AN EXISTENCE AND UNIQUENESS RESULT FOR LINEAR SEQUENTIAL FRACTIONAL BOUNDARY VALUE PROBLEMS (BVPS) VIA LYAPUNOV TYPE INEQUALITY

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ABSTRACT. A sufficient condition for the existence and uniqueness of solution of nonhomogenous fractional boundary value problem involving sequential fractional derivative of Riemann Liouville type is established by using a new Lyapunov type inequality and disconjugacy criterion. Green's function and some of its properties are also presented. Our approach is quite new and to the best of our knowledge, the uniqueness of solution of nonhomogenous fractional boundary value problems is proved by employing Lyapunov type inequality for the first time and this Lyapunov type inequality improves and generalizes the previous ones.

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1. INTRODUCTION

In this paper we prove an existence and uniqueness criterion for the solution of nonhomogenous boundary value problem of sequential fractional differential equations of the form

$$(1.1a) \qquad (_aD^\alpha(_aD^\beta y))(t) + f(t)y(t) = g(t), \quad a < t < b, \quad \frac{1}{2} < \alpha, \beta \le 1,$$

$$(1.1b) \qquad y(a) = 0, \quad y(b) = B,$$

by deriving Lyapunov type inequality and disconjugacy criterion for the following associated homogenous boundary value problem

$$(1.2a) \qquad (_aD^\alpha(_aD^\beta y))(t) + f(t)y(t) = 0, \quad a < t < b, \quad \frac{1}{2} < \alpha, \beta \le 1,$$

$$(1.2b) \qquad y(a) = 0, \quad y(b) = 0,$$

where $(_aD^\alpha)$ and $(_aD^\beta)$ are Riemann Liouville fractional derivatives of order α and β, respectively, $f, g : [a, b] \to \mathbb{R}$ are given continuous functions and a, b, B are given real constants.

Fractional differential equations have attracted a great deal of attention and the theory of which has been developed rapidly in the last three decades because they are not only generalization of integer order derivatives but also modelling tools for many real world phenomena occurring in physical and technical sciences, see [23, 25, 18].

On the other hand, sequential fractional derivative arises naturally in many fields of science and engineering because modelling procedure of a real world phenomena by using differential equations originates from substituting equation containing derivatives into another one. If the derivatives in both equations are fractional, then the resulting equation will be sequential fractional derivative. Besides, Riemann Liouville and Caputo fractional derivatives can be considered as particular cases of the sequential fractional derivative.

Although there are few results for existence and uniqueness of boundary value problems of sequential fractional derivative by using standart fixed point theorems, such as the method of upper and lower solutions and Schauder fixed point theorem [28], contraction principle [19], the method of upper and lower solutions and its associated monotone iterative method [4], Banach's contraction mapping principle, Krasnoselskii's fixed point theorem and nonlinear alternative of Leray-Schauder type [3, 2], as far as we know there is not much done by using Lyapunov type inequality and disconjugacy criterion. The proof of the main theorem of the present paper is based on the method, which arises in [17] for the first time showing the connection between linear impulsive boundary value problems and Lyapunov type inequality.

The history of Lyapunov type inequalities was started by Lyapunov [22] with the following result.

Theorem 1.1 ([22]). *If the boundary value problem*

$$(1.3) \qquad \begin{aligned} y'' + q(t)y &= 0, \quad a < t < b \\ y(a) = y(b) &= 0 \end{aligned}$$

has a nontrivial solution, where q is a real and continuous function with $q(t) \geq 0$, $q(t) \not\equiv 0$, then the so-called Lyapunov inequality

$$(1.4) \qquad \int_a^b q(t)dt > \frac{4}{b-a},$$

holds.

After the initiated work of Lyapunov [22], many authors have paid a considerable attention to Lyapunov type inequalities and various proofs and generalizations or improvements have appeared in the literature. For a comprehensive exibition of these results we refer two surveys [6, 27] and references therein. The results for (1.3) in [5, 20] are worth mentioning due to their contribution to this subject. Borg [5] changed the nonnegativity condition of $q(t)$ by nonnegative integral of $q(t)$ and improved inequality (1.4).

