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STABILITY OF TIME-DEPENDENT DYNAMIC MONOPOLY WITH CONCENTRATED AND DISTRIBUTED DELAYS

ELENA BRAVERMAN AND MICHELLE MICHELLE

Dept. of Math. and Stats., University of Calgary, Calgary, AB, Canada T2N 1N4
Dept. of Math. and Stats. Sciences, University of Alberta,
Edmonton, AB, Canada T6G 2G1

ABSTRACT. A possible combination of continuously distributed and concentrated delays is incorporated in a monopoly model with either bounded or unbounded time window for the past data, while its length and the adjustment speed may vary. In this general setting, we obtain sufficient stability conditions. Sharper tests are established for autonomous equations with finite or infinite distributed delays. A similar stability analysis is implemented for the output of the leader firm in the Stackelberg duopoly model.

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Dedicated to Prof. Ravi Agarwal on the occasion of his 70th birthday

1. INTRODUCTION

Our purpose is to investigate the output stability of a profit maximizing behavior for a firm in the presence of delays and bounded rationality. This type of analysis was implemented in some recent papers [2, 10, 11, 18, 19, 20, 21, 22, 23, 24, 26, 27].

As discussed in these studies, one of the main assumptions that leads to the usage of delays is the idea of bounded rationality. In most cases, it is hard for the firm to use real-time quality demanded by the consumers to maximize its profits, because the data collection and analysis process could take some time: by the time there is data for period t, we are in the period $t+s$. Moreover, the firm may not have perfect and complete information on the structure or the functional form of the market demand, which is why it may use some approximation using the past information to estimate the actual market demand.

Another assumption that justifies the application of delay is the adjustment cost. If there is a positive demand shock, i.e. a steep growth in the consumers' demand, it may not be optimal for a firm (say, a monopolist) to respond instantaneously to this abrupt change. Commonly, a firm should invest in labor or additional capital (e.g., machinery) to accommodate the growth in demand. However, rapid purchases and recruitments can be very costly, may lead further to excess idle capacity and pose

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inefficiencies if there is a subsequent and persistent negative demand shock. Thus, even though a firm may know what the actual demand is at real time, it may choose to use the information on the demand in the past and to gradually adjust its output to control the costs.

The developed theory of delay differential equations (DDE) allows to extend the analysis done in [2, 10, 11, 18, 19, 20, 21, 22, 23, 24, 26] to more general models.

In this paper, we use a linear demand function, and the revenue r as a function of the supply quantity q is $r(q) = (A - Bq)q$, with a demand function $p(q) = A - Bq$ (the unit price), where $A, B > 0$. Similarly to [18], we assume the production (labor and capital) and inventory costs being clumped together into one, represented by $c(q)$ and use the quadratic cost function $c(q) = Cq^2$, where $C > 0$, see also [2, 3, 4, 9, 10, 12, 24]. In principle, we could have added a constant term to denote the presence of fixed cost, but this does not contribute anything to the output stability analysis, as we differentiate the expected profit. Based on the above, the profit $\Pi(q)$ can be computed as

$$\Pi(q) = p(q)q - c(q) = (A - Bq)q - Cq^2, \tag{1.1}$$

the marginal profit

$$\Pi'(q) = A - 2(B + C)q = 0 \tag{1.2}$$

is achieved at the positive equilibrium

$$q^* = \frac{A}{2(B + C)}. \tag{1.3}$$

Following [18], we introduce the gradient dynamics to (1.2) as

$$\dot{q}(t) = \alpha \Pi'(q_t),$$

where the adjustment speed α can be time-dependent, and q_t includes information on the present and past quantity demanded $q(s)$ for $s \le t$. The interpretation of this gradient dynamics is that we assume that the firm adjusts its output in proportion to and in the same direction as the average past marginal profit.

In a more general setting, such gradient dynamics can be described by the equation with a distributed delay

$$\dot{q}(t) = \alpha(t) \left[A - 2(B + C) \int_{h(t)}^{t} K(t,s) q(s) ds \right], \tag{1.4}$$

where $K(t,s)$ is a non-negative kernel of either a convolution $K(t,s) = K(t-s)$ or a non-convolution type, continuous in t and integrable in s. In the present paper, we always assume that $K(t,s)$ is normalized in the sense that $\int_{-\infty}^{t} K(t,s)ds = 1$ for all t if the delay is unbounded, and $\int_{h(t)}^{t} K(t,s)ds = 1$ for all t if the delay is bounded. This is not a limitation, as the positive value of the integral just becomes a part of $\alpha(t)$. In fact, we consider a more general type of the delay model which will be described later.

After a change of the variable $z(t) = q(t) - q^* = q(t) - \frac{A}{2(B+C)}$, (1.4) takes the form

$$\dot{z}(t) + 2\alpha(t)(B+C)\int_{h(t)}^{t} K(t,s)z(s)ds = 0. \tag{1.5}$$

The purpose of this paper is two-fold. First, we are going to consider a model generalizing (1.5) and deduce sufficient conditions guaranteeing exponential convergence of solutions to a positive equilibrium q^*. For several particular cases of autonomous equations with either bounded or unbounded delays, we obtain sharper exponential stability tests. Second, we analyze output stability of the leader firm in a Stackelberg game.

Compared to most of the previous publications, for example, [18, 22], the present paper presents a different approach from the following points of view.

1. We investigate non-autonomous models where more general types of present and past data are involved in the analysis, and the adjustment speed can vary. The considered model involves [18, 22] and many other equations as special cases. Moreover, a firm can adapt its data analysis to the current situation, for example, disregard older data when some quick demand changes occur. However, such flexibility of the general framework comes at a cost: unlike [18, 22] where sharp stability tests are obtained, we have only sufficient stability conditions.
2. Also, unlike [18, 22] where a complete bifurcation analysis is implemented, we reduce ourselves to the study of the stability domain for the parameters.
3. To the best of our knowledge, for both a monopolist and a Stackelberg game, the models where historical and immediate data are incorporated, have not been earlier considered.

