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STOCHASTIC MULTICULTURAL DYNAMIC NETWORKS

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ABSTRACT. In this work, we seek to study the cohesive properties of a dynamic multi-cultural network under random environmental perturbations. By considering a multi-agent dynamic network, we seek to model a social structure and find conditions under which cohesion and coexistence is maintained. Utilizing Lyapunov's Second Method and the comparison method, we present a prototype illustration which serves the significance of the framework and approach. Moreover, the explicit sufficient conditions in terms of system parameters are given to exhibit when the network is cohesive. The sufficient conditions are algebraically simple, easy to verify, and robust. Further, we decompose the cultural state domain into invariant sets and consider the behavior of members within each set. We also demonstrate how conservative the estimates are using Euler-Maruyama type numerical approximation schemes based on the given illustration.

AMS (MOS) Subject Classification. 37H10.

1. INTRODUCTION

The goal of this work is to explore the cohesive properties of a dynamic network under random environmental perturbations [6]. Cohesion within a social network is a current topic of great interest, and many authors have done research in this area [8, 4]. The concepts of cohesion, coordination, and cooperation within a group are often multi-faceted, dynamic and complex, but are important concepts when trying to better understand how nations or communities function [3]. As Knoke and Yang note [15], it is social cohesion which enables information to spread and allows a group to act as a unit rather than individuals.

Another concept studied using a dynamic social network is that of consensus [7, 9, 21, 1]. In such models, the conditions under which a group collectively comes to an agreement are studied. Another question of interest for such a network is when the group may divide into subgroups with an agreement reached within the subgroup but never reaching a consensus at an overall group level. Such dynamic network models are useful in many areas. For example, studies in economics, finance, engineering, management sciences, and biological networks consider such large scale dynamic models to investigate connectivity, stability and convergence [20, 19, 2, 22].

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Using these ideas, much of the work done in these areas seek to develop consensus seeking algorithms and consider long term stability of the network in consideration [5, 23, 12, 13].

We use the term multi-cultural social network to describe a social network in which the agents have a diverse cultural background and are actively seeking to maintain diversity. In such a network, the goal of agents is not in arriving at a consensus but rather creating an environment in which members live and work cooperatively with one another. For example, consider a population in an area in which there exists a sub-populace of immigrants. In such a situation, the subgroup of immigrants desire to become an integrated part of the society while retaining their cultural diversity.

We seek to model such a situation and better understand the social dynamics of a group seeking to find such a balance. In doing so, we are interested in better understanding the cohesive properties of a multi-cultural social network. We present a simple stochastic dynamic model for which we explore the features of the a network. The presented example is used to exhibit the quantitative and qualitative properties of the network. Further, the techniques used are computationally attractive and algebraically simple relating with the underlying network parameters.

In Section 2, we present the general problem under consideration and the underlining assumptions. We then present an example of such a network in Section 3. Using an appropriate energy function and the comparison method, upper and lower estimates on cultural states are established in Sections 4 and 5, respectively. In Section 6, the long-term behavior of the solutions to the comparison equations are examined. Then, in Section 7, we explore the study of the cultural state invariant sets in the context of the illustration presented in Section 3 and using the long-term behavior of the comparison solutions described in Section 6. In Section 8, we use numerical simulations to model the network and better understand to what extent the estimates in Sections 4, 5, and 6 are feasible. Using the cohesive property of the network, we examine the dynamic properties of the network in Section 9.

2. **PROBLEM FORMULATION**

The network consists of m agents whose position at time t is represented by $x^i(t)$, $i \in I(1,m) = \{1, 2, \ldots, m\}$, with $x^i(t) \in \mathbb{R}^n$. In our model, this vector does not represent a geographical location but rather a cultural state position of the ith member. That is to say, the vector x^i is a numerical representation of the ith member's beliefs or background on certain cultural or ethnic characteristics relevant to the network and question being considered. Further, we assume that $\xi_{ij}, i, j \in I(1,m)$ is a normalized Wiener process with $\xi_{ij} = \xi ji$ and for $j \neq k$, ξ_{ij} and ξ_{ik} are independent. We then consider a cultural state stochastic dynamic model described by a system of

Itô-Doob type stochastic differential equation:

$$dx_i = \sum_{j=1}^{m} f(t, x_i - x_j)dt + \sum_{j \neq i}^{m} \sigma(t, x_i - x_j, \eta)d\xi_{ij}(t), \tag{2.1}$$

where $i \in I(1, m)$; f and σ are drift and diffusion rate coefficient functions, respectively. We will also make the following assumptions:

Assumption H_1: For $t_0 \in [0, \infty)$,

(i) $x_i(t_0) = x_{i,0}$ is an n-dimensional initial cultural state random vector defined on the complete probability space (Ω, F, P);
(ii) For $t \geq t_0$, F_t is an increasing family of sub-σ algebras of σ-algebra F;
(iii) For $i, j \in I(1, m)$, $\xi_i(t) = (\xi_{i1}, \xi_{i2}, \ldots, \xi_{im})^T$ is a m-dimensional normalized Wiener process of independent increments for $i \in I(1, m)$;
(iv) $\xi_{ij}(t)$ is F_t-measurable for $t \geq t_0$ and $x_i(t_0)$ is F_{t_0} measurable;
(v) $x_i(t_0)$ and $\xi_{ij}(t)$ are independent for each $t \geq t_0$ for $i \neq j$, $i, j \in I(1, m)$.

