)$ in (3.1), we obtain that for $x \in C$,

$$\begin{aligned}
&\alpha_1\phi(T^2x, u) + \alpha_2\phi(Tx, u) + (1 - \alpha_1 - \alpha_2)\phi(x, u)\\
&\qquad + \gamma_1\{\phi(u, T^2x) - \phi(u, x)\} + \gamma_2\{\phi(u, Tx) - \phi(u, x)\}\\
&\quad \le \beta_1\phi(T^2x, u) + \beta_2\phi(Tx, u) + (1 - \beta_1 - \beta_2)\phi(x, u)\\
&\qquad + \delta_1\{\phi(u, T^2x) - \phi(u, x)\} + \delta_2\{\phi(u, Tx) - \phi(u, x)\}
\end{aligned}$$

and hence

$$\begin{aligned}
&(\alpha_1 - \beta_1)\{\phi(T^2x, u) - \phi(x, u)\} + (\alpha_2 - \beta_2)\{\phi(Tx, u) - \phi(x, u)\}\\
&\qquad + (\gamma_1 - \delta_1)\{\phi(u, T^2x) - \phi(u, x)\} + (\gamma_2 - \delta_2)\{\phi(u, Tx) - \phi(u, x)\} \le 0.
\end{aligned}$$

If $\alpha_1 - \beta_1 = 0$, $\gamma_1 \le \delta_1$, $\gamma_2 \le \delta_2$ and $\alpha_2 > \beta_2$, then we have from (4.1) that

$$\begin{aligned}
(\alpha_2-\beta_2)\{\phi(Tx, u) - \phi(x, u)\} &\le (\delta_1 - \gamma_1)\{\phi(u, T^2x) - \phi(u, x)\}\\
&\quad + (\delta_2 - \gamma_2)\{\phi(u, Tx) - \phi(u, x)\} \le 0.
\end{aligned}$$

So, we have that

$$\phi(Tx, u) \le \phi(x, u) \tag{4.2}$$

for all $x \in C$ and $u \in F(T)$. This implies that T is generalized nonexpansive in the sense of [8].

Now using the technique developed by [22] and [28], we can prove a mean convergence theorem without convexity for commutative 2-generalized nonspreading mappings in a Banach space. For proving this result, we need the following lemmas.

Lemma 4.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let S and T be commutative 2-generalized nonspreading mappings of C into itself. If $\{S^kT^lz : k,l \in \mathbb{N}\cup\{0\}\}$ for some $z \in C$ is bounded and*

$$S_nx = \frac{1}{(1+n)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^kT^lx$$

for all $x \in C$ and $n \in \mathbb{N}\cup\{0\}$, then every weak cluster point of $\{S_nx\}$ is a point of $A(S)\cap A(T)$. Additionally, if C is closed and convex, then every weak cluster point of $\{S_nx\}$ is a point of $F(S)\cap F(T)$.

Proof. Since $S : C \to C$ is 2-generalized nonspreading, we have that for all $x, y \in C$, (3.2) holds. Since there exists $z \in C$ such that $\{S^kT^lz : k,l \in \mathbb{N}\cup\{0\}\}$ is bounded, $\{S^kT^lx : k,l \in \mathbb{N}\cup\{0\}\}$ for all $x \in C$ is bounded. Then as in the proof of Theorem 3.2, we have that for any $y \in C$

