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OSCILLATORY BEHAVIOR OF ODD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH A NONPOSITIVE NEUTRAL TERM

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ABSTRACT: In this paper, sufficient conditions are established for oscillation of all solutions of the odd-order nonlinear differential equations with a nonpositive neutral term. When the neutral term is present, all the results are new even for $n = 3$. An example is given to illustrate the main results.

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1. INTRODUCTION

This paper deals with oscillatory behavior of all solutions of the nonlinear odd order differential equations with a nonpositive neutral term of the form

$$y^{(n)}(t) + q(t)x^{\beta}(\tau(t)) = 0, \tag{1.1}$$

where $y(t) = x(t) - p(t)x(\sigma(t))$, $t \geq t_0 > 0$ and $n \geq 3$ is an odd natural number.

Throughout, we assume that the following assumptions hold:

(i) β is a ratio of odd positive integers;

(ii) $p, q \in \mathcal{C}([t_0,\infty),[0,\infty))$, $0 \le p(t) < 1$;

(iii) $\tau, \sigma \in \mathcal{C}^1([t_0,\infty),\mathbb{R})$, $\tau(t) \le t$, $\sigma(t) \le t$, $\tau'(t) > 0$, $\sigma'(t) > 0$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(iv) $h(t) := \sigma^{-1}(\tau(t)) \le t$, $h'(t) \ge 0$, $\lim_{t\to\infty} h(t) = \infty$.

Set $t_x = \min_{t\in[t_0,\infty)}\{\sigma(t),\tau(t)\}$. By a solution of (1.1) we mean a nontrivial function $x(t) \in \mathcal{C}([t_x,\infty),\mathbb{R})$ such that $y(t) \in \mathcal{C}^n([t_0,\infty),\mathbb{R})$ and $x(t)$ satisfies (1.1) on $[t_0,\infty)$. We consider only those solutions $x(t)$ of (1.1) which satisfy

$$\sup\{x(t) : t \ge T\} > 0 \quad \text{for any} \quad T \ge t_0,$$

and we tacitly assume that (1.1) possesses such solutions.

As customary, a solution is said to be oscillatory if it has infinitely many zeros, and otherwise it is called nonoscillatory. Equation itself is termed oscillatory if all its solutions are oscillatory.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various differential equations with linear and nonlinear neutral term, and we refer the reader to [1, 2, 3, 16, 5, 6, 7, 8, 11, 14, 15, 16, 17, 18, 20] and the references cited therein. A commonly employed condition is

$$-1 \le p(t) \le 0$$

as well as the condition

$$-\infty < -p_0 \le p(t) \le 0.$$

There are only few results dealing with the oscillation of differential equations with a nonpositive neutral term. In a pioneering work [19], several oscillation results were obtained for equation (1.1) in the special case $n = 2$ and $\beta = 1$ under the assumptions

$$0 \le p(t) \le p_0 < 1, \quad \tau(t) = t - \tau_0, \quad \sigma(t) = t - \sigma_0.$$

Further contributions for odd-order equations with nonpositive neutral term of type (1.1) and its generalizations were made in [5, 7, 8, 11, 10, 18], where authors established sufficient conditions ensuring that every solution x of (1.1) is either oscillatory or converges to zero as $t \to \infty$. Unfortunately, these results, mostly given for $n = 3$, cannot distinguish solutions with different behaviors and to the best of our knowledge, it seems there is nothing about oscillation of all solutions of (1.1) when n is odd.

In this article, we shall establish new oscillation theorems for all solutions of odd-order nonlinear differential equations with a nonpositive neutral term of type (1.1). When the neutral term is present, all the results are new even for $n = 3$.

2. MAIN RESULTS

In what follows, all functional inequalities are assumed to hold for all t large enough. Without loss of generality, we can deal only with positive solutions of (1.1) since the substitution $z(t) = -x(t)$ transforms (1.1) into an equation of the same form.

First, we state some lemmas which will be useful in the proofs of our main results. Lemma 1 is an adaptation of a well-known Kiguradze lemma (1964), while Lemmas 2 and 3 are due to Philos and Staikos (1981) and Staikos and Stavroulakis (1977), respectively.