The paper is organized as follows. Section 2 involves auxiliary results for delay equations. Section 3 contains our main results. We derive some sufficient stability conditions for a Stackelberg game in Section 4. In Section 5 we discuss the obtained results and outline some directions for future research.

2. PRELIMINARIES

We start with relevant definitions and auxiliary results on linear equations with concentrated and distributed delays which will be further used in the proofs of the main results. For autonomous equations with a distributed delay, stability conditions are obtained by the analysis of appropriate characteristic equations.

Generally, we fix $t_0 \in [0,\infty)$ and consider a non-autonomous equation with a distributed delay

$$\dot{x}(t) + a(t)\int_{h(t)}^{t} x(s) d_s R(t,s) = 0, \quad t \geq t_0. \tag{2.1}$$

In the case when $R(t,\cdot)$ is absolutely continuous, and $\frac{d}{ds}R(t,s) = K(t,s)$, we can rewrite (2.1) as an integro-differential equation

$$\dot{x}(t) + a(t)\int_{h(t)}^{t} K(t,s)x(s)\; ds = 0, \quad t \geq t_0. \tag{2.2}$$

We also investigate particular cases of (2.1) and (2.2) when there is a non-delay term

$$\dot{x}(t) + b(t)x(t) + a(t)\int_{h(t)}^{t} x(s) d_s R(t,s) = 0, \quad t \geq t_0 \tag{2.3}$$

and

$$\dot{x}(t) + b(t)x(t) + a(t)\int_{h(t)}^{t} K(t,s)x(s)\; ds = 0, \quad t \geq t_0, \tag{2.4}$$

as well as equations with both distributed and concentrated delays, for example,

$$\dot{x}(t) + b(t)x(t) + c(t)x(g(t)) + a(t)\int_{h(t)}^{t} K(t,s)x(s)\; ds = 0, \quad t \geq t_0. \tag{2.5}$$

Further, we assume that some of the following conditions are satisfied:
(a1) $a(t), b(t), c(t) : [0,\infty) \to [0,\infty)$ are Lebesgue measurable bounded on $[0,\infty)$ functions;
(a2) $h, g : [0,\infty) \to \mathbb{R}$ are Lebesgue measurable functions, and there exist $\tau > 0$ and $\sigma > 0$ such that $t - \tau \leq h(t) \leq t$, $t - \sigma \leq g(t) \leq t$;
(a3) $R(t,\cdot)$ is a left continuous non-decreasing function for any t, $R(\cdot, s)$ is locally integrable for any s, and $R(t,h(t)) = 0$, $R(t,t^+) = 1$ for any $t \geq 0$;
(a4) $K(t,s) \geq 0$ is Lebesgue integrable, $\int_{h(t)}^{t} K(t,s)ds = 1$ for any $t \geq 0$.

We consider (2.1) with an initial condition

$$x(t) = \varphi(t), \quad t \leq t_0, \tag{2.6}$$

and everywhere further we assume φ to be a continuous function.

Definition 2.1. An absolutely continuous function $x : \mathbb{R} \to \mathbb{R}$ is called **a solution of problem** (2.1), (2.6) if it satisfies equation (2.1) for almost all $t \in [t_0, \infty)$ and (2.6) for $t \leq t_0$.

Definition 2.2 ([17])**.** The zero solution of (2.1), (2.6) is **uniformly stable** if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any t_0, $|\varphi(t)| < \delta$ for $t \leq t_0$ implies $|x(t)| < \varepsilon$ for any $t > t_0$. The zero solution of (2.1), (2.6) is **uniformly asymptotically stable** if it is uniformly stable and also such $b_0 > 0$ exists that for any $\varepsilon > 0$ there is $M > 0$ such that $|\varphi(t)| < b_0$ for $t \leq t_0$ implies $|x(t)| < \varepsilon$ for any $t > t_0 + M$.

Definition 2.3. For each $t \geq 0$ the solution $X(t,s)$ of the problem

$$\dot{x}(t) + a(t) \int_{h(t)}^{t} x(s)\ d_s R(t,s) = 0, \quad t \geq s,\ x(t) = 0,\ t < s,\ x(s) = 1,$$

is called **the fundamental function** of equation (2.1). Note that $X(t,s) = 0$, $t < s$.

Definition 2.4. Equation (2.1) is **uniformly exponentially stable**, if there exist $M > 0$ and $\lambda > 0$ such that its fundamental function $X(t,s)$ has the estimate

$$|X(t,s)| \leq M\ e^{-\lambda(t-s)}, \quad t \geq s \geq 0. \tag{2.7}$$

We will further apply the following lemma, which is based on classical results (see, for example, [14, 15] and [7, Lemma 3.1]).

Lemma 2.5. *Consider the scalar equation*

$$\dot{x}(t) + b(t)x(t) + a(t)x(h(t)) = 0, \tag{2.8}$$

where a and b are Lebesgue measurable essentially bounded on $[0,\infty)$ functions, and there exists $\tau > 0$ such that the measurable function h satisfies $0 \leq t - h(t) \leq \tau$.

1) If $b(t) \equiv 0$, $a(t) \geq a_0 > 0$ and $\limsup\limits_{t\to\infty} \int_{h(t)}^{t} a(s)ds < \frac{3}{2}$ then (2.8) is uniformly exponentially stable.

2) If $b(t) \geq b_0 > 0$ and $\limsup\limits_{t\to\infty} \frac{|a(t)|}{b(t)} < 1$ then (2.8) is uniformly exponentially stable.

Consider a more general version of equation (2.8)

$$\dot{x}(t) + b(t)x(t) + a(t)x(h(t)) + c(t)x(g(t)) = 0, \tag{2.9}$$

where a, b, c are Lebesgue measurable essentially bounded on $[0,\infty)$ functions, and the delays satisfy (a2).