We wish to consider the cohesive properties of such a network. Further, we will explore the behavior of a member of the network based on the distance between the cultural state of a network member and the center of the network. To this end, we introduce the following definitions. We say a network is *cohesive with probability* 1 *(or almost surely)* if there exist non-negative functions $r_1(t), r_2(t)$ for $t \in [0, \infty)$, such that

$$r_1(t) \leq \|x_i(t) - x_j(t)\| \leq r_2(t), \tag{2.2}$$

for all $i, j \in I(1, m)$. We further use the term *relative cultural state affinity* to be the value $\|x_i(t) - x_j(t)\|$ for $t \geq t_0$.

3. PROTOTYPE DYNAMIC MODEL

Let us define a prototype multicultural network dynamic model under the stochastic environmental perturbations described by the Itô-Doob type stochastic system of differential equations

$$\begin{cases} dx_i & = \Big[a \sum_{j=1}^{m} x_{ij} - q \,\|\|x_i - \bar{x}\|^2 \sum_{j=1}^{m} x_{ij} \\ & \quad + b \sin \|x_i - \bar{x}\| \sum_{j=1}^{m} x_{ij} \exp\left[-\frac{\|x_{ij}\|^2}{c} \right] \Big] dt \\ & \quad + \beta \sin \|x_i - \bar{x}\| \sum_{j=1}^{m} x_{ij} \exp\left[-\frac{\|x_{ij}\|^2}{c} \right] d\xi_{ij}, \\ x_i(t_0) & = x_{i0}, \end{cases} \tag{3.1}$$

where a, q, b, c and β are positive real numbers; and ξ_{ij}'s are Weiner processes that are mutually independent for $i \neq j$, for $i, j \in I(1, m)$, and

$$x_{ij} = x_i - x_j. \tag{3.2}$$

Here, $\bar{x}$ is the center of the multicultural dynamic system (3.1) defined by:

$$\bar{x} = \frac{1}{m}\sum_{j=1}^{m} x_j, \tag{3.3}$$

and note that by substituting $x_i = \bar{x}$ into (3.1),

$$\begin{aligned} d\bar{x} &= \Bigg[a\sum_{j=1}^{m}(\bar{x}-x_j) - q\,\|\bar{x}-\bar{x}\|^2\sum_{j=1}^{m}(\bar{x}-x_j) \\ &\quad + b\sin\|\bar{x}-\bar{x}\|\sum_{j=1}^{m}(\bar{x}-x_j)\exp\left[-\frac{\|\bar{x}-x_j\|^2}{c}\right]\Bigg]dt \\ &\quad + \beta\sin\|\bar{x}-\bar{x}\|\sum_{j=1}^{m}(\bar{x}-x_j)\exp\left[-\frac{\|\bar{x}-x_j\|^2}{c}\right]d\xi_{\bar{x}j}, \\ &= am\bar{x} - a\sum_{j=1}^{m}x_j \\ &= am\bar{x} - am\bar{x} \\ &= 0, \end{aligned} \tag{3.4}$$

and thus $\bar{x}$ defined in (3.3) is a stationary center of the multicultural dynamic network. We define the transformation $z_i = x_i - \bar{x}$ and observe that $x_{ij} = z_i - z_j = z_{ij}$. Then the transformed network dynamic model corresponding to (3.1) is reduced to:

$$\begin{cases} dz_i &= \left[amz_i - q\,\|z_i\|^2\,mz_i + b\sin\|z_i\|\sum_{j=1}^{m} z_{ij}\exp\left[-\frac{\|z_{ij}\|^2}{c}\right]\right]dt \\ & \quad +\beta\sin\|z_i\|\sum_{j=1}^{m} z_{ij}\exp\left[-\frac{\|z_{ij}\|^2}{c}\right]d\xi_{ij}, \\ z_i(t_0) &= z_{i0}. \end{cases} \tag{3.5}$$

The center $\bar{x}$ of the multicultural dynamic model (3.1) is reduced to the center zero in (3.5). It exhibits both attractive and repulsive forces that are centered at the center of the network. The magnitude of the repulsive force is described by $am\,\|z_i\|$. Repulsive forces are attributes that create some desire for individuals to leave or be less involved in the group or to preserve some personal identity from one other with their individual magnitude of inner repulsive force. A desire to retain a sense of individuality, economic or emotional cost, interpersonal conflict within the group, or disagreement with parts of the overall philosophies of the group are forces that may be considered as repulsive forces. The magnitude of the long range deterministic attractive force is characterized by $b\left\|\sum z_{ij}\exp\left[-\frac{\|z_{ij}\|^2}{c}\right]\right\|$. Attractive influences can be thought of as attributes that bring people to active membership within the group. Social acceptance, gaining social status, economic opportunity, career growth, common purpose and membership, personal development, and a sense of mutual respect, trust and understanding are examples of attractive influences within a social cultural network. Further, $\sin\|z_i\|$ is the sine-cyclical influence of the ith member's relative distance to the center of the network. The stochastic term represents the environmental influence due to long-range attractive forces. In particular, in the case of a multi-cultural network, the noise captures the uncertainty generated due to the membership interactions and deliberations under the influence of the long-range cultural forces.

In order to study the multicultural dynamics (3.5), we use Lyapunov's Second Method in conjunction with the comparison method [18]. These methods are computationally attractive and provide a means of better understanding the movement and behavior of the cultural state memberships of the network. By utilizing these methods, we are able to establish conditions for which we have both upper and lower estimates on the members cultural state positions.