Lemma 1 (See [12]). *Let u be a positive and k-times differentiable function on an interval $[t_a, \infty)$ with its k-th derivative $u^{(k)}$ nonpositive on $[t_a, \infty)$ and not identically zero on any subray of $[t_a, \infty)$. Then there exist a $t_b \geq t_a$ and an integer l, $0 \leq l \leq k-1$, with $k + l$ odd so that*

$$\begin{cases} (-1)^{l+j} u^{(j)} > 0 \quad \text{on} \quad [t_b, \infty) \quad (j = l, \dots, k-1), \\ u^{(i)} > 0 \quad \text{on} \quad [t_b, \infty) \quad (i = 1, \dots, l-1), \quad \text{when} \quad l > 1. \end{cases}$$

Lemma 2 (See [15]). *Let u be as in Lemma 1, $t_b \geq t_a$ be assigned to u by Lemma 1 and assume that $\lim_{t\to\infty} u(t) \neq 0$. Moreover, let θ be a number with $0 < \theta < 1$. Then there exists a $t_c \geq t_b/\theta$ such that*

$$u(t) \geq \frac{\theta}{(k-1)!} t^{k-1} u^{(k-1)}(t), \quad \text{for every} \quad t \geq t_c. \tag{2.1}$$

Lemma 3 (See [17]). *Let $u(t)$ be a bounded k-times differentiable function on an interval $[t_a, \infty)$ with*

$$u(t) > 0 \quad (-1)^k u^{(k)}(t) \geq 0 \quad \text{for} \quad t \geq t_a.$$

Then there exists a $t_b \geq t_a$ such that

$$(-1)^i u^{(i)}(t) \geq 0 \quad \text{for every} \quad t \geq t_b, \quad i = 1, 2, \dots, k$$

and

$$u(\xi) \geq \frac{(-1)^{k-1} u^{(k-1)}(\eta)}{(k-1)!} (\eta - \xi)^{k-1} \quad \text{for every} \quad t \geq t_b, \quad t_b \leq \xi \leq \eta. \tag{2.2}$$

Now, we are able to state the following new result, which is based on the comparison principle with first-order delay differential equations.

Theorem 4. *Let conditions (i) – (iv) hold and assume that there exists a nonincreasing function* $g \in \mathcal{C}^1([t_0,\infty),(0,\infty))$ *such that* $\tau(t) \le g(t) \le t$ *and* $\lim_{t\to\infty} g(t) = \infty$. *If the first-order delay differential equations*

$$X'(t) + \left(\frac{c_1}{(n-1)!}\tau^{n-1}(t)\right)^\beta q(t)X^\beta(\tau(t)) = 0, \tag{2.3}$$

$$Y'(t) + \left(\frac{(g(t)-\tau(t))^{n-1}}{(n-1)!}\right)^\beta q(t)Y^\beta(g(t)) = 0, \tag{2.4}$$

and

$$Z'(t) + \left(\frac{c_2}{(n-2)!}h^{n-2}(t)\right)^\beta \left(\int_t^\infty \frac{q(s)}{p^\beta(h(s))}\mathrm{d}s\right) Z^\beta(h(t)) = 0 \tag{2.5}$$

are oscillatory for some c_1, $c_2 \in (0,1)$, *then* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. By Eq. (1.1) and the definition of $y(t)$, we have

$$y^{(n)}(t) = -\,q(t)x^\beta(\tau(t)) \le 0. \tag{2.6}$$

Hence $y^{(n-1)}(t)$ is nonincreasing and of one sign eventually. That is, there exists $t_2 \ge t_1$ such that either $y^{(n-1)}(t) > 0$ or $y^{(n-1)}(t) < 0$ for $t \ge t_2$. We claim that $y^{(n-1)}(t) > 0$ for $t \ge t_2$. To see this, suppose on the contrary that $y^{(n-1)}(t) < 0$ for $t \ge t_2$. Then $\lim_{t\to\infty} y(t) = -\infty$. Since $y(t) > -x(\sigma(t))$, $x(t)$ must be unbounded, and so there exists a sequence $\{T_k\}_{k=0}^\infty$ such that $x(T_k) = \max\{x(s) : T_0 \le s \le T_k\}$ with $\lim_{k\to\infty} T_k = \infty$ and $\lim_{k\to\infty} x(T_k) = \infty$. Furthermore, since $\sigma(T_k) > T_0$ for all k sufficiently large and $\sigma(t) \le t$, we see that

$$x(\sigma(T_k)) \le \max\{x(s) : T_0 \le s \le T_k\} = x(T_k)$$

Therefore, for all large k,

$$y(T_k) = x(T_k) - p(T_k)x(\sigma(T_k)) \ge (1 - p(T_k))x(T_k) > 0,$$

which contradicts the fact that $\lim_{t\to\infty} y(t) = -\infty$. Hence, we have proven the claim.