Lemma 2.6 ([5, Corollary 1.6])**.** *Suppose at least one of the following conditions holds:*

1. $b(t) \geq b_0 > 0$ and $\limsup\limits_{t\to\infty} \frac{|a(t)|+|c(t)|}{b(t)} < 1$;

2. $a(t) + b(t) + c(t) \geq b_0 > 0$ and

$$\limsup_{t\to\infty} \left[\frac{a(t)}{a(t)+b(t)+c(t)} \int_{h(t)}^{t} (|a(s)| + |b(s)| + |c(s)|)\, ds \right.$$
$$\left. + \frac{c(t)}{a(t)+b(t)+c(t)} \int_{g(t)}^{t} (|a(s)| + |b(s)| + |c(s)|)\ ds \right] < 1;$$

3. $a(t) + b(t) \geq b_0 > 0$ and

$$\limsup_{t\to\infty} \left[\frac{|a(t)|}{a(t)+b(t)} \int_{h(t)}^{t} (|a(s)| + |b(s)| + |c(s)|)\ ds + \frac{|c(t)|}{a(t)+b(t)} \right] < 1;$$

4. $c(t)+b(t) \geq b_0 > 0$ *and*

$$\limsup_{t\to\infty} \left[\frac{|c(t)|}{b(t)+c(t)} \int_{g(t)}^{t} \left(|a(s)|+|b(s)|+|c(s)|\right)\, ds + \frac{|a(t)|}{b(t)+c(t)} \right] < 1.$$

Then equation (2.9) *is uniformly exponentially stable.*

However, the above results are for equations with concentrated delays. Since our main object is an equation with either a distributed delay or incorporating both types of delay, we need the following auxiliary transformations from [6].

Lemma 2.7 ([6, Corollary 9, p. 302]). *Assume that $R(t,s)$ is a Lebesgue measurable non-decreasing function in s, $h : [0,\infty) \to \mathbb{R}$ is a measurable function satisfying $h(t) \leq t$, $\lim_{t\to\infty} h(t) = \infty$, and y is continuous on $[t_0,\infty)$. Then there exists a measurable function $h_0 : [0,\infty) \to \mathbb{R}$ satisfying $h(t) \leq h_0(t) \leq t$ such that*

$$\int_{h(t)}^{t} y(s) d_s R(t,s) = \left(\int_{h(t)}^{t} d_s R(t,s) \right) y(h_0(t)).$$

For $\frac{d}{ds}R(t,s) = K(t,s)$ we obtain the following result.

Lemma 2.8. *Assume that $K(t,s) \geq 0$ is locally integrable, $h : [0,\infty) \to \mathbb{R}$ is a measurable function satisfying $h(t) \leq t$, $\lim_{t\to\infty} h(t) = \infty$, and y is continuous on $[t_0,\infty)$. Then there exists a function $h_0 : [0,\infty) \to \mathbb{R}$, $h(t) \leq h_0(t) \leq t$ such that*

$$\int_{h(t)}^{t} K(t,s) y(s) ds = \left(\int_{h(t)}^{t} K(t,s) ds \right) y(h_0(t)).$$

For autonomous equations with a distributed delay, we present stability conditions which will be further applied.

Lemma 2.9 ([16, Theorem 1 and Remark 3]). *Consider the following equation*

$$\dot{x}(t) + \psi_1 x(t) + \psi_2 \int_{t-r}^{t} K(t-s) x(s) ds = 0. \tag{2.10}$$

Suppose $\psi_1 \geq 0$, $\psi_2 > 0$, $t \geq r > 0$, and $\int_0^r K(s)ds = 1$ with $\int_0^u K(s)ds$ being a non-decreasing, non-negative, and continuous function for all $u \in (0,r)$. If

$$\psi_1 + \psi_2 > 0, \quad 0 < \psi_2 \int_0^r sK(s)ds < \frac{\pi}{2},$$

then the zero solution of (2.10) *is uniformly asymptotically stable.*

Lemma 2.10 ([25, Theorem 1]). *Consider the problem*

$$\dot{x}(t) + \psi_2 \int_{t-\tau-r}^{t-\tau} x(s) ds = 0, \quad t \geq 0,\ r > 0, \quad x(\zeta) = 0,\ \zeta < 0. \tag{2.11}$$

where $\tau \geq 0$, and its characteristic equation

$$g(p) = p + \frac{\psi_2}{p} e^{-p\tau}(1 - e^{-pr}) = 0. \tag{2.12}$$

All zeros p of (2.12) satisfy $Re(p) < 0$, and the zero steady state solution of (2.11) is uniformly exponentially stable if and only if the following inequality is true:

$$0 < \frac{\psi_2 r^2}{2} < \left(\frac{\pi/2}{2\tau/r+1}\right)^2 \left(\sin\left(\frac{\pi/2}{2\tau/r+1}\right)\right)^{-1}.$$

Lemma 2.11 ([8, Theorems 2.4 and 5.3])**.** *Consider the following equation*

$$\dot{x}(t) + \psi_1 x(t) + \psi_2 \int_{-\infty}^{t} K(t-s)x(s)ds = 0. \tag{2.13}$$

Suppose $\psi_1, \psi_2 \in \mathbb{R}$, and $\int_0^\infty K(s)ds = 1$ with $\int_0^u K(s)ds$ being a monotone non-decreasing, non-negative, and continuous function for all $u \in (0,\infty)$. The zero solution of (2.13) is uniformly asymptotically stable if $\psi_1 > -\psi_2$ and $\psi_1 \geq |\psi_2|$, or if $\psi_2 > |\psi_1|$ and the mean of the kernel $\mathbb{E}[s]$ satisfies

$$\mathbb{E}[s] := \int_0^\infty sK(s)ds < \frac{\arccos(-\psi_1/\psi_2)}{\sqrt{\psi_2^2 - \psi_1^2}}. \tag{2.14}$$

Suppose that there exists $\nu > 0$ such that the following inequality is satisfied

$$\int_0^\infty e^{\nu s} K(s)ds < \infty. \tag{2.15}$$

The zero solution of (2.13) is uniformly exponentially stable if and only if all roots of its characteristic equation have negative real parts.