4. UPPER COMPARISON EQUATION

Using Lyapunov's Second Method and differential inequalities, we first seek a function $r(t, t_0, u_0)$ such that

$$\|z_i(t)\| \leq r(t, t_0, r_0). \tag{4.1}$$

From (2.2), relation (4.1) generates a concept of a upper-cohesive cultural network in the almost sure sense.

To this end, let us choose an energy function V as:

$$V(z_i) = \|z_i\| = \left(z_i^T z_i\right)^{\frac{1}{2}}, \tag{4.2}$$

and let us denote

$$\phi_1(z_i) = amz_i - q\,\|z_i\|^2\, mz_i + b\sin\|z_i\| \sum_{j=1}^{m} z_{ij} \exp\left[-\frac{\|z_{ij}\|^2}{c}\right], \tag{4.3}$$

and

$$\phi_2(z_{ij}) = \beta \sin\|z_i\|\, z_{ij} \exp\left[-\frac{\|z_{ij}\|^2}{c}\right]. \tag{4.4}$$

Then applying Itô-Doob differential formula [16]to (4.2), the differential of V in the direction of the vector field represented by (3.5) is

$$\begin{aligned} dV &= \frac{z_i^T dz_i}{\|z_i\|} + \frac{1}{2}\left[\frac{dz_i^T dz_i}{\|z_i\|} - \frac{\left(z_i^T dz_i\right)^2}{\|z_i\|^3}\right] \\ &= \frac{z_i^T\left(\phi_1(z_i)dt + \sum_{j=1}^m \phi_2(z_{ij})d\xi_{ij}\right)}{\|z_i\|} \\ &\quad + \frac{\left(\phi_1^T(z_i)dt + \sum_{j=1}^m \phi_2^T(z_{ij})d\xi_{ij}\right)\left(\phi_1(z_i)dt + \sum_{j=1}^m \phi_2(z_{ij})d\xi_{ij}\right)}{2\,\|z_i\|} \end{aligned} \tag{4.5}$$

$$-\frac{\left(z_i^T\left(\phi_1(z_i)dt+\sum_{j=1}^m\phi_2(z_{ij})d\xi_{ij}\right)\right)^2}{2\left\|z_i\right\|^3}$$
$$=\frac{z_i^T\sum_{j=1}^m\phi_2(z_{ij})d\xi_{ij}}{\left\|z_i\right\|}+LV(z_i)dt,$$

where

$$LV(z_i)=\frac{z_i^T\phi_1(z_i)dt}{\left\|z_i\right\|}+\frac{\sum_{j=1}^m\phi_2^T(z_{ij})\phi_2(z_{ij})}{2\left\|z_i\right\|}-\frac{\left(z_i^T\sum_{j=1}^m\phi_2(z_{ij})\right)^2}{2\left\|z_i\right\|^3}$$
$$=\Bigg[am\left\|z_i\right\|-qm\left\|z_i\right\|^3+\frac{b\sin\left\|z_i\right\|}{\left\|z_i\right\|}\sum_{j=1}^m z_i^Tz_{ij}\exp\left[-\frac{\left\|z_{ij}\right\|^2}{c}\right]$$
$$+\frac{\beta^2\sin^2\left\|z_i\right\|\sum_{j=1}^m z_{ij}^Tz_{ij}\exp\left[-\frac{2\left\|z_{ij}\right\|^2}{c}\right]}{2\left\|z_i\right\|}$$
$$-\frac{\beta^2\sin^2\left\|z_i\right\|\sum_{j=1^m}(z_i^Tz_{ij})^2\exp\left[-\frac{2\left\|z_i\right\|^2}{c}\right]}{2\left\|z_i\right\|^3}\Bigg]dt. \tag{4.6}$$

We seek constraints on the parameters a, b, c, q and β for which we have an upper estimate on the first moment of $V(z_i)$. Thus, let us consider an upper estimate on LV defined in (4.6). We first note that the function

$$f(r)=r\exp\left[-\frac{r^2}{c}\right] \tag{4.7}$$

has a maximum value of $\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]$ when $r=\sqrt{\frac{c}{2}}$. Further the function

$$g(r)=r^2\left(\exp\left[-\frac{r^2}{c}\right]\right)^2=r^2\exp\left[-\frac{2r^2}{c}\right] \tag{4.8}$$

that has a maximum value of $\frac{c}{2}\exp\left[-1\right]$ when $r=\sqrt{\frac{c}{2}}$. Therefore, from (4.6),

(4.9)

$$LV\le am\left\|z_i\right\|-qm\left\|z_i\right\|^3+\frac{b\sin\left\|z_i\right\|}{\left\|z_i\right\|}\sum_{j\neq i}^m z_i^Tz_{ij}\exp\left[-\frac{\left\|z_{ij}\right\|^2}{c}\right]$$
$$+\frac{\beta^2\sin^2\left\|z_i\right\|}{2\left\|z_i\right\|}\sum_{j\neq i}^m\left\|z_{ij}\right\|^2\exp\left[-\frac{\left\|z_{ij}\right\|^2}{c}\right]$$
$$\le am\left\|z_i\right\|-qm\left\|z_i\right\|^3+b\sum_{j\neq i}^m\left\|z_i\right\|\left\|z_{ij}\right\|\exp\left[-\frac{\left\|z_{ij}\right\|^2}{c}\right]$$
$$+\frac{\beta^2\left\|z_i\right\|\sin^2\left\|z_i\right\|}{2\left\|z_i\right\|^2}\sum_{j\neq i}^m\left\|z_{ij}\right\|^2\exp\left[-\frac{\left\|z_{ij}\right\|^2}{c}\right]$$
$$\le am\left\|z_i\right\|-qm\left\|z_i\right\|^3+b\left\|z_i\right\|\sum_{j\neq i}^m\left\|z_{ij}\right\|\exp\left[-\frac{\left\|z_{ij}\right\|^2}{c}\right]$$

