Dynamic Systems and Applications, **27**, No. 2 (2018), 367-386 ISSN: 1056-2176

LYAPUNOV EXPONENTS FOR DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT: In this paper, we deal with a concept of Lyapunov exponents for functions defined on time scales and study some its basic properties. We also establish the relationship between Lyapunov exponents and the stability of linear dynamic equations. This work can be considered as a unification and generalization of investigations of Lyapunov exponents on continuous and discrete times.

AMS Subject Classification: 37D25, 34N05, 26E70

Key Words: time scales, Lyapunov exponent, Lyapunov spectrum, exponential function, exponential stability, linear dynamic equation

Received: November 30, 2017 ; **Accepted:** February 19, 2018 ;
Published: April 5, 2018 **doi:** 10.12732/dsa.v27i2.10

Dynamic Publishers, Inc., Acad. Publishers, Ltd. https://acadsol.eu/dsa

1. INTRODUCTION AND PRELIMINARIES

In 1988, the theory of analysis on time scales was introduced by Stefan Hilger [11] in his Ph.D thesis in order to unify and extend continuous and discrete calculus. One of the most important problems in analysis on time scales is to consider the stability of dynamic equations. There have been many papers dealing with this topic. However, as far as we know, authors have used only the second Lyapunov method (method of Lyapunov functions) to investigate whether a dynamic equation is stable or not, see [6, 9, 12, 16, 17]. Meanwhile, the first Lyapunov method (method of Lyapunov exponents to know the growth rate of a function) was a quite classical and basic concept for differential and difference equations [14, 16, 17] and it is a strong tool to study the linear systems. But so far there have been no works dealing with the concept of Lyapunov exponents for functions defined on time scales. The main reason for this situation is that the traditional approach to Lyapunov exponents via logarithm function is no longer valid because there is no reasonable definition for logarithm function, which one regards as the inverse of the exponent function on the time scale, even if there were some works trying to approach this notion, see [3].

In this paper, we introduce an approach to the first Lyapunov method for dynamic equations on time scales. Although we can not define the logarithm function on time scales, the idea of comparing the growth rate of a function with exponential functions in the definition of the Lyapunov exponent is still useful on the time scales. Therefore, instead of considering the limit

$$\limsup_{t\to\infty} \frac{1}{t} \ln \frac{|f(t)|}{t},$$

we can study the oscillation of the ratio

$$\frac{|f(t)|}{e_\alpha(t,t_0)} \quad \text{as} \;\; t\to\infty$$

in the parameter α to define the Lyapunov exponent of the function f. Where $e_\alpha(t,t_0)$ is the exponential function with a certain parameter α.

This paper is organized as follows. In Section 1 we give a brief survey on the theory of time scales. Section 2 defines Lyapunov exponent for functions defined on time scales and establishes its fundamental properties. Section 3 deals with the Lyapunov exponents of the solutions of linear dynamic equations. The relation between Lyapunov spectrum and the stability of a linear dynamic equation on time scales is considered in Section 4.

Firstly, we introduce some basic concepts on time scales. A time scale is a nonempty closed subset of the real numbers $\mathbb{R}$, and we usually denote it by $\mathbb{T}$. We assume that a time scale $\mathbb{T}$ has the induced topology from the real numbers with the standard topology. We define the *forward jump operator* and the *backward jump operator* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ (supplemented by $\inf\emptyset = \sup\mathbb{T}$) and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ (supplemented by $\sup\emptyset = \inf\mathbb{T}$). The *graininess* $\mu : \mathbb{T} \to \mathbb{R}^+ \cup \{0\}$ is given by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be *right-dense* if $t < \sup\mathbb{T}$ and $\sigma(t) = t$; *right-scattered* if $\sigma(t) < t$; *left-dense* if $t > \inf\mathbb{T}$ and $\rho(t) = t$; *left-scattered* if $\rho(t) < t$, and *isolated* if t is simultaneuosly right-scattered and left-scattered. For every $a, b \in \mathbb{T}$, by $[a, b]$, we mean the set $\{t \in \mathbb{T} : a \leqslant t \leqslant b\}$. The set $\mathbb{T}^k$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise it is $\mathbb{T}$ without this left-scattered maximum. Now, let f be a function defined on $\mathbb{T}$. We say that f is *delta differentiable* (or simply, *differentiable*) at $t \in \mathbb{T}^k$ provided there exists an α such that for all $\varepsilon > 0$ there is a neighborhood V around t with $|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leqslant \varepsilon|\sigma(t) - s|$ for all $s \in$V. In this case we denote the α by $f^\Delta(t)$, and if f is differentiable for every $t \in \mathbb{T}^k$, then f is said to be *differentiable on* $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$ then delta derivative is $f'(t)$ from continuous calculus; if $\mathbb{T} = \mathbb{Z}$ then the delta derivative is the forward difference, Δf, from discrete calculus. A function f defined on $\mathbb{T}$, valued in a Banach space Y, is *rd-continuous* if it is continuous at every right-dense point and if the left-sided limit exists at every left-dense point. The set of all rd-continuous function from $\mathbb{T}$ to Y is denoted by $\mathrm{C_{rd}}(\mathbb{T}, Y)$. A function $f \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$ is *regressive* (resp. *positively regressive*) if $1 + \mu(t)f(t) \neq 0$ (resp. $1 + \mu(t)f(t) > 0$) for every $t \in \mathbb{T}$. We denote $\mathcal{R}$ (resp. $\mathcal{R}^+$) the set of the regressive functions (resp. positively regressive). A matrix function $A \in \mathrm{C_{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ is regressive (resp. positively regressive) if $\det(I + \mu(t)A(t)) \neq 0$ (resp. $\det(I + \mu(t)A(t)) > 0$) for all $t \in \mathbb{T}$.

Theorem 1 (see [5]). *Let $A(t)$ be a regressive $n \times n$ matrix function. Then, the initial value problem (IVP) $x^\Delta = A(t)x, x(t_0) = x_0$ has a unique solution x defined on $\mathbb{T}$.*

By this theorem, if $A(t)$ is regressive then the IVP $X^\Delta = A(t)X, X(t_0) = I$ has a unique matrix-valued solution, says $\Phi_A(t, t_0)$.

Theorem 2 (see [5]). *Let $A(t)$ be regressive. Then the following statements hold,*

1. *Any solution $x(\cdot)$ of the IVP $x^\Delta = A(t)x$, $x(t_0) = x_0$ can be written as $x(\cdot) = \Phi_A(\cdot, t_0)x_0$.*
2. *The cocycle property is valid $\Phi_A(t, \tau) = \Phi_A(\tau, s)\Phi_A(s, t)$ for all $t, s, \tau \in \mathbb{T}$.*
3. *$\Phi_A(t, t_0)$ is invertible.*

Remark 3. When $A(t)$ is not regressive, the solution of the corresponding matrix-valued IVP $X^\Delta = A(t)X, X(t_0) = I$ $(t \geqslant t_0)$ also exists uniquely but in general it does not exist for $t < t_0$ and $\Phi_A(t, t_0)$ may not invertible (see [10, 19]).

We are concerned with the one dimension case. Let $p : \mathbb{T} \to \mathbb{R}$ be a regressive function. The unique solution of IVP $x^\Delta = p(t)x, x(t_0) = 1$ is called *exponential* function on the time scale $\mathbb{T}$. We denote this function by $e_p(\cdot, t_0)$.

In the following, one lists some fundamental properties of the exponential functions which will be used in this paper.

Theorem 4 (see [1, 5]). *Given $p(\cdot), q(\cdot) \in \mathrm{C_{rd}}\mathcal{R}(\mathbb{T}, \mathbb{C})$, for all $s, t \in \mathbb{T}$, we have*

1. *$e_p(t, t) = 1$, $e_0(t, s) = 1$.*

2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$.

3. $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$, *where* $(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t)$.

4. $\dfrac{e_p(t, s)}{e_q(t, s)} = e_{p\ominus q}(t, s)$, *where* $(p \ominus q)(t) := \frac{p(t)-q(t)}{1+\mu(t)q(t)}$.

5. $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$.