For some other results on stability of equations with a distributed delay, refer to [17].

3. MAIN RESULTS

First, we consider some generalizations, as well as particular cases, of main model equation (1.5). For all these equations, we assume that (a1)–(a4) hold. We study the most general equation with a distributed delay

$$\dot{z}(t) + 2\alpha(t)(B+C)\int_{h(t)}^{t} z(s)d_sR(t,s) = 0, \tag{3.1}$$

as well as its integral counterpart (1.5) and equations with non-delay terms

$$\dot{z}(t) + 2\alpha(t)(B+C)\left[(1-a(t))z(t) + a(t)\int_{h(t)}^{t} z(s)d_sR(t,s)\right] = 0, \quad 0 \leq a(t) \leq 1, \tag{3.2}$$

$$\dot{z}(t) + 2\alpha(t)(B+C)\left[(1-a(t))z(t) + a(t)\int_{h(t)}^{t} K(t,s)z(s)ds\right] = 0, \quad 0 \leq a(t) \leq 1, \tag{3.3}$$

$$\dot{z}(t) + 2\alpha(t)(B+C)\left[(1-a(t)-b(t))z(t) + b(t)z(g(t)) + a(t)\int_{h(t)}^{t} z(s)d_sR(t,s)\right] = 0, \tag{3.4}$$

with $0 \le a(t) \le 1$, $0 \le b(t) \le 1$, $0 \le a(t)+b(t) \le 1$,

(3.5)

$$\dot z(t)+2\alpha(t)(B+C)\left[(1-a(t)-b(t))z(t)+b(t)z(g(t))+a(t)\int_{h(t)}^t K(t,s)z(s)ds\right]=0,$$

with $0 \le a(t) \le 1$, $0 \le b(t) \le 1$, $0 \le a(t)+b(t) \le 1$.

Theorem 3.1. *If*

$$\limsup_{t\to\infty} \int_{h(t)}^t \alpha(s)\ ds < \frac{3}{4(B+C)}, \tag{3.6}$$

then (3.1) *and* (1.5) *are uniformly exponentially stable.*

Proof. Let continuous function $z(t)$ be a solution of (3.1). Then, by Lemma 2.7 and assumption (a3), for some $h_0(t) \in [h(t),t]$,

$$\int_{h(t)}^t z(s)d_sR(t,s) = \left(\int_{h(t)}^t d_sR(t,s)\right) z(h_0(t)) = z(h_0(t)).$$

Thus, $z(t)$ is also a solution of the equation

$$\dot z(t)+2\alpha(t)(B+C)z(h_0(t))=0, \tag{3.7}$$

which by Part 1) of Lemma 2.5 is uniformly exponentially stable if

$$\limsup_{t\to\infty} 2(B+C)\int_{h_0(t)}^t \alpha(s)\ ds < \frac{3}{2}.$$

Since $h_0(t) \in [h(t),t]$, we have that

$$\limsup_{t\to\infty} 2(B+C)\int_{h_0(t)}^t \alpha(s)\ ds \le \limsup_{t\to\infty} 2(B+C)\int_{h(t)}^t \alpha(s)\ ds < \frac{3}{2},$$

which implies uniform exponential stability of (3.1).

Next, by Lemma 2.8, if $z(t)$ is a solution of (1.5), it also satisfies (3.7) for some $h_0(t)$ such that $h(t) \le h_0(t) \le t$. The rest of the proof coincides with the case for equation (3.1). □

Similarly, Part 2) of Lemma 2.5 and either Lemma 2.7 or Lemma 2.8, imply the following result for equations (3.2) and (3.3).

Theorem 3.2. *If*

$$\limsup_{t\to\infty} a(t) < \frac{1}{2}, \tag{3.8}$$

then (3.2) *and* (3.3) *are uniformly exponentially stable.*

Lemma 2.6, together with either Lemma 2.7 or Lemma 2.8 imply stability conditions for equations (3.4) and (3.5).

Theorem 3.3. *Suppose $\alpha(t) \geq \alpha_0 > 0$ is a Lebesgue measurable function, and at least one of the following conditions is satisfied:*
1. $\limsup_{t\to\infty}[a(t)+b(t)] < \frac{1}{2}$;
2. $\limsup_{t\to\infty}\left[a(t)\int_{h(t)}^t \alpha(s)ds + b(t)\int_{g(t)}^t \alpha(s)ds < \frac{1}{2(B+C)}\right]$;
3. $b(t) \leq b_0 < \frac{1}{2}$ *and*

$$\limsup_{t\to\infty}\left[2(B+C)\frac{a(t)}{1-b(t)}\int_{h(t)}^t \alpha(s)ds + \frac{b(t)}{1-b(t)}\right] < 1;$$

4. $a(t) \leq a_0 < \frac{1}{2}$ *and*

$$\limsup_{t\to\infty}\left[2(B+C)\frac{b(t)}{1-a(t)}\int_{g(t)}^t \alpha(s)ds + \frac{a(t)}{1-a(t)}\right] < 1.$$

Then (3.4) and (3.5) are uniformly exponentially stable.

Further, we obtain sharper stability conditions for equation (1.5) with various convolution kernels and $\alpha(t) = \alpha$, where α is a fixed positive constant.

Proposition 3.4. *Consider the kernel*

$$K(t-s) = \begin{cases} \frac{1}{h}, & t-\tau-h \leq s < t-\tau, \\ 0, & \text{otherwise}, \end{cases} \tag{3.9}$$

where $\tau \geq 0$ and $h > 0$. The zero steady state solution of (1.5) with $\alpha(t) = \alpha > 0$ is uniformly exponentially stable if and only if

$$2\alpha(B+C) < \left(\frac{2}{h^2}\right)\left(\frac{\pi/2}{2\tau/h+1}\right)^2\left(\sin\left(\frac{\pi/2}{2\tau/h+1}\right)\right)^{-1} \tag{3.10}$$

Proof. Inequality (3.10) follows directly from Lemma 2.10. □

Proposition 3.5. *Consider a kernel $K(t,s) = K(t-s)$, where $\int_0^r K(s)ds = 1$ with $\int_0^u K(s)ds$ being a non-decreasing, non-negative, and continuous function for all $u \in (0,r)$. If*

$$2\alpha(B+C) < \frac{\pi}{2}\left(\int_0^r sK(s)ds\right)^{-1}, \tag{3.11}$$

then the zero solution of (1.5) with $\alpha(t) = \alpha > 0$ is uniformly asymptotically stable.