$$
\begin{aligned}
&+\frac{\beta^2\|z_i\|}{2}\sum_{j\neq i}^{m}\|z_{ij}\|^2\exp\left[-\frac{\|z_{ij}\|^2}{c}\right]\\
&\leq am\|z_i\| - qm\|z_i\|^3 + b\|z_i\|(m-1)\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]\\
&+\frac{\beta^2\|z_i\|(m-1)c\exp[-1]}{4}\\
&= \|z_i\|\left(am + b(m-1)\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \frac{\beta^2(m-1)c\exp[-1]}{4}\right)\\
&-qm\|z_i\|^3\\
&= qm\|z_i\|\left(\frac{a}{q} + \frac{4b(m-1)\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2(m-1)c\exp[-1]}{4qm} - \|z_i\|^2\right)\\
&= qmV\left(\eta^2 - V^2\right)\\
&= qmV(\eta - V)(\eta + V),
\end{aligned}
$$

where

$$
\eta = \left(\frac{a}{q} + \frac{4b(m-1)\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2(m-1)c\exp[-1]}{4qm}\right)^{\frac{1}{2}}. \tag{4.10}
$$

In the following, we present a fundamental result that will be used subsequently.

Lemma 4.1. *Let V be the energy function defined in* (4.2) *and z_i be a solution of the initial value problem defined in* (3.5)*. Then, for each $i \in I(1,m)$,*

$$
E\left[V(z_i(t+\Delta t)) - V(z_i(t))|F_t\right] = LV(z_i(t))\Delta t, \tag{4.11}
$$

where E stands for the conditional expected value for given F_t and Δt, a positive increment to t.

Proof. Let $z_i(t, t_0, z_i(t_0))$ be the solution process of (3.5). Let F_t be an increasing family of sub-σ algebras as previously defined and set

$$
m(t) = E\left[V(z_i(t))|F_t\right] = V(z_i(t)), \tag{4.12}
$$

where the last equality holds as $z_i(t)$ is F_t measurable. Similarly, we set

$$
m(t+\Delta t) = E\left[V(z_i(t+\Delta t))|F_t\right], \tag{4.13}
$$

for all $\Delta t > 0$. We consider

$$
\begin{aligned}
m(t+\Delta t) - m(t) &= E\left[V\left(z_i(t+\Delta t) - V\left(z_(t)\right)\right)|F_t\right] \\
&= E\Bigg[\frac{\partial V}{\partial z}(z_i(t))\Delta z_i(t)\\
&\quad +\frac{1}{2}tr\left(\frac{\partial^2 V}{\partial z^2}(\Delta z_i(t))(\Delta z_i(t))^T\right)|F_t\Bigg]
\end{aligned} \tag{4.14}
$$

$$= E\left[dV\left(z_i(t)\right)|F_t\right].$$

This together with (4.5) yields

$$m\left(t+\Delta t\right) - m\left(t\right) = E\left[LV\left(z_i(t)\right)\Delta t|F_t\right] = LV\left(z_i(t)\right)\Delta t, \tag{4.15}$$

as $z_i(t)$ is F_t measurable. We note that for small Δt, we have

$$dm(t) = LV\left(z_i(t)\right)dt. \tag{4.16}$$

From (4.12) and (4.13), (4.16) reduces to (4.11). This completes the proof of the Lemma. □

From inequality (4.9) in conjunction with the comparison method [18] and Lemma 4.1, we establish the following lemma. The presented result establishes not only an upper bound but also the upper cohesive property almost surely. Hereafter, all inequalities and equalities are assumed to be valid with probability one.

Lemma 4.2. *Let V be the energy function defined in* (4.2) *and z_i be a solution of the initial value problem defined in* (3.5). *Let*

$$du = \left[qmu\left(\eta - u\right)\left(\eta + u\right)\right]dt, \qquad r(t_0) = u_0, \tag{4.17}$$

where η is as defined in (4.10). *For each $V(z_i)$, $i \in I(1,m)$ satisfying the differential inequality* (4.9) *and $V(z_i(t_0)) \leq u_0$, it follows that the multicultural dynamic network* (3.1) *is upper cohesive and*

$$V(z_i(t)) \leq r(t,t_0,u_0), \tag{4.18}$$

where $r(t)$ is the maximal solution of the comparison random initial value problem (4.17).

Proof. From (4.9), Lemma 4.1, and following the standard argument used in proofs of comparison theorems in the frame-work of the Lyapunov method, with probability 1, it follows that

$$V(z_i(t)) \leq r(t,t_0,u_0), \tag{4.19}$$

whenever $V(z_i(t_0)) \leq u_0$. We note that the maximal solution of (4.17) is an upper bound. Hence, the network is upper cohesive almost surely. □

5. **LOWER COMPARISON EQUATION**

Using Lyapunov's Second Method and differential inequalities, we next seek a function $\rho(t, t_0, u_0)$ such that

$$\|z_i(t)\| \geq \rho(t, t_0, \rho_0). \tag{5.1}$$

Again, from (2.2), relation (5.1) initiates a notion of a lower cohesive cultural dynamic network in the almost sure sense.