6. *If* $p(\cdot), q(\cdot) \in \mathcal{R}^+$ *and* $p \leqslant q$ *then* $0 < e_p(t, s) \leqslant e_q(t, s)$, *for all* $t \geqslant s$.

Lemma 5 (Gronwall's Inequality (see [1, 18])). *Let* $u, a, b \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$, $b(t) \geqslant 0$ *for all* $t \in \mathbb{T}$. *Then, the inequality*

$$u(t) \leqslant a(t) + \int_{t_0}^{t} b(s)u(s)\,\Delta s \quad \textit{for all } t \geqslant t_0$$

implies

$$u(t) \leqslant a(t) + \int_{t_0}^{t} a(s)b(s)e_b(t, \sigma(s))\,\Delta s \quad \textit{for all } t \geqslant t_0.$$

We refer to [5, 10, 13] for more information on analysis on time scales.

From now on, we fix $t_0 \in \mathbb{T}$ and denote $\mathbb{T}_{t_0} := [t_0, \infty) \cap \mathbb{T}$. For our purpose, we assume that the time scale $\mathbb{T}$ is unbounded above, i.e., $\sup \mathbb{T} = \infty$ and the graininess $\mu(t)$ is bounded on $\mathbb{T}$, that is, $\mu_* = \sup_{t\in\mathbb{T}} \mu(t) < \infty$. This is equivalent to the existence positive numbers m_1, m_2 such that for every $t \in \mathbb{T}$, there exists $c = c(t) \in \mathbb{T}$ satisfying $m_1 \leqslant c - t < m_2$ (also see [19, p. 319]). Furthermore, by definition, if $\alpha \in \mathcal{R}^+ \cap \mathbb{R}$ then $\alpha > -\frac{1}{\mu(t)}$ for all $t \in \mathbb{T}$. As a consequence we have

$$\inf(\mathcal{R}^+ \cap \mathbb{R}) = -\frac{1}{\mu_*}, \text{ supplemented by } \frac{1}{0} = \infty.$$

2. LYAPUNOV EXPONENTS: DEFINITION AND BASIC PROPERTIES

2.1. DEFINITION OF LYAPUNOV EXPONENTS

The idea of comparing a considered function with exponential function motivates us to use $\mathcal{R}^+ \cap \mathbb{R}$ as possible values of Lyapunov exponents. Moreover, similar to the case of analysis on the real line $\mathbb{R}$, Lyapunov exponent of the zero function is $-\infty$, the left extreme exponent, that is infimum of all possible values of Lyapunov exponents. Of course, Lyapunov exponent may be $+\infty$, the right extreme exponent.

Definition 6. Lyapunov exponent of the function f defined on $\mathbb{T}_{t_0}$, valued in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), is a real number $a \in \mathcal{R}^+$ such that for all arbitrary $\varepsilon > 0$

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{a\oplus\varepsilon}(t,t_0)} = 0, \tag{2.1}$$

$$\limsup_{t\to\infty} \frac{|f(t)|}{e_{a\ominus\varepsilon}(t,t_0)} = \infty. \tag{2.2}$$

The Lyapunov exponent of the function f is denoted by $\Upsilon[f]$.

We introduce the concept of extreme exponents. If (2.1) is true for all $a \in \mathcal{R}^+ \cap \mathbb{R}$ then we say by convention that f has *left extreme exponent*, $\Upsilon[f] = -1/\mu_* = \inf(\mathcal{R}^+ \cap \mathbb{R})$. If (2.2) is true for all $a \in \mathcal{R}^+ \cap \mathbb{R}$, we say the function f has *right extreme exponent*, $\Upsilon[f] = +\infty$. If $\Upsilon[f]$ is neither left extreme exponent nor right extreme exponent, then we call $\Upsilon[f]$ by *normal Lyapunov exponent*.

The next lemma shows a necessary and sufficient condition for the existence of the normal Lyapunov exponent.

Lemma 7. *Let $f : \mathbb{T}_{t_0} \to \mathbb{K}$ be a function. Then, f has a normal Lyapunov exponent if and only if there exist two real numbers $\lambda, \gamma \in \mathcal{R}^+$ with $\lambda \neq \inf(\mathcal{R}^+ \cap \mathbb{R})$ such that*

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{\gamma}(t,t_0)} = 0; \quad \limsup_{t\to\infty} \frac{|f(t)|}{e_{\lambda}(t,t_0)} = \infty. \tag{2.3}$$

Proof. Let the Lyapunov exponent $\Upsilon[f]$ of f be normal, this means that $-\frac{1}{\mu^*} < \Upsilon[f] < \infty$. Choose $-\frac{1}{\mu^*} < \lambda < \Upsilon[f] < \gamma < \infty$. Since for small enough $\varepsilon > 0$, $\lambda < \Upsilon[f] \ominus \varepsilon(t) < \Upsilon[f] \oplus \varepsilon(t) < \gamma$ for any $t \in \mathbb{T}$. From (2.1) and (2.2) it follows (2.3).

Suppose that there are λ and γ such that (2.3) takes place. Set

$$\begin{aligned} A &= \left\{\lambda_0 \in \mathcal{R}^+ \cap \mathbb{R} : \tfrac{|f(t)|}{e_{\lambda_0}(t,t_0)} \text{ is unbounded on } \mathbb{T}_{t_0}\right\}, \\ B &= \left\{\lambda_1 \in \mathcal{R}^+ \cap \mathbb{R} : \lim_{t\to\infty} \tfrac{|f(t)|}{e_{\lambda_1}(t,t_0)} = 0\right\}. \end{aligned} \tag{2.4}$$

Since $\lambda \in A$ and $\gamma \in B$, hence $A, B \neq \emptyset$. Furthermore, if $x \in A$ and $y \in B$ then $x \leqslant y$. As a consequence, A is bounded from above, B is bounded from below and $\sup A < \gamma$, $\inf B > \lambda$. It is easily seen that $\sup A = \inf B$ and we denote this common value by a. For every $\varepsilon > 0$, let ε_1 be a positive number satisfying $a \oplus \varepsilon \geqslant a + \varepsilon_1$. By the definition of a,

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{a\oplus\varepsilon}(t,t_0)} \leqslant \lim_{t\to\infty} \frac{|f(t)|}{e_{a+\varepsilon_1}(t,t_0)} = 0,$$

which deduces $\lim_{t\to\infty} \frac{|f(t)|}{e_{a\oplus\varepsilon}(t,t_0)} = 0$. In addition, by setting $\varepsilon_2 = \frac{\inf_t\{1+a\mu(t)\}}{1+\varepsilon \sup_t \mu(t)}$ we have $a \ominus \varepsilon = \frac{a-\varepsilon}{1+\mu(t)\varepsilon} \leqslant a - \varepsilon_2 \in \mathcal{R}^+$. Therefore, $\frac{|f(t)|}{e_{a\ominus\varepsilon}(t,t_0)} \geqslant \frac{|f(t)|}{e_{a-\varepsilon_2}(t,t_0)}$ is unbounded from above since $a - \varepsilon_2 \in A$. Thus a satisfies Definition 6.

For the uniqueness, let b be a real number satisfying (2.1) and (2.2). We show that $a = b$. Suppose on the contrary that $a < b$. By choosing $\varepsilon > 0$ satisfying $\mu_*(1+\mu_*|a|)\varepsilon^2 + 2(1+\mu_*|a|)\varepsilon + (a-b) \leqslant 0$ we see that $a \oplus \varepsilon \leqslant b \ominus \varepsilon$.

Hence, $e_{a\oplus\varepsilon}(t,t_0) \leqslant e_{b\ominus\varepsilon}(t,t_0)$ which implies $\frac{|f(t)|}{e_{a\oplus\varepsilon}(t,t_0)} \geqslant \frac{|f(t)|}{e_{b\ominus\varepsilon}(t,t_0)}$ and we have a contradiction. Lemma is proved. $\square$

Example 8.

1. In case $\mathbb{T} = \mathbb{R}$, the definition 6 leads to the classical one of Lyapunov exponent, i.e.,

$$\Upsilon[f] = \chi[f] = \limsup_{t\to\infty} \frac{\ln|f(t)|}{t}.$$

2. In case $\mathbb{T} = \mathbb{Z}$, it is easy to see that

$$\ln(1+\Upsilon[f]) = \limsup_{n\to\infty} \frac{\ln|f(n)|}{n} = \chi[f].$$

Furthermore, the left extreme exponent is $\inf(\mathcal{R}^+ \cap \mathbb{R}) = -1$.