Proof. Inequality (3.11) follows directly from Lemma 2.9. □

Example 3.6. Consider the kernel

$$K(t-s) = \begin{cases} \frac{2m}{n^2}(-m(t-s)+n), & t-\frac{n}{m} \leq s < t \quad (n,m>0), \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 3.5, if

$$2\alpha(B+C) < \frac{\pi}{2}\left(\frac{2m}{n^2}\int_0^{\frac{n}{m}} s(-ms+n)ds\right)^{-1} = \frac{3m\pi}{2n},$$

then the zero solution of (1.5) with $\alpha(t) = \alpha > 0$ is uniformly asymptotically stable.

Example 3.7. Consider a truncated exponential distribution. Let $0 < \delta < 1$ describe how much area under the exponential is used, i.e.,

$$K(t-s) = \begin{cases} \frac{\sigma}{\delta}\exp(-\sigma(t-s)), & t + \frac{\ln(1-\delta)}{\sigma} \le s < t \quad (\sigma > 0), \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 3.5, if

$$2\alpha(B+C) \le \frac{\pi\sigma\delta}{2((1-\delta)\ln(1-\delta)+\delta)},$$

then the zero solution of (1.5) with $\alpha(t) = \alpha > 0$ is uniformly asymptotically stable.

As another example of a truncated exponential distribution, consider

$$K(t-s) = \begin{cases} \frac{\sigma}{1-e^{-\sigma t}}\exp(-\sigma(t-s)), & 0 \le s \le t \quad (\sigma > 0), \\ 0, & \text{otherwise,} \end{cases}$$

then, as proved in [1, Lemma 4.1], (1.5) with $\alpha(t) = \alpha > 0$ is uniformly exponentially stable as long as $\sigma > 0$ and $2\alpha(B+C) > 0$. Thus, the adjustment speed and the parameters that determine the demand and cost functions of the firm are irrelevant in guaranteeing the stability of the equation.

Proposition 3.8. *Consider a kernel $K(t,s) = K(t-s)$, where $\int_0^\infty K(s)ds = 1$ with $\int_0^u K(s)ds$ being a monotone non-decreasing, non-negative, and continuous function for all $u \in (0,\infty)$. If for some $\nu > 0$*

$$\int_0^\infty e^{\nu s}K(s)ds < \infty, \quad 2\alpha(B+C) < \frac{\pi}{2}\left(\int_0^\infty sK(s)ds\right)^{-1}, \tag{3.12}$$

then the zero solution of equation (1.5) with $\alpha(t) = \alpha > 0$ is uniformly exponentially stable.

Proof. Inequality (3.12) follows from Lemma 2.11 by setting $\psi_1 = 0$ and $\psi_2 = 2\alpha(B+C)$. □

Example 3.9. Consider some popular distributions that decay slowly to zero as $t \to \infty$, such as gamma, half normal (with $\mu = 0$), or Weibull. Consider

$$
\begin{aligned}
K_{\text{Gamma}}(t-s) &= \frac{\phi^j}{\Gamma(j)}(t-s)^{j-1}e^{-\phi(t-s)}, \quad \text{where } \phi, j > 0,\\
K_{\text{Normal}}(t-s) &= \frac{\sqrt{2}}{\sqrt{\pi\sigma^2}}\exp\left(\frac{-(t-s)^2}{2\sigma^2}\right), \quad \text{where } \sigma > 0,\\
K_{\text{Weibull}}(t-s) &= (\phi j)(\phi(t-s))^{j-1}e^{-(\phi(t-s))^j}, \quad \text{where } \phi > 0, j \geq 1,
\end{aligned}
\tag{3.13}
$$

where $\Gamma(\cdot)$ is the gamma function. The expected values of the above three distributions are $\frac{j}{\phi}$, $\sigma\sqrt{\frac{2}{\pi}}$ and $\frac{\Gamma(1+1/j)}{\phi}$, respectively. Recall that each of the above kernels integrates to 1.

Let us briefly verify that condition (2.15) holds for some positive ν. Consider K_{Gamma}, for any ν such that $\phi > \nu$, we have

$$\int_0^\infty e^{\nu s}\frac{\phi^j s^{j-1}e^{-\phi s}}{\Gamma(j)}ds = \int_0^\infty \frac{\phi^j}{(\phi-\nu)^j}\frac{(\phi-\nu)^j s^{j-1}e^{-(\phi-\nu)s}}{\Gamma(j)}ds = \left(\frac{\phi}{\phi-\nu}\right)^j < \infty.$$

Consider K_{Normal}. By completing the squares, for any $\nu > 0$, we get

$$
\begin{aligned}
\int_0^\infty e^{\nu s}\frac{\sqrt{2}}{\sqrt{\pi\sigma^2}}\exp\left(\frac{-s^2}{2\sigma^2}\right)ds &= \exp\left(\frac{(\nu\sigma)^2}{2}\right)\int_0^\infty \frac{\sqrt{2}}{\sqrt{\pi\sigma^2}}\exp\left(\frac{-(s-\nu\sigma^2)^2}{2\sigma^2}\right)ds\\
&= \exp\left(\frac{(\nu\sigma)^2}{2}\right) < \infty.
\end{aligned}
$$