Using the energy function defined in (4.2) and relation (4.6), it follows that

$$\begin{aligned} LV &\geq am\|z_i\| - qm\|z_i\|^3 - b\sum_{j\neq i}^{m}\|z_i\|\,\|z_{ij}\|\exp\left[-\frac{\|z_{ij}\|^2}{c}\right] \\ &\quad -\frac{\beta^2}{2\|z_i\|}\sum_{j\neq i}^{m}\|z_i\|^2\|z_{ij}\|^2\exp\left[-\frac{2\|z_{ij}\|^2}{c}\right] \\ &= am\|z_i\| - qm\|z_i\|^3 - b\|z_i\|\sum_{j\neq i}^{m}\|z_{ij}\|\exp\left[-\frac{\|z_{ij}\|^2}{c}\right] \\ &\quad -\frac{\beta^2\|z_i\|}{2}\sum_{j\neq i}^{m}\|z_{ij}\|^2\exp\left[-\frac{2\|z_{ij}\|^2}{c}\right] \\ &\geq amV - qmV^3 - V(m-1)b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] \\ &\quad -\frac{\beta^2(m-1)c\exp[-1]}{4}V \\ &= qmV\left(\frac{a}{q} - \frac{4(m-1)b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2c(m-1)\exp[-1]}{4qm} - V^2\right). \end{aligned} \tag{5.2}$$

Assumption H_2: Assume there exists a positive number α such that

$$\alpha \leq \left(\frac{a}{q} - \frac{4(m-1)b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2(m-1)c\exp[-1]}{4qm}\right)^{\frac{1}{2}}. \tag{5.3}$$

From (5.2), and noticing the fact that assumption H_2 implies

$$\frac{a}{q} > \frac{4\,(m-1)\,b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2\,(m-1)\exp[-1]}{4qm}, \tag{5.4}$$

it follows that

$$LV \geq qmV(\alpha - V)(\alpha + V). \tag{5.5}$$

From inequality (5.5) in conjunction with the comparison method [18] and Lemma 4.1, we establish the following lemma. The presented result provides the lower estimate which in turn establishes the lower cohesive property of (3.5).

Lemma 5.1. *Let V be the energy function defined in* (4.2) *and z_i be a solution of the initial value problem defined in* (3.5). *Let*

$$du = qmu\,(\alpha - u)\,(\alpha + u)\,dt, \qquad u(t_0) = u_0, \tag{5.6}$$

where α as defined in (5.3). *For each $V(z_i)$, $i \in I(1,m)$ satisfying the differential inequality* (5.5) *and $\|z_i(t_0)\| \geq u_0$, it follows that*

$$V(z_i(t)) \geq \rho(t, t_0, u_0), \tag{5.7}$$

where $\rho(t)$ is the minimal solution of the comparison random initial value problem (5.6).

Proof. From equation (5.5) in conjunction with Lemma 4.1 and the imitating the outline of the proof of Lemma 4.2, it follows that

$$V(z_i(t)) \geq \rho(t, t_0, u_0) \tag{5.8}$$

provided that $V(z_i(t_0)) \geq u_0$. As the minimal solution of (5.6) is a lower bound, the network is lower cohesive almost surely. □

We note that comparison differential equations (4.17) and (5.6) each have a unique solution process. Therefore the maximal and minimal solutions of (4.17) and (5.6) are indeed the unique solution of the respective random initial value problems.

6. LONG-TERM BEHAVIOR OF THE COMPARISON DIFFERENTIAL EQUATION

To appreciate the role and scope of Lemmas 4.2 and 5.1, we seek to better understand the long-term behavior of the network. For this purpose, we find the closed form solutions of the comparison random initial value problems (4.17) and (5.6). Moreover, we analyze the qualitative properties of the solutions to the comparison equations. Using the comparison method [18], we are able to establish, computationally, the overall long-term behavior of both individual member cultural dynamic states within the network as well as multicultural network state as a whole.

Let us first begin with the solution of the comparison differential equation

$$du = qmu\,(\nu - u)\,(\nu + u)\,dt, \qquad u(t_0) = u_0, \tag{6.1}$$

where ν is a positive real number. Following the method of finding the closed form solution process of the initial value problem [16], the solution of (6.1) is represented by

$$u(t, t_0, u_0) = \frac{u_0 \nu}{\sqrt{u_0^2 + (\nu^2 - u_0^2)\exp\left[-2\nu^2 qm(t - t_0)\right]}}. \tag{6.2}$$

We note that both ν and u_0 in (6.2) are positive. If $\nu > u_0$, then $\nu^2 > u_0^2$, and hence the term under the radical is positive. Suppose it is the case that $\nu < u_0$. Then we note that

$$0 < \exp\left[-2\nu^2 qm(t-t_0)\right] < 1, \tag{6.3}$$

for $t > t_0$. From (6.2) and (6.3), it follows that

$$\begin{aligned} u_0^2 + \left(\nu^2 - u_0^2\right)\exp\left[-2\nu^2 qm(t-t_0)\right] &= u_0^2 - u_0^2 \exp\left[-2\nu^2 qm(t-t_0)\right] \\ &\quad + \nu^2 \exp\left[-2\nu^2 qm(t-t_0)\right] \\ &= u_0^2\left(1 - \exp\left[-2\nu^2 qm(t-t_0)\right]\right) \\ &\quad + \nu^2 \exp\left[-2\nu^2 qm(t-t_0)\right] \\ &> 0. \end{aligned} \tag{6.4}$$