Theorem 1.2 ([5]). *If the boundary value problem* (1.3) *has a nontrivial solution, where q is a real and continuous function with $q(t) \not\equiv 0$, then we have the Lyapunov*

type inequality

(1.5)
$$\int_a^b |q(t)|dt > \frac{4}{b-a}.$$

Krein [20] used the same conditions of Theorem 1.2 and obtained the following better inequality by replacing $|q(t)|$ by $q^+(t) = \max\{q(t), 0\}$.

Theorem 1.3 ([20]). *If the boundary value problem* (1.3) *has a nontrivial solution, where q is a real and continuous function with $q(t) \not\equiv 0$, then we have the Lyapunov type inequality*

(1.6)
$$\int_a^b q^+(t)dt > \frac{4}{b-a}.$$

For $\alpha \in (1, 2]$, the fractional counterparts of Lyapunov type inequality is obtained in [10, 11, 26, 14, 15, 16] and the results of [10] will be improved and extended in the present paper.

The theory of disconjugacy is well developed for ordinary differential equations, the history of which starts with [12, 13, 21, 7, 24]. However, generalization of this theory to the fractional case is not considered much, see [1, 8].

This paper is organized as follows: In Section 2 we recall some preliminary facts that we will use in the sequel. Section 3 contains auxiliary tools, which are Green's function and its properties, Lyapunov type inequality and disconjugacy criterion, used to prove the main result. Section 4 is devoted to the main result, which is the existence and uniqueness theorem for nonhomogenous boundary value problem (1.1a)–(1.1b). To the best of our knowledge although many results have been obtained for fractional boundary value problems by using different techniques, there is little known about the connection of fractional boundary value problems and Lyapunov type inequality.

2. **PRELIMINARIES**

Before going further, let us start with basic definitions and some facts about Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and sequential fractional derivative and give definition of disconjugacy for fractional differential equations.

Definition 2.1 ([23, 25, 18]). Let $\alpha \geq 0$ and ϕ be a continuous function defined on $[a, b]$. The Riemann Lioville fractional integral of order α is defined by

$$(_aI^\alpha\phi)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\phi(s)ds \quad \text{for } \alpha > 0$$

and $_aI^0\phi(t) = \phi(t)$ for $\alpha = 0$.

Definition 2.2 ([23, 25, 18])**.** The Riemann Liouville fractional derivative of order $\alpha \geq 0$ is defined by

$$({}_aD^\alpha\phi)(t) = \begin{cases} ({}_aD^m \, {}_aI^{m-\alpha}\phi)(t), & \alpha > 0 \\ \phi(t), & \alpha = 0 \end{cases}$$

where m is the smallest integer greater or equal than α.

The definition of the sequential derivative proposed by Miller [23] is as follows.

Definition 2.3 ([23])**.** The sequential fractional derivative of order $\alpha \geq 0$ is defined by

$$({}_a\mathcal{D}^\alpha\phi)(t) = ({}_aD^{\alpha_1} \, {}_aD^{\alpha_2} \cdots {}_a D^{\alpha_n}\phi)(t),$$

where $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \alpha$ and ${}_aD^{\alpha_k}$, $k = 1, 2, \ldots, n$ denote Riemann Liouville fractional derivative of order $\alpha_k \geq 0$.

Lemma 2.4 ([23, 25, 18])**.** *Assume that $\phi \in C(a,b) \cap L(a,b)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(a,b) \cap L(a,b)$. Then for some constants $c_i = 1, 2, \ldots, m$, one has*

$${}_aI_a^\alpha D^\alpha\phi(t) = \phi(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_m(t-a)^{\alpha-m},$$

where m is the smallest integer greater or equal than α and ${}_aD^\alpha$ denotes Riemann Liouville fractional derivative of order $\alpha > 0$.

Definition 2.5. Equation (1.2a) is called disconjugate on an interval $[a, b]$ if and only if all solutions of equation (1.2a) have at most one zero on the interval $[a, b]$.

3. PREPARATORY THEOREMS

To obtain an existence uniqueness criterion, we need to establish some auxiliary results in a series of theorems. The first two theorems provide Green's function and its properties, the last two yield Lyapunov type inequality and disconjugacy criterion.

3.1. **Green's function and its properties.** We derive Green's function for the nonhomogenous problem (1.1a)–(1.1b) as follows.