2.2. SOME FUNDAMENTAL PROPERTIES

Property 9. *Let $f, g : \mathbb{T}_{t_0} \to \mathbb{K}$ be the functions, we have*

1. $\Upsilon[|f|] = \Upsilon[f]$.
2. $\Upsilon[0] = \inf(\mathbb{R}^+ \cap \mathbb{R})$ *(left extreme exponent).*
3. $\Upsilon[cf] = \Upsilon[f]$, *where $c \neq 0$ is a constant.*
4. *If $a \in \mathcal{R}^+ \cap \mathbb{R}$ and (2.1) is satisfied for any $\varepsilon > 0$ then $\Upsilon[f] \leqslant a$. Similarly, if $a \in \mathcal{R}^+ \cap \mathbb{R}$ and (2.2) holds for any $\varepsilon > 0$ then $\Upsilon[f] \geqslant a$.*
5. *If $|f(t)| \leqslant |g(t)|$ for all large enough t then $\Upsilon[f] \leqslant \Upsilon[g]$.*
6. *If f is bounded from above (resp. from below) then $\Upsilon[f] \leqslant 0$ (resp. $\Upsilon[f] \geqslant 0$). As a consequence, if f is bounded then $\Upsilon[f] = 0$.*

Proof. The proof immediately follows from the definition of Lyapunov exponents. $\square$

Property 10. *For any $\lambda \in \mathcal{R} \cap \mathbb{C}$, we have*

1. $\Upsilon[e_\lambda(\cdot,t_0)] = \Upsilon[e_{\widehat{\Re}\lambda}(\cdot,t_0)]$.
2. $\Upsilon[e_\lambda(\cdot,t_0)]$ *does not depend on t_0.*
3. *If $q(\cdot) \in \mathcal{R}^+$ then*

$$\Upsilon[e_q(\cdot,t_0)] \leqslant \limsup_{t\to\infty} q(t). \tag{2.5}$$

4.

$$\Upsilon[e_\lambda(\cdot,t_0)]\leqslant \limsup_{t\to\infty}\widehat{\Re}\lambda(t)\leqslant|\lambda|. \tag{2.6}$$

5.

$$\Re\lambda\leqslant \liminf_{t\to\infty}\widehat{\Re}\lambda(t)\leqslant\Upsilon[e_\lambda(\cdot,t_0)]. \tag{2.7}$$

Proof. Since

$$\widehat{\Re}\lambda(t)=\lim_{s\searrow\mu(t)}\frac{|1+s\lambda|-1}{s}=\begin{cases}\Re\lambda & \text{if } \mu(t)=0\\ \dfrac{|1+\mu(t)\lambda|-1}{\mu(t)} & \text{if } \mu(t)\neq 0,\end{cases}$$

it follows that

$$\begin{aligned}&\Re\lambda\leqslant\widehat{\Re}\lambda(t)\leqslant|\lambda| \ \ \forall\, t\in\mathbb{T}\\ \Longrightarrow&\Re\lambda\leqslant\liminf_{t\to\infty}\widehat{\Re}\lambda(t)\leqslant\limsup_{t\to\infty}\widehat{\Re}\lambda(t)\leqslant|\lambda|.\end{aligned} \tag{2.8}$$

1. It is known that $|e_\lambda(t,t_0)|=e_{\widehat{\Re}\lambda}(t,t_0)$ (see [10], Theorem 7.4). Thus, $\Upsilon[e_\lambda(\cdot,t_0)]=\Upsilon[e_{\widehat{\Re}\lambda}(\cdot,t_0)]$.
2. For $t_1>t_0$, we have $e_\lambda(t,t_0)=e_\lambda(t,t_1)e_\lambda(t_1,t_0)$. Furthermore, since $\lambda\in\mathcal{R}\cap\mathbb{C}$, $e_\lambda(t_1,t_0)\neq 0$. Therefore, by Property 1.3 we see that $\Upsilon[e_\lambda(\cdot,t_0)]=\Upsilon[e_\lambda(\cdot,t_1)]$.
3. Set $\alpha=\limsup_{t\to\infty}q(t)=\lim_{T\to\infty}\sup_{t\geqslant T}q(t)$. For any $\varepsilon>0$, we can find $T_0>t_0$ such that $q(t)\leqslant\alpha+\varepsilon$ for all $t\geqslant T_0$, which implies that $0<e_{q(\cdot)}(t,T_0)\leqslant e_{\alpha+\varepsilon}(t,T_0)$. Hence, $\Upsilon\left[e_{q(\cdot)}(\cdot,T_0)\right]\leqslant\Upsilon\left[e_{\alpha+\varepsilon}(\cdot,T_0)\right]=\alpha+\varepsilon$ by Property 1.5. Since $\varepsilon>0$ is arbitrary, $\Upsilon\left[e_\lambda(\cdot,t_0)\right]\leqslant\alpha$.
4. This property follows from (2.8) and the Properties 2.1; 2.3.
5. Let $\beta=\liminf_{t\to\infty}\widehat{\Re}\lambda(t)=\lim_{T\to\infty}\inf_{t\geqslant T}\widehat{\Re}\lambda(t)$. We see that $\Re\lambda\leqslant\beta$ by (2.8). In case $\beta=-\frac{1}{\mu_*}$ the inequality is trivial since $\Upsilon\left[e_\lambda(\cdot,t_0)\right]\geqslant-\frac{1}{\mu_*}$. We consider the case $\beta>-\frac{1}{\mu_*}$. For any sufficiently small $\varepsilon>0$, we can find $T_0>t_0$ such that $-\frac{1}{\mu_*}<\beta-\varepsilon\leqslant\widehat{\Re}\lambda(t)$ for all $t\geqslant T_0$. Hence, $0<e_{\beta-\varepsilon}(\cdot,T_0)\leqslant e_{\widehat{\Re}\lambda}(\cdot,T_0)$ which follows that $\beta-\varepsilon=\Upsilon\left[e_{\beta-\varepsilon}(\cdot,T_0)\right]\leqslant\Upsilon\left[e_{\widehat{\Re}\lambda}(\cdot,T_0)\right]=\Upsilon\left[e_{\widehat{\Re}\lambda}(\cdot,t_0)\right]=\Upsilon\left[e_\lambda(\cdot,t_0)\right]$. Thus, $\beta\leqslant\Upsilon\left[e_\lambda(\cdot,t_0)\right]$. The proof is complete. □

Corollary 11. *1. If $\lambda\in\mathcal{R}^+\cap\mathbb{R}$ then $\widehat{\Re}\lambda(t)=\lambda$, and hence $\Upsilon[e_\lambda(\cdot,t_0)]=\lambda$.*

2. If $\mathbb{T}=\mathbb{R}$ then $\Upsilon[e_\lambda(\cdot,t_0)]=\chi[e^{\lambda(t-t_0)}]=\Re\lambda$ $(\lambda\in\mathbb{C})$.

3. If $\mathbb{T}$ is a homogeneous time scale, i.e., $\mu(t)\equiv h\neq 0$ then $\Upsilon[e_\lambda(\cdot,t_0)]=\Upsilon[(1+h\lambda)^{t-t_0}]=\frac{|1+h\lambda|-1}{h}$. Especially, if $\mathbb{T}=\mathbb{Z}$ then $\Upsilon[e_\lambda(\cdot,t_0)]=|1+\lambda|-1$.

Property 12. $\Upsilon[f+g]\leqslant\max\{\Upsilon[f],\Upsilon[g]\}$ *and if* $\Upsilon[f]\neq\Upsilon[g]$ *then the equality holds.*

Proof. Set $\alpha = \Upsilon[f]$, $\beta = \Upsilon[g]$ and suppose that $\alpha \leqslant \beta$, $\alpha, \beta \in \mathcal{R}^+ \cap \mathbb{R}$. It is seen that

$$\frac{|(f+g)(t)|}{e_{\beta\oplus\varepsilon}(t,t_0)} \leqslant \frac{|f(t)|}{e_{\beta\oplus\varepsilon}(t,t_0)} + \frac{|g(t)|}{e_{\beta\oplus\varepsilon}(t,t_0)} \overset{t\to\infty}{\longrightarrow} 0,$$

which implies $\Upsilon[f+g] \leqslant \beta$.