$$\begin{aligned}
0 \le (\beta_1 - \alpha_1)&\frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(S^{n+2}T^lx, Sy) + \phi(S^{n+1}T^lx, Sy)\\
&\quad - \phi(ST^lx, Sy) - \phi(T^lx, Sy)\}\\
&+ (\beta_2 - \alpha_2)\frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(S^{n+1}T^lx, Sy) - \phi(T^lx, Sy)\} + \phi(Sy, y)\\
&+ 2\Big\langle S_nx + \beta_1\frac{1}{(n+1)^2}\sum_{l=0}^{n}(S^{n+2}T^lx + S^{n+1}T^lx - ST^lx - T^lx)\\
&\quad + \beta_2\frac{1}{(n+1)^2}\sum_{l=0}^{n}(S^{n+1}T^lx - T^lx) - Sy, JSy - Jy\Big\rangle\\
&- \gamma_1\frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(Sy, S^{n+2}T^lx) + \phi(Sy, S^{n+1}T^lx)\\
&\quad - \phi(Sy, ST^lx) - \phi(Sy, T^lx)\}\\
&- \gamma_2\frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(Sy, S^{n+1}T^lx) - \phi(Sy, T^lx)\}\\
&+ \delta_1\frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(y, S^{n+2}T^lx) + \phi(y, S^{n+1}T^lx) - \phi(y, ST^lx) - \phi(y, T^lx)\}\\
&+ \delta_2\frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(y, S^{n+1}T^lx) - \phi(y, T^lx)\}.
\end{aligned}$$

Since $\{S^kT^lx\}$ is bounded, $\{S_nx\}$ is bounded. Thus we have a subsequence $\{S_{n_i}x\}$ of $\{S_nx\}$ such that $\{S_{n_i}x\}$ converges weakly to a point $u \in E$. Letting $n_i \to \infty$, we obtain

$$0 \le \phi(Sy, y) + 2\langle u - Sy, JSy - Jy\rangle.$$

Using (2.4), we have that

$$0 \leq \phi(Sy, y) + \phi(u, y) + \phi(Sy, Sy) - \phi(u, Sy) - \phi(Sy, y)$$

and hence

$$\phi(u, Sy) \leq \phi(u, y).$$

This implies that u is an element of $A(S)$. Similarly, we have that

$$\phi(u, Ty) \leq \phi(u, y).$$

and hence $u \in A(T)$. Therefore we have $u \in A(S) \cap A(T)$. Additionally, if C is closed and convex, we have that $\{S_n x\} \subset C$ and then

$$u \in \overline{co}\{S_n x : n \in \mathbb{N}\} \subset C.$$

Since $u \in A(S) \cap A(T)$ and $u \in C$, we have that

$$\phi(u, Su) \leq \phi(u, u) = 0 \quad \text{and} \quad \phi(u, Tu) \leq \phi(u, u) = 0$$

and hence

$$\phi(u, Su) = 0 \quad \text{and} \quad \phi(u, Tu) = 0.$$

Since E is strictly convex, we have $u \in F(S) \cap F(T)$. This completes the proof. □

Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $B(T)$ the set of *skew-attractive points* of T, i.e., $B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z),\ \forall x \in C\}$. The following result was proved by Lin and Takahashi [20].

Lemma 4.2 ([20]). *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $B(T)$ is closed.*

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E. Define a mapping T^* as follows:

$$T^* x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the duality mapping of T; see also [31] and [7]. It is easy to show that if T is a mapping of C into itselt, then T^* is a mapping of JC into itself. In fact, for $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$. So, we have

$$T^* x^* = JTJ^{-1}x^* \in JC.$$

Then, T^* is a mapping of JC into itself. Using Lemma 2.4, we have the following result.

Lemma 4.3 ([20]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E and let T^* be the duality mapping of T. Then, the following hold:*

(1) $JB(T) = A(T^*)$;
(2) $JA(T) = B(T^*)$.

In particular, $JB(T)$ is closed and convex.

Let $D = \{(k,l) : k,l \in \mathbb{N} \cup \{0\}\}$. Then D is a directed set by the binary relation:

$$(k,l) \leq (i,j) \quad \text{if } k \leq i \text{ and } l \leq j.$$

Theorem 4.4. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let $S,T : C \to C$ be commutative 2-generalized nonspreading mappings such that $\{S^kT^lz : k,l \in \mathbb{N} \cup \{0\}\}$ for some $z \in C$ is bounded, $A(S) = B(S)$ and $A(T) = B(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $B(S) \cap B(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $A(S) \cap A(T)$, where $q = \lim_{(k,l)\in D} RS^kT^lx$.