Theorem 3.1. *$y \in C[a, b]$ is a solution of the boundary value problem (1.1a)–(1.1b) if and only if y satisfies the following integral equation*

$$(3.1) \qquad y(t) = B\left(\frac{t-a}{b-a}\right)^{\alpha+\beta-1} + \int_a^b G(t,s)\left[g(s) - f(s)y(s)\right]ds$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha+\beta)} \begin{cases} -\left(\dfrac{t-a}{b-a}\right)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1} + (t-s)^{\alpha+\beta-1}, & a \leq s < t \leq b \\ -\left(\dfrac{t-a}{b-a}\right)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1}, & a \leq t \leq s \leq b \end{cases}$$

Proof. The proof is similar to that of [10] but for the completeness of this paper, we will give all the proofs in detail.

By applying the fractional integral operators $_aI^\alpha$ and $_aI^\beta$ to the both sides of equation (1.1a), respectively and using the semigroup property

$$(_aI_a^\alpha I^\beta y)(t) = (_aI^{\alpha+\beta} y)(t)$$

and Lemma 2.4, one can obtain that y is a solution of (1.1a) if and only if

(3.2)
$$y(t) = c_1 \frac{\Gamma(\alpha)(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + c_2(t-a)^{\beta-1}$$
$$+ \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} [g(s) - f(s)y(s)] \, ds$$

for some real constants c_1, c_2. In order to have $y(a) = 0$, we will find $c_2 = 0$. By imposing the second boundary condition, $y(b) = B$, we have

$$c_1 = \frac{B\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)(b-a)^{\alpha+\beta-1}} - \frac{1}{\Gamma(\alpha)(b-a)^{\alpha+\beta-1}} \int_a^b (b-s)^{\alpha+\beta-1} [g(s) - f(s)y(s)] \, ds$$

and the solution y becomes as

$$y(t) = -\left(\frac{t-a}{b-a}\right)^{\alpha+\beta-1} \frac{1}{\Gamma(\alpha+\beta)} \int_a^b (b-s)^{\alpha+\beta-1} [g(s) - f(s)y(s)] \, ds$$
$$+ \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} [g(s) - f(s)y(s)] \, ds + B\left(\frac{t-a}{b-a}\right)^{\alpha+\beta-1},$$

which implies the desired result. □

Remark 3.2. Let us consider the boundary value problem

(3.3a) $(_aD^\alpha y)(t) + f(t)y(t) = 0 \quad a < t < b, \quad 1 < \alpha \le 2,$

(3.3b) $y(a) = 0, \quad y(b) = 0,$

and its solution and Green's function

(3.4)
$$y(t) = \int_a^b G(t,s)f(s)y(s)ds,$$

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(\dfrac{t-a}{b-a}\right)^{\alpha-1} (b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \le s < t \le b \\ \left(\dfrac{t-a}{b-a}\right)^{\alpha-1} (b-s)^{\alpha-1}, & a \le t \le s \le b \end{cases},$$

respectively, given in [10]. Although $(_aD^\alpha(_aD^\beta y))(t) \ne (_aD^{\alpha+\beta} y))(t)$, our result can be considered as a generalization of this result to the sequential differential equation.

Properties of Green's functions, which will be used in the main theorem, are given in the next theorem.

Theorem 3.3. *Green's functon has the following features:*

(1) $G(t, s) \leq 0$ *for all* $a \leq t, \, s \leq b.$

(2) $\min_{t \in [a,b]} G(t, s) = G(s, s), \, s \in [a, b].$

(3) $\min_{s \in [a,b]} G(s, s) = G\left(\frac{a+b}{2}, \frac{a+b}{2}\right) = -\frac{1}{\Gamma(\alpha+\beta)}\left(\frac{b-a}{4}\right)^{\alpha+\beta-1}.$

Proof. Let us define two functions $g_1(t, s) = -\left(\frac{t-a}{b-a}\right)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1} + (t-s)^{\alpha+\beta-1}$, $a \leq s < t \leq b$ and $g_2(t, s) = -\left(\frac{t-a}{b-a}\right)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1}$, $a \leq t \leq s \leq b.$

(1) It is obvious that $g_2(t, s) \leq 0$, $g_2(t, s)$ is a decreasing function of t, and so, $\min_{t \in [a,b]} g_2(t, s) = g_2(s, s) = -\left(\frac{s-a}{b-a}\right)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1}$, $s \in [a, b].$