Let $\alpha < \beta$. Choosing any small enough $\varepsilon > 0$ to $\alpha \oplus \varepsilon \leqslant \beta \ominus \varepsilon$ gets

$$\begin{aligned}\limsup_{t\to\infty} \frac{|(f+g)(t)|}{e_{\beta\ominus\varepsilon}(t,t_0)} &\geqslant \limsup_{t\to\infty} \left(\frac{|g(t)|}{e_{\beta\ominus\varepsilon}(t,t_0)} - \frac{|f(t)|}{e_{\beta\ominus\varepsilon}(t,t_0)} \right) \\ &\geqslant \limsup_{t\to\infty} \frac{|g(t)|}{e_{\beta\ominus\varepsilon}(t,t_0)} - \limsup_{t\to\infty} \frac{|f(t)|}{e_{\alpha\oplus\varepsilon}(t,t_0)} = \infty.\end{aligned}$$

This means that $\Upsilon[f+g] \geqslant \beta$. The proof is complete. $\square$

Remark 13.

1. If either α or β or both is the left extreme exponent or ∞ then the above inequality is also valid.
2. We always have $\Upsilon[\sum_{i=1}^n c_i f_i] \leqslant \max_{1\leqslant i\leqslant n} \Upsilon[f_i]$, where f_i is continuous on $[t_0,\infty)_{\mathbb{T}}$, $c_i \neq 0$. Moreover, if there exists an index i such that $\Upsilon[f_i] > \Upsilon[f_j]$, $\forall j \neq i$ then $\Upsilon[\sum_{i=1}^n c_i f_i] = \Upsilon[f_i]$.

Because $\alpha, \beta \in \mathcal{R}^+ \cap \mathbb{R}$ does not imply $\alpha + \beta \in \mathcal{R}^+ \cap \mathbb{R}$, we can not expect $\Upsilon[fg] \leqslant \Upsilon[f] + \Upsilon[g]$ as in the case $\mathbb{T} = \mathbb{R}$. However, we have

Property 14. $\Upsilon[fg] \leqslant \Upsilon[e_{\Upsilon[f]\oplus\Upsilon[g]}(\cdot,t_0)]$.

Proof. Denote $\alpha = \Upsilon[f]$ and $\beta = \Upsilon[g]$. For all $\varepsilon > 0$ one has

$$\begin{aligned}\frac{|\,(fg)(t)\,|}{e_{\Upsilon[e_{\alpha\oplus\beta}(\cdot,t_0)]\oplus\varepsilon}(t,t_0)} &= \frac{|f(t)|}{e_{\alpha\oplus\varepsilon_1}(t,t_0)} \times \frac{|g(t)|}{e_{\beta\oplus\varepsilon_2}(t,t_0)} \\ &\quad\times \frac{e_{\alpha\oplus\beta}(t,t_0)e_{\varepsilon_1\oplus\varepsilon_2\oplus\varepsilon_3}(t,t_0)}{e_{\Upsilon[e_{\alpha\oplus\beta}(\cdot,t_0)]\oplus\varepsilon_3}(t,t_0)e_\varepsilon(t,t_0)},\end{aligned}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ are chosen such that $(\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3)(t) \leqslant \varepsilon$ for all $t \in \mathbb{T}_{t_0}$. Since

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{\alpha\oplus\varepsilon_1}(t,t_0)} = 0, \lim_{t\to\infty} \frac{|g(t)|}{e_{\alpha\oplus\varepsilon_2}(t,t_0)} = 0$$

and

$$\lim_{t\to\infty} \frac{e_{\alpha\oplus\beta}(t,t_0)}{e_{\Upsilon[e_{\alpha\oplus\beta}(\cdot,t_0)]\oplus\varepsilon_3}(t,t_0)} = 0,$$

it follows that $\lim_{t\to\infty} \dfrac{|(fg)(t)|}{e_{\Upsilon[e_{\alpha\oplus\beta}(\cdot,t_0)]\oplus\varepsilon}(t,t_0)} = 0$. According to Property 1.4 we have $\Upsilon[fg] \leqslant \Upsilon[e_{\Upsilon[f]\oplus\Upsilon[g]}(\cdot,t_0)]$. The proof is complete. $\square$

Definition 15. The function f is said to have exact Lyapunov exponent (shortly, exact exponent) α if

$$\lim_{t\to\infty}\frac{|f(t)|}{e_{\alpha\oplus\varepsilon}(t,t_0)}=0 \quad\text{and}\quad \lim_{t\to\infty}\frac{|f(t)|}{e_{\alpha\ominus\varepsilon}(t,t_0)}=\infty,$$

for any $\varepsilon>0$.

Property 16. *If at least one of the functions f and g has exact Lyapunov exponent then $\Upsilon[fg]=\Upsilon[e_{\Upsilon[f]\oplus\Upsilon[g]}(\cdot,t_0)]$.*

Proof. Suppose that f has exact exponent. For any $\varepsilon>0$, there is a sequence $t_n\uparrow\infty$ such that $\lim\limits_{n\to\infty}\frac{|g(t_n)|}{e_{\Upsilon[g]\ominus\varepsilon}(t_n,t_0)}=\infty$. Since f has exact exponent, $\lim\limits_{n\to\infty}\frac{|f(t_n)|}{e_{\Upsilon[f]\ominus\varepsilon}(t_n,t_0)}=\infty$. Therefore,

$$\begin{aligned}\limsup_{t\to\infty}\frac{|(fg)(t)|}{e_{\Upsilon[f]\oplus\Upsilon[g]\ominus\varepsilon}(t,t_0)} &\geqslant \lim_{n\to\infty}\frac{|f(t_n)|}{e_{\Upsilon[f]\ominus\varepsilon/2}(t_n,t_0)}\lim_{n\to\infty}\frac{|g(t_n)|}{e_{\Upsilon[g]\ominus\varepsilon/2}(t_n,t_0)}\\ &=\infty,\end{aligned}$$

which implies that $\Upsilon[fg]\geqslant\Upsilon[e_{\Upsilon[f]\oplus\Upsilon[g]}(\cdot,t_0)]$. The proof of Property 5 is complete. $\square$

Remark 17. If both functions f and g have exact exponents then so does the function fg, and $\Upsilon[fg]=\Upsilon[e_{\Upsilon[f]\oplus\Upsilon[g]}(\cdot,t_0)]$. Generally, if all of functions $f_1,f_2,...,f_m$ have exact exponents then

$$\Upsilon[f_1f_2...f_m]=\Upsilon[e_{\Upsilon[f_1]\oplus\Upsilon[f_2]\oplus...\oplus\Upsilon[f_m]}(\cdot,t_0)].$$

Remark 18.

1. In case $\mathbb{T}=\mathbb{R}$, $\Upsilon[fg]\leqslant\Upsilon[e_{\Upsilon[f]\oplus\Upsilon[g]}(\cdot,t_0)]=\Upsilon[f]+\Upsilon[g]$.

2. In case $\mathbb{T}=\mathbb{Z}$, $\Upsilon[fg]\leqslant\Upsilon[e_{\Upsilon[f]\oplus\Upsilon[g]}(\cdot,t_0)]=\Upsilon[f]+\Upsilon[g]+\Upsilon[f]\Upsilon[g]$ (or equivalently $\chi[fg]\leqslant\chi[f]+\chi[g]$).