Proof. We have from Theorem 3.2 that $A(S) \cap A(T) = B(S) \cap B(T)$ is nonempty. We know from Lemmas 2.11, 4.2 and 4.3 that $B(S) \cap B(T)$ is closed, and

$$J(B(S) \cap B(T)) = JB(S) \cap JB(T)$$

is closed and convex. So, from Lemma 2.5 and Lemma 2.7 there exists the sunny generalized nonexpansive retraction R of E onto $B(S) \cap B(T)$. From Lemma 2.8, this retraction R is characterized by

$$Rx = \arg \min_{u \in B(S)\cap B(T)} \phi(x,u).$$

We also know from Lemma 2.6 that

$$0 \leq \langle v - Rv, JRv - Ju \rangle, \quad \forall u \in B(S) \cap B(T),\ v \in C.$$

Adding up $\phi(Rv,u)$ to both sides of this inequality, we have

$$\begin{aligned} \phi(Rv,u) &\leq \phi(Rv,u) + 2\langle v - Rv, JRv - Ju \rangle \\ &= \phi(Rv,u) + \phi(v,u) + \phi(Rv,Rv) - \phi(v,Rv) - \phi(Rv,u) \\ &= \phi(v,u) - \phi(v,Rv). \end{aligned} \tag{4.3}$$

Since $\phi(Sz,u) \leq \phi(z,u)$ and $\phi(Tz,u) \leq \phi(z,u)$ for any $u \in B(S) \cap B(T)$ and $z \in C$, it follows that for any $(k,l),(i,j) \in D$ with $(k,l) \leq (i,j)$,

$$\phi(S^iT^jx, RS^iT^jx) \leq \phi(S^iT^jx, RS^kT^lx)$$

$$\leq \phi(S^kT^lx, RS^kT^lx).$$

Hence the net $\phi(S^kT^lx, RS^kT^lx)$ is nonincreasing. Putting $u = RS^kT^lx$ and $v = S^iT^jx$ with $(k,l) \leq (i,j)$ in (4.3), we have from Lemma 2.3 that

$$\begin{aligned} g(\|RS^iT^jx - RS^kT^lx\|) &\leq \phi(RS^iT^jx, RS^kT^lx) \\ &\leq \phi(S^iT^jx, RS^kT^lx) - \phi(S^iT^jx, RS^iT^jx) \\ &\leq \phi(S^kT^lx, RS^kT^lx) - \phi(S^iT^jx, RS^iT^jx), \end{aligned}$$

where g is a strictly increasing, continuous and convex real-valued function with $g(0) = 0$. From the properties of g, $\{RS^kT^lx\}$ is a Cauchy net; see [19]. Therefore $\{RS^kT^lx\}$ converges strongly to a point $q \in B(S) \cap B(T)$. Next, consider a fixed $x \in C$ and an arbitrary subsequence $\{S_{n_i}x\}$ of $\{S_nx\}$ which converges weakly to a point v. From the proof of Lemma 4.1, we know that $v \in A(S) \cap A(T) = B(S) \cap B(T)$. Rewriting the characterization of the retraction R, we have that for any $u \in B(S) \cap B(T)$,

$$0 \leq \left\langle S^kT^lx - RS^kT^lx, JRS^kT^lx - Ju \right\rangle$$

and hence

$$\begin{aligned} \left\langle S^kT^lx - RS^kT^lx, Ju - Jq \right\rangle &\leq \left\langle S^kT^lx - RS^kT^lx, JRS^kT^lx - Jq \right\rangle \\ &\leq \|S^kT^lx - RS^kT^lx\| \cdot \|JRS^kT^lx - Jq\| \\ &\leq K\|JRS^kT^lx - Jq\|, \end{aligned}$$

where K is an upper bound for $\|S^kT^lx - RS^kT^lx\|$. Summing up these inequalities for $k = 0, 1, \ldots, n$ and $l = 0, 1, \ldots, n$ and dividing by $(n+1)^2$, we arrive to

$$\left\langle S_nx - \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} RS^kT^lx, Ju - Jq \right\rangle \leq K\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} \|JRS^kT^lx - Jq\|,$$

where $S_nx = \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^kT^lx$. Letting $n_i \to \infty$ and remembering that J is continuous, we get

$$\langle v - q, Ju - Jq \rangle \leq 0.$$

This holds for any $u \in B(S) \cap B(T)$. Therefore $Rv = q$. But because $v \in B(S) \cap B(T)$, we have $v = q$. Thus the sequence $\{S_nx\}$ converges weakly to the point $q \in A(S) \cap A(T)$. □

Using Theorem 4.4, we obtain the following theorems.