Hence, the term under the radical in (6.2) is positive in both cases: $\nu > u_0$ and $\nu < u_0$. Thus, under either of the conditions, $\nu > u_0$ or $\nu < u_0$,

$$\begin{aligned} \lim_{t\to\infty} u(t, t_0, u_0) &= \lim_{t\to\infty} \frac{u_0 \nu}{\sqrt{u_0^2 + (\nu^2 - u_0^2)\exp\left[-2\nu^2 qm(t-t_0)\right]}} \\ &= \nu \end{aligned} \tag{6.5}$$

From (4.10) and Lemma 4.2 in conjunction with (6.5), for $\nu = \eta$ it follows that the limit of the upper comparison solution $r(t)$ as t grows large is

$$\eta = \left(\frac{a}{q} + \frac{4b\,(m-1)\sqrt{\frac{1}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2\,(m-1)\,c\exp\left[-1\right]}{4qm}\right)^{\frac{1}{2}}. \tag{6.6}$$

Similarly, from (5.3) and Lemma 5.1 in conjunction with (6.5), for $\nu = \alpha$, the limit of the lower comparison solution $\rho(t)$ as t grows large is α, where,

$$\alpha \leq \left(\frac{a}{q} - \frac{4(m-1)b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2(m-1)c\exp\left[-1\right]}{4qm}\right)^{\frac{1}{2}}. \tag{6.7}$$

From the solution of the comparison equations in conjunction with Lemmas 4.2 and 5.1, we establish the following theorem.

Theorem 6.1. *Let the hypotheses of Lemmas* 4.2 *and* 5.1 *be satisfied. Then the network is cohesive in the almost sure sense.*

Proof. From Lemmas 4.2 and 5.1,

$$\rho(t, t_0, \rho_0) \leq V(z_i(t)) \leq r(t, t_0, r_0) \tag{6.8}$$

with probability 1. Moreover, as the solution to the upper comparison equation is bounded above by η and the solution to the lower comparison equation is bounded below by α, the network is cohesive almost surely. □

In the following section, we provide various characterizations of cultural state dynamics. This is achieved by the nature of the initial cultural state parameters and the behavior of the upper and lower comparison cultural state dynamic processes.

7. INVARIANT SETS

In this section, we analyze various types of invariant states of the multicultural dynamic network. This is achieved by using the behavior of the solutions to both the upper and lower comparison equations. Let us denote

$$r_2 = \left(\frac{a}{q} - \frac{4(m-1)b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2(m-1)c\exp\left[-1\right]}{4qm} \right)^{\frac{1}{2}} \tag{7.1}$$

and

$$r_1 = \left(\frac{a}{q} + \frac{4b\,(m-1)\,\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2\,(m-1)\,c\exp\left[-1\right]}{4qm} \right)^{\frac{1}{2}} \tag{7.2}$$

We note that the parameters a, b, q, c, and β imply the following relation:

$$r_2 < r_1. \tag{7.3}$$

Further, let us define the following sets:

$$\begin{cases} A = B\,(0, r_2) \\ B = B^c\,(0, r_2) \cap B\,(0, r_1) \\ C = B^c\,(0, r_1) \end{cases} \tag{7.4}$$

Under the obvious relation (7.3), we develop and establish the following result.

Theorem 7.1. *Let the hypotheses of Lemmas* 4.2 *and* 5.1 *be satisfied. Then, in the almost sure sense,*

(i) *the set $A \cup B$ is conditionally invariant relative to A;*
(ii) *the set B is self-invariant;*
(iii) *the set $B \cup C$ is conditionally invariant relative to C.*

Proof. For $z_i \in C$, $i \in I(1, m)$, the hypotheses of Lemmas 4.2 and 5.1 are satisfied. Thus by the application of these Lemmas, we have

$$\rho(t, t_0, \rho_0) \leq V(z_i(t, t_0, z_0)) \leq r(t, t_0, r_0), \tag{7.5}$$

for $t > t_0$, $z_{i0} \in \bar{B}^c(0, r_1)$, and $\rho(t, t_0, , \rho_0)$ and $r(t, t_0, r_0)$ are the minimal and maximal solutions of the comparison differential equations (5.6) and (4.17) respectively. Moreover, for $z_i \in B^c(0, r_1)$, with $r_0 = \rho_0 = V(z_{i0}) = \|z_{i0}\|$, the solutions $r(t, t_0, r_0)$ and $\rho(t, t_0, \rho_0)$ are both monotonically decreasing and approaching to r_1 and r_2 respectively. Hence, we have

$$\rho(t, t_0, \rho_0) \leq \|z_i(t, t_0, z_{i,0})\| \leq r(t, t_0, r_0), \tag{7.6}$$

for $t \geq t_0$. From the definitions of self-invariant and conditionally invariant sets [17], it follows that statement (iii) is valid. The proofs of (i) and (ii) follow by imitating the argument used in the proof of (iii). For $z_{i0} \in B$, we note that $\rho(t, t_0, \rho_0)$ is monotonically decreasing and $r(t, t_0, r_0)$ is monotonically increasing to r_2 and r_1 as $t \to \infty$, respectively. This establishes that $z_i(t, t_0, z_{i0}) \in B$ proving statement (ii). For $z_{i0} \in A$, the solutions to the comparison equation (5.6), $\rho(t, t_0, \rho_0)$ is monotonically increasing to r_2 as $t \to \infty$. Therefore $z_i(t, t_0, z_{i0}) \in A \cup B$ proving statement (i). $\square$

Let us examine the results of Theorems 7.1. First, we note that this theorem provides sufficient conditions for the qualitative and quantitative behavior of the cultural state dynamics. In particular, the model is cohesive and simultaneously, it does not reach a cultural consensus.