3. Since $\Upsilon[f]\oplus\Upsilon[g](\cdot)\in\mathcal{R}^+$, by the relation (2.5) we have

$$\begin{aligned}\Upsilon[fg]&\leqslant\limsup_{t\to\infty}\{\Upsilon[f]\oplus\Upsilon[g](t)\}\\ &=\limsup_{t\to\infty}\{(\Upsilon[f]+\Upsilon[g]+\mu(t)\Upsilon[f]\Upsilon[g])\,(t)\}\\ &=\begin{cases}\Upsilon[f]+\Upsilon[g]+\Upsilon[f]\Upsilon[g]\limsup\limits_{t\to\infty}\mu(t) & \text{if } \Upsilon[f]\Upsilon[g]\geqslant 0\\ \Upsilon[f]+\Upsilon[g]+\Upsilon[f]\Upsilon[g]\liminf\limits_{t\to\infty}\mu(t) & \text{if } \Upsilon[f]\Upsilon[g]<0.\end{cases}\end{aligned}$$

2.3. EXPONENTS OF MATRIX FUNCTIONS

The Lyapunov exponent of a matrix function $F(t) = [f_{ij}(t)]_{m\times n}$, where $f_{ij} : \mathbb{T}_{t_0} \to \mathbb{R}$, is defined by $\Upsilon[F] = \max_{i,j} \Upsilon[f_{ij}]$. It is easy to see that $\Upsilon[\|F\|] = \Upsilon[F]$ and $\Upsilon[F]$ has all properties 1, 3 and 4 as the case of Lyapunov exponent of one dimensional functions.

2.4. EXPONENTS OF INTEGRALS

Theorem 19. *Given a continuous function f defined on $\mathbb{T}_{t_0}$. Let*

$$F(t) = \begin{cases} \int_t^\infty f(s)\,\Delta s & \text{if } \Upsilon[f] < 0 \\ \int_{t_0}^t f(s)\,\Delta s & \text{if } \Upsilon[f] \geqslant 0. \end{cases}$$

Then $\Upsilon[F] \leqslant \Upsilon[f]$.

Proof. Set $\lambda = \Upsilon[f]$ and suppose $\lambda \in \mathcal{R}^+$. By definition, for any $\varepsilon_1 > 0$ there exist $C > 0$ and $T_0 > t_0$ such that

$$|f(t)| \leqslant C e_{\lambda\oplus\varepsilon_1}(t, t_0), \quad \forall\ t \geqslant T_0. \tag{2.9}$$

Suppose that $\lambda < 0$. Let $\varepsilon > 0$ and choose $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$ such that, $\lambda \oplus \varepsilon_1 \leqslant \lambda + \varepsilon_2 \leqslant \lambda \oplus \varepsilon \ominus \varepsilon_3$ and $\lambda + \varepsilon_2 < 0$ for all $t > t_0$. For all $t \geqslant T_0$, we have

$$\begin{aligned} |F(t)| &\leqslant C \int_t^\infty e_{\lambda\oplus\varepsilon_1}(s, t_0)\Delta s \\ &\leqslant C \int_t^\infty e_{\lambda+\varepsilon_2}(s, t_0)\,\Delta s = \frac{-C}{\lambda+\varepsilon_2} e_{\lambda+\varepsilon_2}(t, t_0) \\ &\leqslant \frac{-C}{\lambda+\varepsilon_2} e_{\lambda\oplus\varepsilon\ominus\varepsilon_3}(t, t_0) = \frac{-C}{\lambda+\varepsilon_2} \frac{e_{\lambda\oplus\varepsilon}(t, t_0)}{e_{\varepsilon_3}(t, t_0)}. \end{aligned}$$

Hence,

$$\frac{|F(t)|}{e_{\lambda\oplus\varepsilon}(t, t_0)} \leqslant \frac{-C}{\lambda+\varepsilon_2} \frac{1}{e_{\varepsilon_3}(t, t_0)} \overset{t\to\infty}{\longrightarrow} 0 \text{ (for all } \varepsilon > 0).$$

Using Property 1.4 gets $\Upsilon[F] \leqslant \lambda$.

The case $\lambda \geqslant 0$ can be proved by a similar way. If $\lambda = \Upsilon[f]$ is the left extreme exponent or ∞, then we also have $\Upsilon[F] \leqslant \Upsilon[f]$. The theorem is proved. □

3. LYAPUNOV EXPONENTS OF THE SOLUTIONS OF LINEAR EQUATIONS

3.1. LYAPUNOV SPECTRUM OF A LINEAR EQUATION

Consider the linear equation

$$x^{\Delta} = A(t)x, \tag{3.1}$$

where $A(t)$ is a regressive and rd-continuous $n \times n$-matrix. It is known that the equation (3.1) with the initial value $x(t_0) = x_0$ has a unique solution $x(t) = x(t; t_0, x_0)$ on $\mathbb{T}$.

Theorem 20. *Let $\mathcal{M} = \limsup_{t\to\infty} \|A(t)\|$. If $x(\cdot)$ is a nontrivial solution of the equation (3.1), then $\Upsilon[x(\cdot)] \leqslant \mathcal{M}$. Furthermore, if $\limsup_{t\to\infty} \mu(t) < \frac{1}{\mathcal{M}}$, then one has the appreciation $-\mathcal{M} \leqslant \Upsilon[x(\cdot)] \leqslant \mathcal{M}$.*

Proof. The first assertion can be proved by a similar way as in the continuous case [8, Chapter III, Section 3] by using Gronwall's Lemma.

We prove the second one. Let $T_1 > T_0$ satisfy $\mu(t) < \frac{1}{\mathcal{M}}$, for all $t \geqslant T_1$. It is easy to see that $\Phi_A^{-1}(t, T_1)$ satisfies the adjoint dynamic equation

$$\begin{aligned}[\Phi_A^{-1}(t, T_1)]^{\Delta} &= -\Phi_A^{-1}(\sigma(t), T_1)A(t)\\ &= -\Phi_A^{-1}(t, T_1)(I + \mu(t)A(t))^{-1}A(t).\end{aligned}$$

Therefore,

$$\Phi_A^{-1}(t, T_1) = I - \int_{T_1}^{t} \Phi_A^{-1}(s, T_1)[I + \mu(s)A(s)]^{-1}A(s)\,\Delta s.$$

Hence,

$$\|\Phi_A^{-1}(t, T_1)\| \leqslant 1 + \int_{T_1}^{t} \|(I + \mu(s)A(s))^{-1}\|\|A(s)\|\|\Phi_A^{-1}(s, T_1)\|\,\Delta s.$$

Using Gronwall's Lemma gets

$$\|\Phi_A^{-1}(s, T_1)\| \leqslant e_{\|(I+\mu(t)A(t))^{-1}\|\|A(t)\|}(t, T_1),$$

which implies that $\|\Phi_A^{-1}(s, T_1)\|^{-1} \geqslant e_{\ominus\|(I+\mu(t)A(t))^{-1}\|\|A(t)\|}(t, T_1)$.

Since $\mu(t)\|A(t)\| < 1$, by Hille-Yosida Theorem we have

$$\|(I + \mu(t)A(t))^{-1}\| \leqslant \frac{1}{1 - \mu(t)\|A(t)\|}.$$

This deduces $\ominus\|(I + \mu(t)A(t))^{-1}\|\|A(t)\| \geqslant -\|A(t)\| \geqslant -\mathcal{M} \in \mathcal{R}^+ \cap \mathbb{R}$, for all $t \geqslant T_1$.

Furthermore,

$$\frac{\|x(t)\|}{\|x(T_1)\|} \geqslant \|\Phi_A^{-1}(s,T_1)\|^{-1}$$
$$\geqslant (e_{\ominus\|(I+\mu(t)A(t))^{-1}\|\|A(t)\|}(t,T_1) \geqslant e_{-\mathcal{M}}(t,T_1).$$

Thus, $\Upsilon[x(\cdot)] \geqslant \Upsilon[e_{-\mathcal{M}}(t,T_1)] = -\mathcal{M}$. The proof is complete. □

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) \equiv 0$ we find again a popular inequality

$$-\mathcal{M} \leqslant \Upsilon[x(\cdot)] = \chi[x(\cdot)] \leqslant \mathcal{M}.$$

The set of all finite Lyapunov exponents of the solutions to the equation (3.1) is called Lyapunov spectrum of this equation.