Theorem 4.5. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $S, T : E \to E$ be commutative $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$ and $(\alpha_1', \alpha_2', \beta_1', \beta_2', \gamma_1', \gamma_2', \delta_1', \delta_2')$-generalized nonspreading mappings such that $\alpha_1 - \beta_1 = 0$, $\gamma_1 \leq \delta_1$, $\gamma_2 \leq \delta_2$, $\alpha_2 > \beta_2$ and $\alpha_1' - \beta_1' = 0$, $\gamma_1' \leq \delta_1'$, $\gamma_2' \leq \delta_2'$, $\alpha_2' > \beta_2'$, respectively.*

Assume that $\{S^kT^lz : k,l \in \mathbb{N}\cup\{0\}\}$ for some $z \in C$ is bounded. Let R be the sunny generalized nonexpansive retraction of E onto $F(S)\cap F(T)$. Then, for any $x \in E$,

$$S_nx = \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^kT^lx$$

converges weakly to an element q of $F(S)\cap F(T)$, where $q = \lim_{(k,l)\in D} RS^kT^lx$.

Proof. Since $\{S^kT^lz : k,l \in \mathbb{N}\cup\{0\}\}$ for some $z \in C$ is bounded, we have that $A(S)\cap A(T) = F(S)\cap F(T)$ is nonempty. We also know that $\alpha_2 > \beta_2$ together with $\alpha_1 - \beta_1 = 0$, $\gamma_1 \le \delta_1$ and $\gamma_2 \le \delta_2$ implies that

$$\phi(Sx,u) \le \phi(x,u)$$

for all $x \in E$ and $u \in F(S)$. Similarly, $\alpha_2' > \beta_2'$ together with $\alpha_1' - \beta_1' = 0$, $\gamma_1' \le \delta_1'$ and $\gamma_2' \le \delta_2'$ implies that

$$\phi(Tx,v) \le \phi(x,v)$$

for all $x \in E$ and $v \in F(T)$. Thus, we have that $F(S) = B(S)$ and $F(T) = B(T)$. Therefore, we have the desired result from Theorem 4.4. □

Theorem 4.6 ([6]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let S and T be commutative 2-generalized hybrid mappings of C into itself such that $\{S^kT^lz : k,l \in \mathbb{N}\cup\{0\}\}$ for some $z \in C$ is bounded. Let P be the metric projection of H onto $A(S)\cap A(T)$. Then, for any $x \in C$,*

$$S_nx = \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^kT^lx$$

converges weakly to an element q of $A(S)\cap A(T)$, where $q = \lim_{(k,l)\in D} PS^kT^lx$. In particular, if C is closed and convex, $\{S_nx\}$ converges weakly to an element q of $F(S)\cap F(T)$.

Proof. We have from Theorem 3.2 that $A(S)\cap A(T)$ is nonempty. We also have that $A(S) = B(S)$ and $A(T) = B(T)$. Since $A(S)\cap A(T)$ is a nonempty, closed and convex subset of H, there exists the metric projection of H onto $A(S)\cap A(T)$. In a Hilbert space, the metric projection of H onto $A(S)\cap A(T)$ is equivalent to the sunny generalized nonexpansive retraction of H onto $A(S)\cap A(T)$. On the other hand, commutative 2-generalized hybrid mappings $S,T : C \to C$ are commutative 2-generalized nonspreading mappings. So, we have the desired result from Theorem 4.6. Furthermore, if C is closed and convex, we have that $q \in F(S)\cap F(T)$ and then $\{S_nx\}$ converges weakly to $q \in F(S)\cap F(T)$. □

Remark We do not know whether a mean convergence theorem of Baillon's type for nonspreading mappings in a Banach space holds or not.

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