Now, we have two cases to consider: (I) $y(t) > 0$ or (II) $y(t) < 0$ for $t \ge t_2$.

Case I Assume first that $y(t) > 0$ for $t \ge t_2$. From the definition of $y(t)$, we see that $x(t) \ge y(t)$. Thus, we have

$$y^{(n)}(t) = -\,q(t)y^\beta(\tau(t)) < 0, \quad t \ge t_3 := \tau^{-1}(t_2), \tag{2.7}$$

By Lemma 1, we distinguish the following two cases:

(a) $y(t) > 0, \quad y'(t) > 0, \quad \ldots, \quad y^{(n-2)}(t) > 0, \quad y^{(n-1)}(t) > 0,$

(b) $y(t) > 0, \quad y'(t) < 0, \quad y^{(n-2)}(t) < 0, \quad y^{(n-1)}(t) > 0,$

for $t \geq t_3$.

Suppose that (a) holds. By Lemma 2, there exists $t_4 \in [t_3, \infty)$ such that

$$y(\tau(t)) \geq \frac{c_1}{(n-1)!}\tau^{n-1}(t)y^{(n-1)}(\tau(t)), \tag{2.8}$$

for any $c_1 \in (0,1)$ and $t \geq t_4$. Using (2.8) in (2.7), we get

$$X'(t) + \left(\frac{c_1}{(n-1)!}\tau^{(n-1)}(t)\right)^{\beta} q(t)X^{\beta}(\tau(t)) \leq 0,$$

where we set $X(t) := y^{(n-1)}(t) > 0$. As in [16], it is easy to conclude that there exists a positive solution $X(t)$ of equation (2.3) with $\lim_{t\to\infty} X(t) = 0$, which contradicts the fact that Eq. (2.3) is oscillatory.

Next, we consider (b). By Lemma 3, there exists $t_4 \in [t_3, \infty)$ such that

$$y(\tau(t)) \geq \frac{(g(t)-\tau(t))^{n-1}}{(n-1)!}y^{(n-1)}(g(t)) \tag{2.9}$$

for any $t \geq t_4$, where the function g satisfies the assumptions of the Theorem. Similarly as in the proof of Case I (a), one can show that (2.4) has a positive solution, which is a contradiction.

Case II Suppose that $y(t) < 0$ for $t \geq t_2$. Let $z(t) := -y(t) > 0$ for $t \geq t_2$. Then, by virtue of (1.1) and the definition of $y(t)$, we see that

$$z^{(n)}(t) = q(t)x^{\beta}(\tau(t)) \tag{2.10}$$

and

$$z(t) = -y(t) = p(t)x(\sigma(t)) - x(t) \leq p(t)x(\sigma(t)),$$

i.e.,

$$x(t) \geq \frac{1}{p(\sigma^{-1}(t))}z(\sigma^{-1}(t)), \quad t \geq t_2.$$

In view of (iv), it is obvious that

$$\begin{aligned} x(\tau(t)) &\geq \frac{1}{p(\sigma^{-1}(\tau(t)))}z(\sigma^{-1}(\tau(t))) \\ &= \frac{1}{p(h(t))}z(h(t)), \quad t \geq t_3 := \tau^{-1}(t_2). \end{aligned} \tag{2.11}$$

Using (2.11) in (2.10) yields

$$z^{(n)}(t) \geq \frac{q(t)}{p^{\beta}(h(t))}z^{\beta}(h(t)), \quad t \geq t_3. \tag{2.12}$$

Clearly, $z(t)$ satisfies

$$z(t) > 0, \quad z'(t) > 0, \quad z^{(n-2)}(t) > 0, \quad z^{(n-1)}(t) < 0, \quad z^{(n)}(t) > 0,$$

for $t \geq t_3$. Integrating (2.12) from t to u and letting $u \to \infty$, we get

$$\begin{aligned} -z^{(n-1)}(t) &\geq \int_t^\infty \frac{q(s)}{p^\beta(h(s))} z^\beta(h(s))\mathrm{d}s \\ &\geq z^\beta(h(t)) \int_t^\infty \frac{q(s)}{p^\beta(h(s))}\mathrm{d}s. \end{aligned} \tag{2.13}$$