Consider K_{Weibull}. For our case we simply assume $j \geq 1$, even though the parameter j of Weibull distribution can also take a value between 0 and 1. Also, for any ν such that $\phi^j > \nu$, we have

$$
\begin{aligned}
&\int_0^\infty e^{\nu s}(\phi j)(\phi s)^{j-1}e^{-(\phi s)^j}ds\\
&= \int_0^1 e^{\nu s}(\phi j)(\phi s)^{j-1}e^{-(\phi s)^j}ds + \int_1^\infty e^{\nu s}(\phi j)(\phi s)^{j-1}e^{-(\phi s)^j}ds\\
&< \int_0^1 e^{\nu s}(\phi j)(\phi s)^{j-1}e^{-(\phi s)^j}ds + \int_1^\infty (\phi j)(\phi s)^{j-1}e^{-(\phi^j-\nu)s}ds\\
&= \int_0^1 e^{\nu s}(\phi j)(\phi s)^{j-1}e^{-(\phi s)^j}ds + \frac{\phi^j j\Gamma(j)}{(\phi^j-\nu)^j}\int_1^\infty \frac{(\phi^j-\nu)^j}{\Gamma(j)}s^{j-1}e^{-(\phi^j-\nu)s}ds\\
&< \int_0^1 e^{\nu s}(\phi j)(\phi s)^{j-1}e^{-(\phi s)^j}ds + \frac{\phi^j j\Gamma(j)}{(\phi^j-\nu)^j} < \infty,
\end{aligned}
$$

where we use the fact that $\exp(-s^j) < \exp(-s)$ for all $s > 1$ to move from the second line to the third line, and the fact that the second integral in the fourth line involves integrating a truncated gamma distribution.

By Proposition 3.8, given that we are using either K_{Gamma}, K_{Normal}, or K_{Weibull}, we respectively have the following inequalities

$$(3.14)\quad 2\alpha(B+C) < \frac{\pi}{2}\frac{\phi}{j},\quad 2\alpha(B+C) < \frac{1}{\sigma}\left(\frac{\pi}{2}\right)^{3/2},\quad 2\alpha(B+C) < \frac{\pi}{2}\frac{\phi}{\Gamma(1+\frac{1}{j})}$$

to guarantee the uniform exponential stability of the zero solution of equation (1.5) with $\alpha(t) = \alpha > 0$.

4. OUTPUT STABILITY ANALYSIS OF A STACKELBERG GAME

There are many papers on Cournot duopoly with bounded rationality (e.g., see [11, 13, 23] and references therein). However, there are significantly fewer papers on a Stackelberg duopoly with bounded rationality. Shi, Le, and Sheng in [27] conducted price stability and bifurcation analysis by considering a discrete setting of a Stackelberg game. We aim to follow the way the authors set up the model in [27], with some modifications. Instead of choosing a price level, we assume that both firms involved in this game choose output levels to maximize their profits. As a result, unlike [27], we are allowed to assume both firms produce an identical good. As explained in [12], if two firms selling the same good and having the same cost functions, compete over price, they will eventually earn no profits. This is because a firm has an incentive to pick a price lower than the competitor. Having a lower price than the rival allows a firm to win the entire market, because we assume both firms make the same product. Since each firm continuously tries to undercut the other, they will both pick a price that allows them to just break-even. The aforementioned phenomenon is called *Bertrand Paradox*. In [27] the product heterogeneity assumption is introduced to avoid Bertrand Paradox. In this section, we apply the approach of Section 3 to analyze the leader's output stability in a continuous setting. The follower's output stability analysis is a more challenging task, which is not in the framework of the present paper.

4.1. **Stackelberg duopoly model.** Let us describe some basic assumptions of a Stackelberg game, as explained in [12]. Suppose there are only two firms in the market with identical demand and cost functions. Unlike in Cournot duopoly where both firms maximize their profits by choosing the outputs they want to produce simultaneously, both firms have their own turns in the Stackelberg model. One of the firms is the leader and has a *first-mover advantage*: the leader earns more profit than the follower even though they produce the same product, and face identical demand and cost functions, because the leader gets to choose its output first. The other firm is the follower. So, it chooses its output by observing the amount of output the leader has produced. In mathematical terms, the optimization procedure is set up below and uses *backward induction*. We start by maximizing the follower's profit by taking the output produced by the leader, q_L, as a given and solving for q_F, which is also referred to as *the best response function* of the follower

$$\Pi_F(q_F|q_L) = p(q_L, q_F)q_F - c(q_F) = (A - B(q_L + q_F))q_F - (Cq_F + F),$$

$$\frac{\partial \Pi_F}{\partial q_F} = A - Bq_L - 2Bq_F - C = 0 \implies q_F = \frac{A - Bq_L - C}{2B}, \tag{4.1}$$

where $A, B, C, F > 0$. Next, we plug q_F into the leader's profit function

$$\Pi_L(q_F) = p(q_L, q_F)q_L - c(q_L) = \left(A - B\left(q_L + \frac{A - Bq_L - C}{2B}\right)\right)q_L - (Cq_L + F),$$

$$\frac{\partial \Pi_L}{\partial q_L} = \frac{(A - C)}{2} - Bq_L.$$

In order to avoid non-positive outputs, we assume that $A > C$. Setting $\frac{\partial \Pi_L}{\partial q_L} = 0$, we have the positive steady state solution for the leader firm

$$q^*_{L_1} = \frac{A - C}{2B}.$$

As the last step, we introduce our choice of the gradient dynamics

$$\dot{q}_L(t) = \alpha\left(\frac{\partial \Pi_L}{\partial q_L}(q_{L,\ell}(t))\right) = \alpha\left(\frac{(A - C)}{2} - B\int_{h(t)}^{t} K(t, s)q_L(s)ds\right), \tag{4.2}$$

where $q_{L,\ell}(t)$ represents some lagged terms of $q_L(t)$. Equation (4.2) incorporates a number of assumptions. Firstly, we assume that both firms approximate the market demand by an identical linear function. More importantly, the leader uses past average quantity produced for its profit maximization. Secondly, even though both firms do not know the actual market demand at real time, the follower knows the output produced by the leader instantaneously. It is common to assume that the leader publicly announces its output [27]. In [23], the authors also introduced a non-delayed term for the the rival's output in the context of simultaneous Cournot duopoly.