Now, consider $g_1(t, s)$. Since $s \geq a$, we have $a + \frac{(s-a)(b-a)}{t-a} \geq s$ and

$$
\begin{aligned}
g_1(t, s) &= -\left(\frac{t-a}{b-a}\right)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1} + (t-s)^{\alpha+\beta-1} \\
&= -\left(\frac{t-a}{b-a}\right)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1} \\
&\quad + \left(\frac{t-a}{b-a}\right)^{\alpha+\beta-1}\left[b - \left(a + \frac{(s-a)(b-a)}{t-a}\right)\right]^{\alpha+\beta-1} \\
&\leq -\left(\frac{t-a}{b-a}\right)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1} + \left(\frac{t-a}{b-a}\right)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1} = 0
\end{aligned}
$$

(2) To prove (2), let us differentiate $g_1(t, s)$ with respect to t for every fixed s. Observe that

$$
\begin{aligned}
\frac{\partial g_1(t, s)}{\partial t} &= (\alpha + \beta - 1)\left[-\left(\frac{b-s}{b-a}\right)^{\alpha+\beta-1}(t-a)^{\alpha+\beta-2} + (t-s)^{\alpha+\beta-2}\right] \\
&\geq (\alpha + \beta - 1)\left[-(t-a)^{\alpha+\beta-2} + (t-s)^{\alpha+\beta-2}\right] \geq 0
\end{aligned}
$$

which yields $g_1(t, s)$ is a increasing function of t. Therefore it attains its minimum at $t = s$. Hence we get

$$
\min_{t \in [a,b]} g_1(t, s) = g_1(s, s) = g_2(s, s) = -\left(\frac{s-a}{b-a}\right)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1}.
$$

(3) Let us define

$$
f(s) := g_1(s, s) = g_2(s, s) = -\left(\frac{s-a}{b-a}\right)^{\alpha+\beta-1}(b-s)^{\alpha+\beta-1}, \quad s \in [a, b].
$$

Then for $s \in (a, b)$, we have

$$
f'(s) = (\alpha + \beta - 1)\frac{[(s-a)(b-s)]^{\alpha+\beta-2}}{(b-a)^{\alpha+\beta-1}}(-b + 2s - a)
$$

which yields that $f'(s) = 0$ if $s = \frac{a+b}{2}$. It is easy to show that $f'(s) < 0$ when $s < \frac{a+b}{2}$ and $f'(s) > 0$ when $s > \frac{a+b}{2}$.

$\square$

3.2. **Lyapunov type inequality for homogenous problem.** In this section we will give Lyapunov type inequality and disconjugacy criterion for the corresponding homogenous boundary value problem (1.2a)–(1.2b) in order to prove the uniqueness of the solution of nonhomogenous boundary value problem (1.1a)–(1.1b).

Theorem 3.4. *If the homogenous boundary value problem* (1.2a)–(1.2b) *has a non-trivial solution* $y(t) \neq 0$ *on* (a, b)*, then we have Lyapunov type inequality*

$$(3.5) \qquad \int_a^b f^+(s)ds > \Gamma(\alpha + \beta) \left(\frac{4}{b-a} \right)^{\alpha+\beta-1},$$

where $f^+(t) = \max\{f(t), 0\}$.

Proof. Let $y(t)$ be a nontrivial, real and continuous solution of homogenous boundary value problem (1.2a)–(1.2b). Since $_aD^\alpha$ is a linear operator, without loss of generality we may assume that $y(t) > 0$ on (a, b). Since $y(t)$ is continuos on $[a, b]$, there exist a point c in $[a, b]$ such that $\max_{t \in [a,b]} y(t) = y(c)$. Moreover since $G(t, s) \leq 0$ for all $t, s \in [a, b]$, $\min_{s \in [a,b]} G(s, s) = \max_{s \in [a,b]} -G(s, s)$. Then by using (3.1), we obtain

$$(3.6) \qquad y(c) = -\int_a^b G(c, s)f(s)y(s)ds \leq -\int_a^b G(c, s)f^+(s)y(s)ds,$$

where $f^+(t) = \max\{f(t), 0\}$.