Now, by Lemma 2, there exists $t_4 \in [t_3, \infty)$ such that

$$z(h(t)) \geq \frac{c_2}{(n-2)!} h^{n-2}(t) z^{(n-2)}(h(t)) \tag{2.14}$$

for every $c_2 \in (0,1)$. Using (2.14) in (2.13) yields

$$-z^{(n-1)}(t) \geq \left(z^{(n-2)}(h(t))\right)^\beta \left(\frac{c_2}{(n-2)!} h^{n-2}(t)\right)^\beta \int_t^\infty \frac{q(s)}{p^\beta(h(s))}\mathrm{d}s \tag{2.15}$$

or

$$Z'(t) + \left(\frac{c_2}{(n-2)!} h^{n-2}(t)\right)^\beta \left(\int_t^\infty \frac{q(s)}{p^\beta(h(s))}\mathrm{d}s\right) Z^\beta(h(t)) \leq 0,$$

where $Z(t) := z^{(n-2)}(t)$. Similarly as in the proof of Case I (a), one can show that (2.11) has a positive solution, which is a contradiction.

The proof is complete. □

Applying known oscillation criteria to first-order delay differential equations (2.3), (2.4) and (2.5), one obtains sufficient conditions for oscillation of (1.1). In particular, using the results reported in [6] and [13], respectively, we arrive at the following propositions.

Corollary 5. *Let conditions (i) – (iv) hold, $\beta = 1$ and assume that there exists a nonincreasing function $g \in \mathcal{C}^1([t_0,\infty),(0,\infty))$ such that $\tau(t) \leq g(t) \leq t$ and $\lim_{t\to\infty} g(t) = \infty$. If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^t \tau^{n-1}(s) q(s)\mathrm{d}s > \frac{(n-1)!}{\mathrm{e}},$$

$$\liminf_{t\to\infty} \int_{g(t)}^t (g(s) - \tau(s))^{n-1} q(s)\mathrm{d}s > \frac{(n-1)!}{\mathrm{e}},$$

and

$$\liminf_{t\to\infty} \int_{h(t)}^t h^{n-2}(s) \int_s^\infty \frac{q(u)}{p(h(u))}\mathrm{d}u\, \mathrm{d}s > \frac{(n-2)!}{\mathrm{e}}$$

then (1.1) *is oscillatory.*

Corollary 6. *Let conditions* $(i)-(iv)$ *hold and* $\beta \in (0,1)$. *Assume that there exists a nonincreasing function* $g \in \mathcal{C}^1([t_0,\infty),(0,\infty))$ *such that* $\tau(t) \le g(t) \le t$ *and* $\lim_{t\to\infty} g(t) = \infty$. *If*

$$\int_{t_0}^{\infty} \tau^{\beta(n-1)}(s)q(s)\mathrm{d}s = \infty,$$

$$\int_{t_0}^{\infty} (g(s)-\tau(s))^{\beta(n-1)}q(s)\mathrm{d}s = \infty,$$

and

$$\int_{t_0}^{\infty} h^{\beta(n-3)}(s)\int_s^{\infty} \frac{q(u)}{p^{\beta}(h(u))}\mathrm{d}u\,\mathrm{d}s = \infty$$

then (1.1) *is oscillatory.*

Below, we present another oscillation result for (1.1).

Theorem 7. *Let conditions* $(i)-(iv)$ *hold and* $\beta \in (0,1]$. *If*

$$\limsup_{t\to\infty}\left(\tau^{\beta(n-1)}(t)\int_t^{\infty} q(s)\mathrm{d}s\right) > \begin{cases} (n-1)! & \text{if} \quad \beta = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.16}$$

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} q(s)(\tau(t)-\tau(s))^{\beta(n-1)}\mathrm{d}s > \begin{cases} (n-1)! & \text{if} \quad \beta = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.17}$$

and

$$\limsup_{t\to\infty}\int_{h(t)}^{t} q(s)\left(\frac{h^{n-2}(s)(h(t)-h(s))}{p(h(s))}\right)^{\beta}\mathrm{d}s > \begin{cases} (n-2)! & \text{if} \quad \beta = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.18}$$

then (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. As in the proof of Theorem 4, we have two cases to consider: (I) $y(t) > 0$ or (II) $y(t) < 0$ for $t \ge t_2$.