We introduce the definition of *cultural threshold bound* to describe the boundary between two cultural state sets. It is based on the degree of individual versus community level interaction domains of the cultural states. Suppose $z_i \in A$. It is the case that the individual member cultural state is pushed out/repulsed from the cultural state center $\bar{x}$ at some time $T \geq t_0$. That is to say, the membership of the social network will support and then maintain a relative cultural affinity between members and the cultural center that is bounded below by the quantity r_2. Once the state of the ith member z_i has moved away from the center, it is the case that the state z_i moves to the cultural state set B, at which time the agent's cultural state behavior will follow that of another category of membership described by the cultural state set B discussed below.

Suppose that the ith member initial cultural state z_i of the transformed social network is such that $z_i \in B$. Then by Theorem 7.1, over time, z_i may stay in B, approaching the cultural threshold bounds of sets C and/or A. However, if $z_i \in B$, even though it may approach the cultural bound of A and/or C, it will never cross either of the boundaries. In terms of a given social network, this implies that members with a distinct enough cultural states from the weighted average of cultural states will retain that distinctiveness of culture while maintaining a certain level of closeness to the average cultural state. Thus, if the relative cultural affinity between a member x_i and the center of the network is at least r_2 and less than r_1, initially, then the relative cultural state affinity will always be at least the quantity r_2 but no more than the value r_1.

If it is the case that z_i is a member of the transformed network such that $z_i \in C$. By Theorem 7.1, z_i may either cross the cultural boundary of B or the members cultural state will approach asymptotically to the cultural state network boundary of B. Thus, for agents x_i within the network whose initial relative cultural state affinity with respect to the cultural state center is sufficiently large, as $t \to \infty$, the relative cultural affinity will remain large and although the agent is attracted back towards the center of the network, the relative cultural state affinity is bounded below by r_2.

8. NUMERICAL SIMULATIONS

In this section, we consider numerical simulations for the multicultural dynamic network governed by the stochastic differential equation (3.5) using a Euler-Maruyama [14, 10, 11] type numerical approximation scheme. We consider a network of fifty members, using the same initial position and varying the parameters a, b, q, c, and β. Further, we consider the case such that $\xi_{ij}(t)$ for $i, j \in I(1, 50)$ are a one dimensional Brownian motion process with mean of zero and variance of 1 over the interval $[0, 1]$. To generate each member state cultural trajectory, we average the position for fifty simulations for each of the various cases, and then plot the average position, $z_i(t)$ for each member.

In order to consider the effects of changing the parametric value β, we consider various models for which $a = 2, b = 1$, and $c = 2$ are held constant and we vary both β and q. First, in Figure 1, we consider a network in which $q = \frac{2}{1.7}$ and $\beta = .5$. With the given parameters, $r_1 \approx 1.5$ and $r_2 \approx 1.1$. Therefore, using the upper and lower limits of the comparison equations, the long run behavior of the network has the approximate bounds given by

$$1.1 \leq \|z_i\| \leq 1.5, \tag{8.1}$$

as demonstrated in Figure 1. In the simulation, we see that members whose cultural state start close to the center shift away from the center over time. Further, in the simulation, members whose cultural state start farther away from the center are attracted back towards the center over time.

Next, in Figure 2, we consider the case with the parameters $q = \frac{2}{5.4}$ and $\beta = 1$. In this case, $r_1 \approx 2.7$, and $r_2 \approx 1.8$. In this case, using the bounds on the limits of the solutions of the comparison equations yield the approximate bounds on the long term behavior of the network

$$1.8 \leq \|z_i\| \leq 2.7. \tag{8.2}$$

We note that in this case by increasing β and decreasing q, the upper and lower bounds, as well as the distance between them, increase. In the simulation, we observe a similar behavior of members within the network; those starting close to the center are repulsed away and those starting away from the center are attracted back towards it. In Figure 3, we consider the case with parameters $q = \frac{1}{12}$ and $\beta = 2$. Further, we note that in this case, $r_1 \approx 6.3$, and $r_2 \approx 2.9$. In this case, the approximate bounds on the long term behavior of the network are given by

$$2.9 \leq \|z_i\| \leq 6.3. \tag{8.3}$$

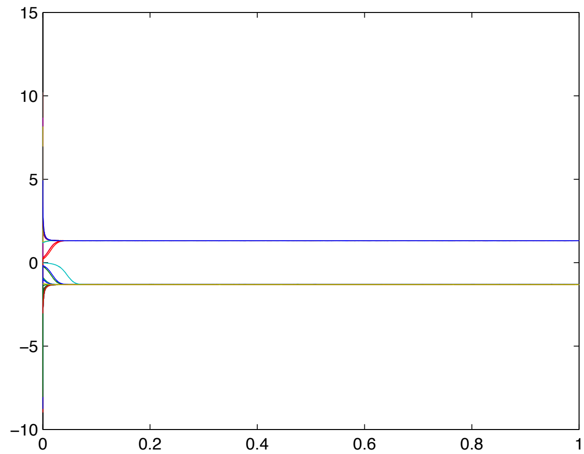

FIGURE 1. Euler-Maruyama approximation of the differential equation given by (3.5) with fifty members and parameters $a = 2$, $b = 1$, $c = 2$, $q = \frac{2}{1.7}$, and $\beta = .5$.