Since $y(t) \leq y(c)$ for all $t \in [a, b]$, we have

$$(3.7) \qquad y(c) < -y(c) \int_a^b G(c, s)f^+(s)ds.$$

Employing the properties of Green's function obtained in Theorem 3.3, (3.7) turns into

$$1 < -\int_a^b G(c, s)f^+(s)ds \leq \frac{1}{\Gamma(\alpha + \beta)} \left(\frac{b-a}{4} \right)^{\alpha+\beta-1} \int_a^b f^+(s)ds,$$

which yields the desired result. $\qquad \square$

Remark 3.5. If $\alpha = \beta = 1$ in (1.2a)–(1.2b), then homogenous fractional boundary value problem (1.2a)–(1.2b) becomes as boundary value problem (1.3) involving integer order derivative considered in [22], [5] and [20]. Then inequality (3.5) reduces to inequality (1.6).

Remark 3.6. Since $f^+(t) \leq |f(t)|$, inequality (3.5) is the fractional generalization of inequality (1.6) and it is an extension and improvement of inequality (1.5).

Remark 3.7. Since $f^+(t) \leq |f(t)|$, Theorem 3.4 can be considered as a generalization and improvement of [10, Theorem 2.1].

3.3. **Disconjugacy criterion for homogenous problem.** Disconjugacy criterion can be considered as an application of Lyapunov type inequality due to the fact that the former is an immediate consequence of the latter. Besides, this criterion will be the sufficient condition for the uniqueness of the solution of the fractional boundary value problem (1.1a)–(1.1b).

Theorem 3.8. *If*

(3.8)
$$\int_a^b f^+(s)ds \le \Gamma(\alpha + \beta)\left(\frac{4}{b-a}\right)^{\alpha+\beta-1}$$

where $f^+(t) = \max\{f(t), 0\}$, then equation (1.2a) is disconjugate on $[a, b]$.

Proof. Proof is made by contradiction. Assume that equation (1.2a) is not disconjugate on $[a, b]$. Then there exist a nontrivial solution y of equation (1.2a) and at least two points $t_1, t_2 \in [a, b]$ such that $y(t_1) = y(t_2) = 0$ for $t \in [a, b]$ and $y(t) \ne 0$ for $t \in [a, b]$. Then by using Lyapunov type inequality on the interval $[t_1, t_2]$, we have

$$\int_{t_1}^{t_2} f^+(s)ds > \Gamma(\alpha + \beta)\left(\frac{4}{t_2 - t_1}\right)^{\alpha+\beta-1}$$

and hence

$$\int_a^b f^+(s)ds > \int_{t_1}^{t_2} f^+(s)ds > \Gamma(\alpha + \beta)\left(\frac{4}{t_2 - t_1}\right)^{\alpha+\beta-1} > \Gamma(\alpha + \beta)\left(\frac{4}{b-a}\right)^{\alpha+\beta-1},$$

which contradicts inequality (3.8). $\square$

4. MAIN RESULT

The main result of the paper providing existence and uniqueness result for the nonhomogenous boundary problem (1.1a)–(1.1b) is as follows:

Theorem 4.1. *If*

(4.1)
$$\int_a^b f^+(s)ds \le \Gamma(\alpha + \beta)\left(\frac{4}{b-a}\right)^{\alpha+\beta-1},$$

then nonhomogenous boundary problem (1.1a)–(1.1b) has a unique solution which is also the unique solution of integral equation (3.1).

Proof. It is shown in the proof of Theorem 3.1 that the solution of nonhomogenous boundary problem (1.1a)–(1.1b) is the solution of integral equation (3.1), and vice versa. To prove the uniqueness, it is sufficient to show that the homogenous boundary value problem (1.2a)–(1.2b) has only trivial solution. Assume on the contrary that $y(t) \not\equiv 0$ is a solution of the homogenous boundary value problem (1.2a)–(1.2b). Then by using Lyapunov type inequality, we have

(4.2)
$$\int_a^b f^+(s)ds > \Gamma(\alpha + \beta)\left(\frac{4}{b-a}\right)^{\alpha+\beta-1}$$

which gives a contradiction to (4.1). Therefore the homogenous boundary value problem (1.2a)–(1.2b) has only trivial solution. Because of the theory of linear fractional boundary value problems, see [9, Theorem 7.19], the nonhomogenous boundary problem (1.1a)–(1.1b) has a unique solution. □

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