Theorem 21. *The Lyapunov spectrum of the equation* (3.1) *has n distinct values at most.*

Proof. The argue is similar to the proof of Theorem 2.1 in [2]. □

3.2. LYAPUNOV'S INEQUALITY

Denote by $X(t,t_0)$ the fundamental solution matrix of the equation (3.1) satisfying $X(t_0,t_0) = X_0 \in \mathbb{R}^{n\times n}$ and $W(t,t_0) = \det(X(t,t_0))$. We see that W is the solution of the equation $W^\Delta = \alpha(t)W$ (see [13]), where $\alpha(t)$ is defined by

$$\alpha(t) = \lim_{s\searrow\mu(t)} \frac{\det(I+sA(t))-1}{s} = \begin{cases} \operatorname{trace} A(t) & \text{if } \mu(t)=0 \\ \frac{\det(I+\mu(t)A(t))-1}{\mu(t)} & \text{if } \mu(t)\neq 0. \end{cases}$$

Since $A(\cdot) \in \mathrm{C_{rd}}\mathcal{R}(\mathbb{T},\mathbb{K}^{\mathrm{n\times n}})$ and $\mu(t) = \sigma(t) - t$ is rd-continuous, $\alpha(\cdot) \in \mathrm{C_{rd}}\mathcal{R}(\mathbb{T},\mathbb{C})$. Therefore, the equation $W^\Delta = \alpha(t)W$ with the initial condition $W(t_0,t_0) = \det(X_0)$ has a unique solution $W(t,t_0) = \det(X_0)e_\alpha(t,t_0)$.

Let $\{x_1(t), x_2(t), ..., x_n(t)\}$ be a system of regular fundamental solutions of the equation (3.1), i.e., a system of fundamental solutions has the property that Lyapunov exponent of some solution combined from arbitrary solutions of this system will be equal to the Lyapunov exponent of a solution attending in the combination. In other words, if $x(t) = k_1x_1(t)+k_2x_2(t)+\cdots+k_nx_n(t)$ then $\Upsilon[x(\cdot)] = \Upsilon[x_i(\cdot)]$ with some i (by the finiteness of the set of Lyapunov spectrum, it is easy to find such a fundamental solution system).

Denote by $S = \{\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n\}$ the set of Lyapunov spectrum of (3.1). In adddition, suppose that $\alpha_i \in \mathcal{R}^+ \cap \mathbb{R}$, for all $i = 1, 2, ..., n$. .

Theorem 22 (Lyapunov's inequality)**.**

$$\Upsilon[e_\alpha(\cdot,t_0)] \leqslant \Upsilon[e_{\alpha_1\oplus\alpha_2\oplus\ldots\oplus\alpha_n}(\cdot,t_0)].$$

Proof. By definition

$$W = \sum_{\sigma \in \Theta} \operatorname{sgn}(\sigma)\, x_{\sigma(1)1} \dots x_{\sigma(n)n},$$

where $x_i = (x_{1i}, x_{2i}, \dots, x_{ni})^T$ and Θ is the set of all permutations of n elements $1, 2, \dots, n$. Therefore,

$$\begin{aligned}
\Upsilon[W] &\leqslant \max_{\sigma \in \Theta} \Upsilon\left[x_{\sigma(1)1} \dots x_{\sigma(n)n}\right] \\
&\leqslant \max_{\sigma \in \Theta} \Upsilon\left[e_{\Upsilon[x_{\sigma(1)1}] \oplus \dots \oplus \Upsilon[x_{\sigma(n)n}]}(\cdot, t_0)\right] \\
&= \max_{\sigma \in \Theta} \Upsilon\left[e_{\Upsilon[x_{\sigma(1)1}]}(\cdot, t_0) \cdots e_{\Upsilon[x_{\sigma(n)n}]}(\cdot, t_0)\right] \\
&\leqslant \Upsilon\left[e_{\alpha_1}(\cdot, t_0) \cdots e_{\alpha_n}(\cdot, t_0)\right] \\
&= \Upsilon\left[e_{\alpha_1 \oplus \dots \oplus \alpha_n}(\cdot, t_0)\right].
\end{aligned}$$

Thus we get $\Upsilon[e_\alpha(\cdot, t_0)] \leqslant \Upsilon[e_{\alpha_1 \oplus \dots \oplus \alpha_n}(\cdot, t_0)]$. The proof of the theorem is complete. □

Example 23. In case $\mathbb{T} = \mathbb{R}$ one has

$$\Upsilon[e_\alpha(\cdot, t_0)] = \limsup_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} (\operatorname{trace} A(s))ds$$

and

$$\Upsilon\left[e_{\alpha_1 \oplus \dots \oplus \alpha_n}(\cdot, t_0)\right] = \alpha_1 + \dots + \alpha_n.$$

Thus, we get the Lyapunov's inequality for ordinary differential equations in [15].

Remark 24. The question of equality if

$$\Upsilon[e_\alpha(\cdot, t_0)] = \Upsilon[e_{\oplus_{i=1}^n \alpha_i}(\cdot, t_0)]$$

is still open even if the matrix A is constant? However, the answer will be positive if we have one more condition.

Consider the equation (3.1) with $A(t) \equiv A$ being a constant and regressive $n \times n$-matrix. Let $\lambda_i, i = 1, 2, ..., n$ be the eigenvalues of A. We show that $\alpha(t) = \lambda_1 \oplus \lambda_2 ... \oplus \lambda_n$. Indeed, let

$$\det(A - \lambda I) = (-1)^n \lambda^n + (-1)^{n-1} a_{n-1} \lambda^{n-1} + \dots - a_1 \lambda + a_0.$$

Then, by Viete's Theorem

$$\sum_{i_1 < i_2 < \dots < i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} = a_{n-k}, \quad \text{for all } k = 1, 2, ..., n.$$

Therefore,

$$\begin{aligned}
\alpha(t) &= \lim_{s \searrow \mu(t)} \frac{\det(I + sA) - 1}{s} \\
&= a_0 \mu(t)^{n-1} + a_1 \mu(t)^{n-2} + \dots + a_{n-2} \mu(t) + a_{n-1}.
\end{aligned}$$

On the other hand, by induction

$$\begin{aligned}\lambda_1 \oplus \lambda_2 ... \oplus \lambda_n(t) &= \sum \lambda_i + \sum_{i<j} \lambda_i \lambda_j \mu(t) \\ &+ \sum_{i<j<k} \lambda_i \lambda_j \lambda_k (\mu(t))^2 + \cdots + \lambda_1 \cdots \lambda_n \mu(t)^{n-1} \\ &= a_0 \mu(t)^{n-1} + a_1 \mu(t)^{n-2} + \cdots + a_{n-1} = \alpha(t).\end{aligned}$$

Hence,

$$\lambda_1 \oplus \lambda_2 ... \oplus \lambda_n(t) = \alpha(t), \quad \forall\, t \in \mathbb{T}. \tag{3.2}$$

Theorem 25. *If for any eigenvalue λ_i of the matrix A the function $e_{\lambda_i}(\cdot, t_0)$ has the exact Lyapunov exponent then*

$$\Upsilon[e_\alpha(\cdot, t_0)] = \Upsilon[e_{\alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_n}(\cdot, t_0)],$$

where $\alpha_i = \Upsilon[e_{\lambda_i}(\cdot, t_0)]$.

Proof. From Remark 17 and (3.2) we have

$$\begin{aligned}\Upsilon[e_\alpha(\cdot, t_0)] &= \Upsilon[e_{\oplus_{i=1}^n \lambda_i}(\cdot, t_0)] = \Upsilon[\Pi_{i=1}^n e_{\lambda_i}(\cdot, t_0)] \\ &= \Upsilon[e_{\oplus_{i=1}^n \Upsilon[e_{\lambda_i}(\cdot, t_0)]}(\cdot, t_0)] = \Upsilon[e_{\oplus_{i=1}^n \alpha_i}(\cdot, t_0)].\end{aligned}$$

The proof is complete. □

4. LYAPUNOV SPECTRUM AND THE STABILITY OF AN EQUATION

We also consider the equation

$$x^\Delta = A(t)x, \tag{4.1}$$

where $A(t)$ is a regressive and rd-continuous $n \times n$-matrix and $\|A(t)\| \leqslant \mathcal{M},\ \forall\, t \in \mathbb{T}_\tau$.