We repeat the same analysis for a quadratic cost function:

$$\Pi_F(q_F|q_L) = (A - B(q_L + q_F))q_F - Cq_F^2,$$

$$\frac{\partial \Pi_F}{\partial q_F} = A - Bq_L - 2Bq_F - 2Cq_F = 0 \implies q_F = \frac{A}{2(B + C)} - \frac{Bq_L}{2(B + C)}, \tag{4.3}$$

$$\Pi_L(q_F) = \left(A - B\left(q_L + \frac{A}{2(B + C)} - \frac{Bq_L}{2(B + C)}\right)\right)q_L - Cq_L^2,$$

$$\frac{\partial \Pi_L}{\partial q_L} = \left(A - \frac{AB}{2(B + C)}\right) - \left(2B - \frac{B^2}{(B + C)} + 2C\right)q_L.$$

In order to avoid non-positive outputs, we assume that $A > \frac{AB}{2(B+C)}$ and $2(B + C) > \frac{B^2}{(B+C)}$. Setting $\frac{\partial \Pi_L}{\partial q_L} = 0$, we have

$$q^*_{L_2} = \frac{AB + 2AC}{4(B + C)^2 - 2B^2}.$$

The last step is to introduce memory-dependent gradient dynamics

$$\dot{q}_L(t) = \alpha\left(\frac{\partial \Pi_L}{\partial q_L}(q_{L,\ell}(t))\right) = \alpha\left[A - \frac{AB}{2(B+C)} - \left(2B - \frac{B^2}{(B+C)} + 2C\right)\int_{h(t)}^{t} K(t,s)q_L(s)ds\right]. \tag{4.4}$$

As above, we apply a change of variables to get the zero equilibrium. Plugging $q_L(t) = z_L(t) + q^*_{L_1}$ into (4.2), we have

$$\dot{z}_L(t) + \alpha B\int_{h(t)}^{t} K(t,s)z_L(s)ds = 0. \tag{4.5}$$

Similarly, let $q(t) = z(t) + q^*_{L_2}$. Substituting into (4.4), we get

$$\dot{z}_L(t) + \alpha\left(2(B+C) - \frac{B^2}{B+C}\right)\int_{h(t)}^{t} K(t,s)z_L(s)ds = 0. \tag{4.6}$$

Assuming time-variable adjustment $\alpha(t)$ and the possibility of both concentrated and distributed delays, we can consider generalizations of (4.5)

$$\dot{z}_L(t) + \alpha(t)B\int_{h(t)}^{t} z_L(s)d_sR(t,s) = 0, \tag{4.7}$$

$$\dot{z}_L(t) + \alpha(t)B\int_{h(t)}^{t} K(t,s)z_L(s)ds = 0, \tag{4.8}$$

$$\dot{z}(t) + \alpha(t)B\left[(1-a(t))z(t) + a(t)\int_{h(t)}^{t} z(s)d_sR(t,s)\right] = 0, \quad 0 \le a(t) \le 1, \tag{4.9}$$

$$\dot{z}(t) + \alpha(t)B\left[(1-a(t))z(t) + a(t)\int_{h(t)}^{t} K(t,s)z(s)ds\right] = 0, \quad 0 \le a(t) \le 1, \tag{4.10}$$

$$\dot{z}(t) + \alpha(t)B\left[(1-a(t)-b(t))z(t) + b(t)z(g(t)) + a(t)\int_{h(t)}^{t} z(s)d_sR(t,s)\right] = 0, \tag{4.11}$$

with $0 \le a(t) \le 1$, $0 \le b(t) \le 1$, $0 \le a(t) + b(t) \le 1$,

$$\dot{z}(t) + \alpha(t)B\left[(1-a(t)-b(t))z(t) + b(t)z(g(t)) + a(t)\int_{h(t)}^{t} K(t,s)z(s)ds\right] = 0, \tag{4.12}$$

with $0 \le a(t) \le 1$, $0 \le b(t) \le 1$, $0 \le a(t) + b(t) \le 1$.

We assume everywhere that (a1)–(a4) are satisfied.

4.2. **Output stability analysis of the leader firm.** The following sufficient stability conditions are obtained by considering equations (4.7)–(4.12) and by a similar fashion as in Section 3.

Theorem 4.1. *If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \alpha(s)\ ds < \frac{3}{2B}, \tag{4.13}$$

then (4.7) *and* (4.8) *are uniformly exponentially stable.*

Theorem 4.2. *If*

$$\limsup_{t\to\infty} a(t) < \frac{1}{2}, \tag{4.14}$$

then (4.9) *and* (4.10) *are uniformly exponentially stable.*

Theorem 4.3. *Suppose $\alpha(t) \geq \alpha_0 > 0$ is a Lebesgue measurable function, and at least one of the following conditions is satisfied:*

1. $\limsup_{t\to\infty}[a(t) + b(t)] < \frac{1}{2}$;

2. $\limsup_{t\to\infty} a(t) \int_{h(t)}^{t} \alpha(s)\ ds + b(t) \int_{g(t)}^{t} \alpha(s)\ ds < \frac{1}{B}$;

3. $b(t) \leq b_0 < \frac{1}{2}$ *and*

$$\limsup_{t\to\infty} \left[B \frac{a(t)}{1-b(t)} \int_{h(t)}^{t} \alpha(s)\ ds + \frac{b(t)}{1-b(t)} \right] < 1;$$

4. $a(t) \leq a_0 < \frac{1}{2}$ *and*

$$\limsup_{t\to\infty} \left[B \frac{b(t)}{1-a(t)} \int_{g(t)}^{t} \alpha(s)\ ds + \frac{a(t)}{1-a(t)} \right] < 1.$$

Then (4.11) *and* (4.12) *are uniformly exponentially stable.*

Next, we obtain sharper stability conditions for equation (4.5) with various convolution kernels.