Case I Suppose first that $y(t) > 0$ for $t \ge t_2$. As in the proof of Theorem 4, we distinguish two cases:

(a) $y(t) > 0, \quad y'(t) > 0, \quad \dots, \quad y^{(n-2)}(t) > 0, \quad y^{(n-1)}(t) > 0,$

(b) $y(t) > 0, \quad y'(t) < 0, \quad y^{(n-2)}(t) < 0, \quad y^{(n-1)}(t) > 0,$

for $t \ge t_3$, $t_3 \in [t_2,\infty)$.

Suppose that (a) holds. Integrating (2.7) from t to u and letting $u \to \infty$, we have

$$y^{(n-1)}(t) \geq \int_t^\infty q(s)y^\beta(\tau(s))\mathrm{d}s \geq y^\beta(\tau(t)) \int_t^\infty q(s)\mathrm{d}s.$$

Using (2.8) and the monotonicity of $y^{(n-1)}(t)$ in the above inequality, we obtain

$$\begin{aligned} y^{(n-1)}(t) &\geq \left(y^{(n-1)}(\tau(t))\right)^\beta \left(\frac{c_1}{(n-1)!}\tau^{n-1}(t)\right)^\beta \int_t^\infty q(s)\mathrm{d}s \\ &\geq \left(y^{(n-1)}(t)\right)^\beta \left(\frac{c_1}{(n-1)!}\tau^{n-1}(t)\right)^\beta \int_t^\infty q(s)\mathrm{d}s, \quad t \geq t_4, \end{aligned}$$

for every $c_1 \in (0,1)$ and $t_4 \in [t_3, \infty)$. Hence,

$$\left(y^{(n-1)}(t)\right)^{1-\beta} \geq \left(\frac{c_1}{(n-1)!}\tau^{n-1}(t)\right)^\beta \int_t^\infty q(s)\mathrm{d}s.$$

Taking lim sup on both sides of this inequality as $t \to \infty$, we get a contradiction with (2.16).

Next, we consider (b). By Lemma 3, there exists $t_4 \in [t_3, \infty)$ such that

$$y(\tau(s)) \geq \frac{(\tau(t) - \tau(s))^{n-1}}{(n-1)!} y^{(n-1)}(\tau(t)) \tag{2.19}$$

for any $t \geq s \geq t_4$. Integrating (2.7) from $\tau(t)$ to t and using (2.19) in the resulting inequality yields

$$\begin{aligned} y^{(n-1)}(\tau(t)) &\geq \int_{\tau(t)}^t q(s)y^\beta(\tau(s))\mathrm{d}s \\ &\geq \left(\frac{y^{(n-1)}(\tau(t))}{(n-1)!}\right)^\beta \int_{\tau(t)}^t q(s)(\tau(t) - \tau(s))^{\beta(n-1)}\mathrm{d}s, \quad t \geq t_3. \end{aligned}$$

Hence,

$$\left(y^{(n-1)}(\tau(t))\right)^{1-\beta} \geq \frac{1}{((n-1)!)^\beta} \int_{\tau(t)}^t q(s)(\tau(t) - \tau(s))^{\beta(n-1)}\mathrm{d}s,$$

which clearly contradicts to (2.17).

Case II Suppose that $y(t) < 0$ for $t \geq t_2$. As in the proof of Theorem 4, we obtain (2.12) with $z(t) = -y(t)$ for $t \geq t_3$, $t_3 \in [t_2, \infty)$, that is,

$$z^{(n)}(t) \geq \frac{q(t)}{p^\beta(h(t))} z^\beta(h(t)). \tag{2.20}$$

Clearly, $z(t)$ satisfies

$$z(t) > 0, \quad z'(t) > 0, \quad z^{(n-2)}(t) > 0, \quad z^{(n-1)}(t) < 0, \quad z^{(n)}(t) > 0,$$

for $t \geq t_3$. Using (2.14) in (2.20), we get

$$z^{(n)}(t) \geq q(t) \left(\frac{c_2}{(n-2)!} \frac{h^{n-2}(t)}{p(h(t))} \right)^{\beta} \left(z^{(n-2)}(h(t)) \right)^{\beta}$$

for every $c_2 \in (0, 1)$ and $t \geq t_3$. Integrating the above inequality from $h(t)$ to t, we see that

$$\begin{aligned} -z^{(n-1)}(h(t)) &\geq z^{(n-1)}(t) - z^{(n-1)}(h(t)) \\ &= \int_{h(t)}^{t} q(s) \left(\frac{c_2}{(n-2)!} \frac{h^{n-2}(s)}{p(h(s))} \right)^{\beta} \left(z^{(n-2)}(h(s)) \right)^{\beta} \mathrm{d}s. \end{aligned}$$