Definition 26. The trivial solution $x(t) \equiv 0$ of the equation (4.1) is said to be exponentially asymptotically stable (shortly, the equation (4.1) is exponentially asymptotically stable) if all solutions $x(t)$ of the equation (4.1) with the initial value $x(t_0)$ satisfy the relation

$$\|x(t)\| \leqslant N \|x(t_0)\| e_{-\alpha}(t, t_0), \ \ t \geqslant t_0, t \in \mathbb{T}_\tau,$$

for some positive constants $N = N(t_0)$ and $\alpha > 0$ with $-\alpha \in \mathcal{R}^+$.

If the constant N can be chosen to be independent of t_0 then this solution is called uniformly exponentially asymptotically stable.

Theorem 27. *Consider the equation* (4.1) *with the stated conditions on* $A(\cdot)$. *Then,*

1. *The equation* (4.1) *is exponentially asymptotically stable if and only if there exists a constant* $\alpha > 0$ *with* $-\alpha \in \mathcal{R}^+$ *such that for every* $t_0 \in \mathbb{T}_\tau$, *there is* $N = N(t_0) \geqslant 1$ *to* $\|\Phi_A(t,t_0)\| \leqslant Ne_{-\alpha}(t,t_0)$ *for all* $t \geqslant t_0, t \in \mathbb{T}_\tau$.
2. *The equation* (4.1) *is uniformly exponentially asymptotically stable if and only if there exist constants* $\alpha > 0,\ N \geqslant 1$ *with* $-\alpha \in \mathcal{R}^+$ *such that* $\|\Phi_A(t,t_0)\| \leqslant Ne_{-\alpha}(t,t_0)$ *for all* $t \geqslant t_0, t, t_0 \in \mathbb{T}_\tau$.

Proof. Every solution of the equation (4.1) satisfying the initial condition $x(t_0) = x_0$ can be expressed $x(t) = \Phi_A(t,t_0)x_0$. Combining with the definition of exponential stability we have the proof. □

Theorem 28 (A spectrum condition for exponential stability)**.** *Let* $-\alpha := \max S$, *where* S *is the set of Lyapunov spectrum of the equation* (4.1). *Then, the equation* (4.1) *is exponentially asymptotically stable if and only if* $\alpha > 0$.

Proof. Let $\{x_i(\cdot) = (x_{1i}(\cdot), x_{2i}(\cdot), \dots, x_{ni}(\cdot))^T\}$, $i = 1,2,...,n$ be a system of fundamental solutions of (4.1). By assumption $\Upsilon[x_i(\cdot)] \leqslant -\alpha < 0$ for all $i = 1,2,...,n$, which implies that

$$\lim_{t\to\infty} \frac{\|x_i(t)\|}{e_{-\alpha/2}(t,t_0)} = 0.$$

Therefore, there is $T_0 > 0$ such that

$$\|x_i(t)\| \leqslant e_{-\alpha/2}(t,t_0), \quad \forall\, t \geqslant T_0, \ \ i = 1,2,...,n.$$

Taking $N^* \geqslant 1$ such that $N^* \geqslant \sup_{1\leqslant i\leqslant n, t_0\leqslant t\leqslant T_0} \frac{\|x_i(t)\|}{e_{-\alpha/2}(t,t_0)}$, we obtain

$$\sup_{1\leqslant i\leqslant n, t_0\leqslant t\leqslant T_0} \|x_i(t)\| \leqslant N^* e_{-\alpha/2}(t,t_0), \quad \forall\, t \geqslant t_0.$$

If $x(\cdot)$ is an arbitrary nontrivial solution of (4.1), then there are constants $a_1, a_2, ..., a_n$ such that

$$x(t) = \sum_{i=1}^n a_i x_i(t).$$

Since $\{x_1(t_0), x_2(t_0), ..., x_n(t_0)\}$ forms a basic of $\mathbb{R}^n$ and the norms are equivalent in $\mathbb{R}^n$, there is a constant c, independent of $x(t_0)$, such that $c\|x(t_0)\| \geqslant \sum_{i=1}^n |a_i|$. Hence,

$$\|x(t)\| \leqslant \sum_{i=1}^n |a_i|\|x_i(t)\| \leqslant N^*(\sum_{i=1}^n |a_i|)e_{-\alpha/2}(t,t_0)$$

$$\leqslant N\|x(t_0)\|e_{-\alpha/2}(t,t_0),$$

where $N = cN^*$. This means that the equation (4.1) is exponentially asymptotically stable.

Conversely, suppose that (4.1) is exponentially asymptotically stable. Then, there exist numbers $N \geqslant 1, \alpha > 0, -\alpha \in \mathcal{R}^+$ such that $\|x(t)\| \leqslant Ne_{-\alpha}(t,t_0)$ for any solution $x(t)$ of (4.1). By Property 1.5 we have $\Upsilon[x(\cdot)] \leqslant -\alpha$. This means that $\max S \leqslant -\alpha < 0$. The proof is complete. □

Consider the case, A is a regressive constant matrix

$$x^{\Delta} = Ax. \tag{4.2}$$

Denote the set of all eigenvalues of the matrix A by $\sigma(A)$. From the regressivity of the matrix A, it follows that $\sigma(A) \subset \mathcal{R}$.

Theorem 29. *If the equation (4.2) is exponentially asymptotically stable then $\Upsilon[e_\lambda(\cdot,t_0)] < 0$ for all $\lambda \in \sigma(A)$. Suppose, in addition, that every $\lambda \in \sigma(A)$ is uniformly regressive, i.e, there is a constant $\delta > 0$ such that $|1+\lambda\mu(t)| \geqslant \delta$ for all $t \in \mathbb{T}^k$. Then, the assumption $\Upsilon[e_\lambda(\cdot,t_0)] < 0$ implies that the equation (4.2) is exponentially asymptotically stable.*

Proof. Suppose that the equation (4.2) is exponentially asymptotically stable. Let $\lambda \in \sigma(A)$ and x_0 be its corresponding eigenvector. Since $x(t;t_0,x_0) = e_\lambda(t,t_0)x_0$ is a solution of (4.2),

$$\|x(t;t_0,x_0)\| = |e_\lambda(t,t_0)|\|x_0\| \leqslant N\|x_0\|e_{-\alpha}(t,t_0),$$

where $N \geqslant 1, \alpha > 0, -\alpha \in \mathcal{R}^+$. Hence, $\Upsilon[e_\lambda(\cdot,t_0)] \leqslant -\alpha < 0$. We define the generalized λ-polynomial by

$$p_0^\lambda(t,\tau) = 1 \text{ and } p_k^\lambda(t,\tau) = \int_\tau^t \frac{1}{1+\lambda\mu(s)} p_{k-1}^\lambda(s,\tau)\Delta s.$$

Using this notation, we obtain an explicit representation for the time scale matrix exponential (see [7])

$$\Phi_A(t,t_0) = \sum_{i=1}^{m}\sum_{k=1}^{s_i} R_{ik} p_{k-1}^{\lambda_i}(t,t_0)e_{\lambda_i}(t,t_0), \tag{4.3}$$

where R_{ik} are constants and $\lambda_1, \lambda_2, ..., \lambda_m$ are the distinct eigenvalues of A with the respective multiples $s_1, s_2, ..., s_m$ $(m \leqslant n)$.

Assume every $\lambda \in \sigma(A)$, $\Upsilon[e_\lambda(\cdot,t_0)] < 0$ and λ is uniformly regressive.