Proposition 4.4. *Suppose $K(t, s)$ is defined as in* (3.9) *and $t \geq \tau + h$. If the following inequality is satisfied*

$$\alpha B < \left(\frac{2}{h^2}\right) \left(\frac{\pi/2}{2\tau/h+1}\right)^2 \left(\sin\left(\frac{\pi/2}{2\tau/h+1}\right)\right)^{-1},$$

then the zero steady state solution of equation (4.5) *is uniformly exponentially stable.*

Proposition 4.5. *Consider a kernel $K(t, s) = K(t - s)$, where $\int_0^r K(s)ds = 1$ with $\int_0^u K(s)ds$ being a non-decreasing, non-negative, and continuous function for all $u \in (0, r)$. If*

$$\alpha B < \frac{\pi}{2} \left(\int_0^r sK(s)ds\right)^{-1},$$

then the zero solution of equation (4.5) *is uniformly exponentially stable.*

Proposition 4.6. *Consider a kernel $K(t,s)=K(t-s)$, where $\int_0^\infty K(s)ds = 1$ with $\int_0^u K(s)ds$ being a monotone non-decreasing, non-negative, and continuous function for all $u \in (0,\infty)$. If for some $\nu > 0$,*

$$\int_0^\infty e^{\nu s} K(s)ds < \infty, \quad \alpha B < \frac{\pi}{2}\left(\int_0^\infty sK(s)ds\right)^{-1},$$

then the zero solution of equation (4.5) *is uniformly exponentially stable.*

Example 4.7. Consider a gamma distribution, a normal distribution with mean 0, and a Weibull distribution as in (3.13). Then, we respectively need

$$\alpha B < \frac{\pi}{2}\frac{\phi}{j}, \quad \alpha B < \frac{1}{\sigma}\left(\frac{\pi}{2}\right)^{3/2}, \quad \alpha B < \frac{\pi}{2}\frac{\phi}{\Gamma\left(1+\frac{1}{j}\right)},$$

to guarantee the uniform exponential stability of the zero solution of (4.5).

Remark 4.8. Similarly, we can consider (4.6) and its generalizations to the cases of variable adjustment and different delay distribution. It suffices for us to replace B with $2(B+C) - \frac{B^2}{B+C}$ in the results we present in Section 4.2.

5. DISCUSSION

As mentioned in the introduction, the present paper focuses on the stability analysis, not on the bifurcation analysis. The reason is that, generally, we consider non-autonomous models which can better describe market models in a quickly changing modern environment. Unlike some of the papers we cite, instead of manually analyzing the roots of the characteristic equations, we have used existing stability theorems for DDE to construct some explicit and easy to interpret bounds on the parameters in the presence of bounded and unbounded delays.

Also, compared to [2, 10, 11, 18, 19, 20, 21, 22, 23, 24, 26, 27], we consider more than one model, both from the point of view of economics (a monopoly and a Stackelberg duopoly) and the type of delays. This gives us an advantage to witness how the preservation of asymptotic stability requires some restrictions on the adjustment speed, and different parameters of the demand and cost function, but not necessarily all at the same time. Moreover, unlike models with constant concentrated delays considered in [2, 10, 19, 20, 21, 24, 26] and integro-differential equations [18], our approach allows to combine both concentrated and distributed delays. The setup of our models can account for all information in a given time period in the past, with possible emphasis of information at certain points in the past or present. Some of the results allow the relation between the history profile used in the model change with time. One of the most general results claims that if the present data used dominates over the past information, the unique positive equilibrium is necessarily uniformly exponentially stable.

For autonomous equations, the recent paper [8] provides an interesting insight. It is known that once an equation with a certain concentrated delay is stable, so is the equation with a distributed delay having this concentrated delay as the maximal possible, and this fact is applied in Section 3. However, Bernard and Crauste [8] compared autonomous equations with a distributed delay to those where the concentrated delay is not at a maximal but at the mean level and, under certain conditions, concluded that asymptotic stability of the latter implies the same property for the former. Thus, distributed delays can be treated as at the very least not more detrimental for stability of the model than concentrated ones.

There are several ways to extend the work done in this paper. Let us describe some possible generalizations and directions of future research.

1. In the present paper, we used positive constants for the parameters of the demand and cost functions, only the adjustment parameter α was allowed to be time-dependent. It would be interesting to investigate the case when instead of the constants A, B, C we have some functions of time $A(t), B(t), C(t)$. In particular, if all these functions $[0,\infty) \to (0,\infty)$ are time periodic, it would also be interesting to study the existence of a periodic solution and explore its stability. In addition to mathematical limitations leading to existence of an asymptotically stable periodic solution, it would be interesting to investigate economic interpretation of these conditions and possible restrictions on the parameters of the equation imposed by a time-dependent economic model.
2. Consider stochastic demand and cost functions, and see what additional insights we can gain. Estimate the size of a stochastic perturbation which keeps stability of a (possibly blurred) equilibrium.
3. All the results of the present paper analyze convergence of solutions to a certain positive equilibrium, without distinguishing between monotone and oscillatory convergence. It would be interesting to obtain sufficient non-oscillation conditions and, in the autonomous case, divide all the domain of parameters where the equation is uniformly exponentially stable into two subdomains, in the first of which we may have eventually monotone convergence, while in the second one all solutions oscillate about the equilibrium solution.
4. In the autonomous cases considered in the present paper, implement a comprehensive bifurcation analysis, complementing, for example, [18].
5. Use optimal control theory to optimize the firm's cumulative profit in the presence of bounded rationality at a given finite/infinite time horizon. This allows us to introduce discount rate and a more sophisticated cost function, which may include adjustment cost, inventory cost, production cost, and distribution cost. It would certainly be interesting to see what kind of additional insights can be obtained if we perform output stability analysis on such a different setup. An

example of the dynamic optimization setup (without taking into account the bounded rationality part) can be found in [28].

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