By increasing β and decreasing q, we have again increased the values of the upper and lower bounds as well as the distance between the bounds. Further, in the simulations, we see a strong repulsion from the center of the network and that over time, the memberships cultural states settle relatively far from the cultural state of the center.

We now consider the case with the parameters $q = \frac{1}{7}$ and $\beta = 2$. In this case, $r_1 \approx 4.8$, and $r_2 \approx 2.2$. We note that using the limits of the upper and lower comparison equations, we compute the long term approximate bounds as

$$2.2 \leq \|z_i\| \leq 4.8, \tag{8.4}$$

as seen in Figure 4. By increasing q in this simulation, the upper and lower bounds are smaller than those from Figure 3. We also note that the distance between the bounds has decreased from that in Figure 3.

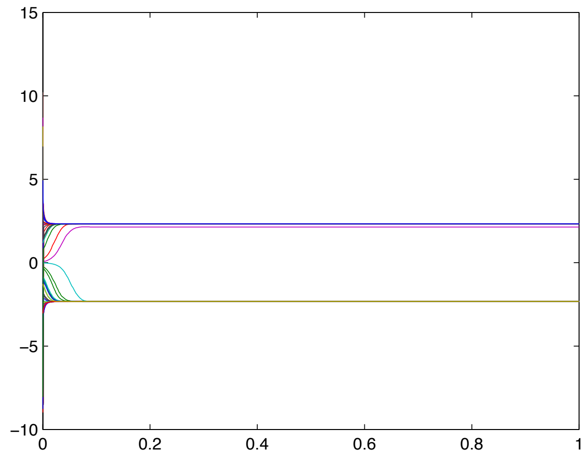

FIGURE 2. Euler-Maruyama approximation of the differential equation given by (3.5) with fifty members and parameters $a = 2$, $b = 1$, $c = 2$, $q = \frac{2}{5.4}$, and $\beta = 1$.

9. CONCLUSION

Maintaining diversity while simultaneously fostering a sense of community membership, individual cultural identity, and cohesion is currently a goal among communities worldwide. It is important for members in society to both feel as a part of the community in which they live and interact as well as feel free to embrace a strong sense of self and individuality. We seek to better understand the factors that play a role in obtaining such a balance by considering the impact of the repulsive and attractive forces influencing the multicultural network. The goal of the presented multicultural dynamic network is model the balance sought by members of the network in achieving these type of objectives. By doing so, we can consider the impact that policies and environmental factors may have on such a network.

We have considered requirements on the parameters that allow the perturbed multicultural dynamic network to remain cohesive while retaining a cultural state that is distinctive from the cultural state center of the network. We established qualitative and quantitative conditions that are computationally attractive and verifiable.

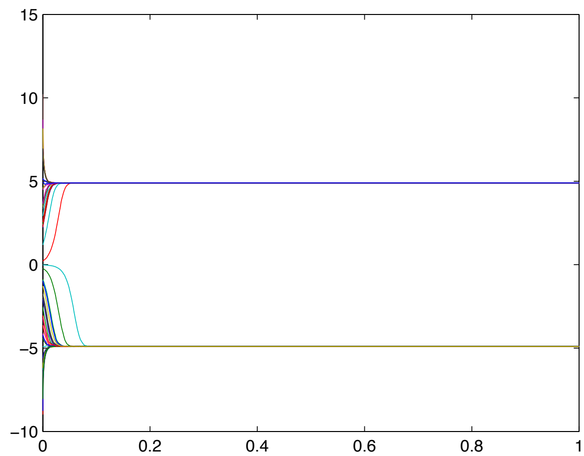

FIGURE 3. Euler-Maruyama approximation of the differential equation given by (3.5) with fifty members and parameters $a = 2$, $b = 1$, $c = 2$, $q = \frac{1}{12}$, and $\beta = 2$.

Further, we have analyzed cultural state invariant sets and long-term cultural states of members within the multicultural dynamic network. We also conducted simulations of the multicultural network that exhibit the influence of the random perturbations as well as demonstrate the long-term behavior of the multicultural network.

We are interested in further exploring similar multicultural networks in the context of better understanding the relative cultural state affinity $\|x_{ij}\|$ between members within the network and not just the relative cultural state affinity between the cultural state of a member relative to the center of the network. The goal is to better understand the environmental factors that help to foster a sense of individuality and diversity between all members within the network while maintaining a cohesive structure.

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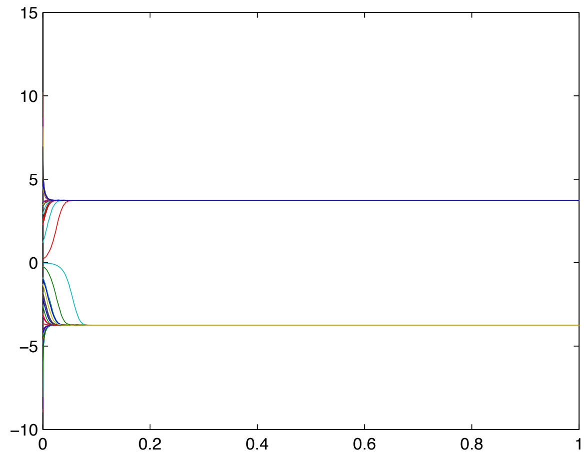

FIGURE 4. Euler-Maruyama approximation of the differential equation given by (3.5) with fifty members and parameters $a = 2$, $b = 1$, $c = 2$, $q = \frac{1}{7}$, and $\beta = 2$.

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