On the other hand, for $t \geq s \geq t_3$, we have

$$\begin{aligned} z^{(n-2)}(h(s)) &\geq -z^{(n-2)}(h(t)) + z^{(n-2)}(h(s)) \\ &= \int_{h(s)}^{h(t)} -z^{(n-1)}(u) \mathrm{d}u \\ &\geq (h(t) - h(s)) \left(-z^{(n-1)}(h(t)) \right). \end{aligned}$$

Thus,

$$\begin{aligned} &- z^{(n-1)}(h(t)) \\ &\quad \geq \left(-z^{(n-1)}(h(t)) \right)^{\beta} \int_{h(t)}^{t} q(s) \left(\frac{c_2}{(n-2)!} \frac{h^{n-2}(s)(h(t) - h(s))}{p(h(s))} \right)^{\beta} \mathrm{d}s \end{aligned}$$

or

$$\left(-z^{(n-1)}(h(t)) \right)^{1-\beta} \geq \int_{h(t)}^{t} q(s) \left(\frac{\theta}{(n-2)!} \frac{h^{n-2}(s)(h(t) - h(s))}{p(h(s))} \right)^{\beta} \mathrm{d}s,$$

which contradicts (2.18) as $t \to \infty$.

The proof is complete. $\square$

Remark 8. If (1.1) is not of neutral type, i.e., if $p(t) \equiv 0$, then Case II in the proof of Theorem 4 (7) cannot occur.

Theorem 9. *Let conditions* (*i*) – (*iii*) *hold with* $p(t) \equiv 0$. *Assume that there exists a function* $g(t) \in \mathcal{C}^1([t_0, \infty), (0, \infty))$ *such that* $g'(t) \geq 0$, $\tau(t) \leq g(t) \leq t$, *and* $\lim_{t\to\infty} g(t) = \infty$. *If the first order delay differential equations* (2.3) *and* (2.4) *are oscillatory for some* c_1, $c_2 \in (0, 1)$, *then* (1.1) *is oscillatory.*

Theorem 10. *Let conditions* (*i*) – (*iii*) *hold,* $\beta \in (0, 1]$, *and* $p(t) \equiv 0$. *If* (2.16) *and* (2.17) *hold, then* (1.1) *is oscillatory.*

Example 11. Consider the Euler type neutral delay differential equation

$$(x(t) - px(\sigma t))^{(n)} + \frac{a}{t^n} x(\lambda t) = 0, \quad t \geq 1, \tag{2.21}$$

where n is odd, $p \in (0,1)$, $\sigma, \lambda \in (0,1)$ are such that $\lambda/\sigma < 1$.

It is easy to verify that conditions $(i) - (iv)$ are satisfied. Now, let us set $g(t) \simeq gt$ and $k(t) \simeq kt$ for any $g \in (\lambda, 1)$ and $k \in (\lambda/\sigma, 1)$. Then, by Corollary 1, we conclude that (2.21) is oscillatory if

$$\lambda^{n-1} a \ln \frac{1}{\lambda} > \frac{(n-1)!}{\mathrm{e}},$$

$$(g-\lambda)^{n-1} a \ln \frac{1}{g} > \frac{(n-1)!}{\mathrm{e}}$$

and

$$\frac{a}{p}\left(\frac{\lambda}{\sigma}\right)^{n-2} \ln \frac{\sigma}{\lambda} > \frac{(n-1)(n-2)!}{\mathrm{e}}. \tag{2.22}$$

Note that none of the results [5, 7, 8, 11, 10, 18] can ensure that (2.21) is oscillatory.

Remark 12. It would be of interest to study the oscillatory behavior of (1.1) without imposing the assumption (iv). This is left for further research.

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