Let $\varepsilon > 0$. Using L'Hôpital's rule (see [4]) we have

$$\lim_{t\to\infty} \frac{|p_1^\lambda(t,t_0)|}{e_\varepsilon(t,t_0)} \leqslant \lim_{t\to\infty} \frac{\int_{t_0}^t \frac{1}{|1+\lambda\mu(s)|}\Delta s}{e_\varepsilon(t,t_0)} = \lim_{t\to\infty} \frac{\left(\int_{t_0}^t \frac{1}{|1+\lambda\mu(s)|}\Delta s\right)^\Delta}{(e_\varepsilon(t,t_0))^\Delta}$$
$$= \lim_{t\to\infty} \frac{1}{\varepsilon|1+\lambda\mu(s)|e_\varepsilon(t,t_0)} \leqslant \lim_{t\to\infty} \frac{1}{\varepsilon\delta e_\varepsilon(t,t_0)} = 0.$$

Since ε is arbitrary, it follows from Property 1.4 that $\Upsilon[p_1^\lambda(\cdot,t_0)] \leqslant 0$. By induction we get $\Upsilon[p_k^\lambda(\cdot,t_0)] \leqslant 0, k = 0,1,2,...$ Therefore,

$$\begin{aligned}\Upsilon[p_k^\lambda(t,t_0)e_\lambda(t,t_0)] &\leqslant \Upsilon[e_{\Upsilon[p_k^\lambda(t,t_0)]\oplus\Upsilon[e_\lambda(t,t_0)]}(t,t_0)] \\ &= \Upsilon[e_{\Upsilon[p_k^\lambda(t,t_0)]}(t,t_0)e_{\Upsilon[e_\lambda(t,t_0)]}(t,t_0)] \\ &\leqslant \Upsilon[e_{\Upsilon[e_\lambda(t,t_0)]}(t,t_0)] = \Upsilon[e_\lambda(t,t_0)] \\ &< 0.\end{aligned}$$

Combining this inequality, the expression (4.3) and Theorem 28 follow the proof. □

Corollary 30. *If for any $\lambda \in \sigma(A)$ we have $\Im\lambda \neq 0$ and $\Upsilon[e_\lambda(\cdot,t_0)] < 0$, then the equation* (4.2) *is exponentially asymptotically stable.*

Proof. The proof follows from the fact that if $\Im\lambda \neq 0$ then λ is uniformly regressive. □

Theorem 31. *Suppose that $\limsup_{t\to\infty} \widehat{\Re}\lambda(t) < 0$ for all $\lambda \in \sigma(A)$. Then, the equation* (4.2) *is exponentially asymptotically stable.*

Proof. From the assumption and the inequality (2.6), we see that $\Upsilon[e_\lambda(\cdot,t_0)] < 0$, for all $\lambda \in \sigma(A)$.

Set $\alpha = \limsup_{t\to\infty} \widehat{\Re}\lambda(t) < 0, \lambda \in \sigma(A)$.

Choose $0 < \varepsilon \leqslant -\frac{\alpha}{2}$. Then, there exists $T_0 \in \mathbb{T}$ such that $\sup_{t\geqslant T_0} \widehat{\Re}\lambda(t) \leqslant \alpha + \varepsilon$ which implies $(\widehat{\Re}\lambda \oplus \varepsilon)(t) \leqslant \frac{\alpha}{2} < 0$, for all $t \geqslant T_0$. Hence, $\lim_{t\to\infty} e_{\widehat{\Re}\lambda\oplus\varepsilon}(t,t_0) = 0$. Applying L'Hôpital's rule ([4]) obtains

$$\begin{aligned}\limsup_{t\to\infty} |p_1^\lambda(t,t_0)e_{\lambda\oplus\varepsilon}(t,t_0)| &\leqslant \limsup_{t\to\infty} \int_{t_0}^t \frac{1}{|1+\lambda\mu(s)|}\Delta s \times e_{\widehat{\Re}\lambda\oplus\varepsilon}(t,t_0) \\ &= \lim_{t\to\infty} \frac{\left(\int_{t_0}^t \frac{1}{|1+\lambda\mu(s)|}\Delta s\right)^\Delta}{\left(e_{\ominus(\widehat{\Re}\lambda\oplus\varepsilon)}(t,t_0)\right)^\Delta} = \lim_{t\to\infty} \frac{e_{\widehat{\Re}\lambda\oplus\varepsilon}(t,t_0)}{\ominus(\widehat{\Re}\lambda\oplus\varepsilon)(t) \times |1+\lambda\mu(t)|} \\ &= -\lim_{t\to\infty} \frac{(1+\varepsilon\mu(t))e_{\widehat{\Re}\lambda\oplus\varepsilon}(t,t_0)}{(\widehat{\Re}\lambda\oplus\varepsilon)(t)} = 0.\end{aligned}$$

Therefore, $p_1^\lambda(t,t_0)e_{\lambda\oplus\varepsilon}(t,t_0)$ is upper bounded by certain constant C when t is large enough, which implies that

$$|p_1^\lambda(t,t_0)e_\lambda(t,t_0)| = |p_1^\lambda(t,t_0)e_{\lambda\oplus\varepsilon}(t,t_0)|e_{\ominus\varepsilon}(t,t_0) \leqslant Ce_{\ominus\varepsilon}(t,t_0).$$

Thus,

$$\begin{aligned}\Upsilon[p_1^\lambda(t,t_0)e_\lambda(t,t_0)] &\leqslant \Upsilon[Ce_{\ominus\varepsilon}(t,t_0)] \\ &\leqslant \sup_t(\ominus\varepsilon) = \sup_t -\frac{\varepsilon}{1+\varepsilon\mu(t)} \leqslant -\frac{\varepsilon}{1+\varepsilon\mu_*} < 0.\end{aligned}$$

By induction we can prove that

$$\Upsilon[p_k^\lambda(t,t_0)e_\lambda(t,t_0)] < 0,$$

for all $k = 0, 1, 2, ...$

Using the expression (4.3) and Theorem 28 we can complete the proof. □

Note that if $\lambda(\cdot) \in \mathcal{R}^+$, $\widehat{\Re}\lambda(t) = \lambda(t)$ for all $t \in \mathbb{T}$. Therefore, in the following we have a corollary of Theorem 31.

Corollary 32. *If $\sigma(A) \subset (-\infty, 0) \cap \mathcal{R}^+$ then the equation (4.2) is exponentially asymptotically stable.*

Example 33. Consider the equation $x^\Delta(t) = Ax(t)$ on the time scale

$$\mathbb{T} = \cup_{k=0}^\infty [2k, 2k+1],$$

with

$$A = \frac{1}{24}\begin{pmatrix} -24 & 0 & 48 \\ 1 & -24 & 24 \\ 33 & -72 & -48 \end{pmatrix}.$$

It is clear that

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \cup_{k=0}^\infty [2k, 2k+1), \\ 1 & \text{if } t \in \cup_{k=0}^\infty \{2k+1\}, \end{cases}$$

the left extreme exponent is -1. Further,

$$\sigma(A) = \left\{-2, -1 + \frac{1}{2}i, -1 - \frac{1}{2}i\right\}$$

and all $\lambda \in \sigma(A)$ are uniformly regressive.

i) With $\lambda_1 = -2$ and $t \in [2k, 2k+1]$, we have

$$e_{-2}(t,0) = \exp\int_0^t -2ds \prod_{\tau\in I_{0,t}} (1 - 2\mu(\tau)) \exp\int_\tau^{\sigma(\tau)} 2ds = e^{-2t}(-1)^k e^{2k}.$$

On the other hand,

$$e_{-\frac{1}{2}}(t,0) = \exp\int_0^t -\frac{1}{2}ds \prod_{\tau\in I_{0,t}}\Big(1-\frac{1}{2}\mu(\tau)\Big)\exp\int_\tau^{\sigma(\tau)}\frac{1}{2}ds$$
$$= e^{-\frac{1}{2}t}\frac{1}{2^k}e^{\frac{1}{2}k}, \quad \text{for all} \;\; t\in[2k,2k+1].$$

By comparing these expressions, we see that there exists $c>0$ such that $|e_{-2}(t,0)| \leqslant ce_{-\frac{1}{2}}(t,0)$. Hence

$$\Upsilon[e_{-2}(\cdot,0)] \leqslant \Upsilon[e_{-\frac{1}{2}}(\cdot,0)] = -\frac{1}{2} < 0.$$

ii) When $\lambda_2 = -1+\frac{1}{2}i$, we have

$$\widehat{\Re}\lambda_2(t) = \lim_{s\searrow\mu(t)}\frac{|1+s(-1+\frac{1}{2}i)|-1}{s} = \begin{cases} -1 & \text{if } \mu(t)=0, \\ \frac{1}{\sqrt{2}}-1 & \text{if } \mu(t)=1, \end{cases}$$

thus

$$\Upsilon[e_{\lambda_2}(\cdot,0)] \leqslant \limsup_{t\to\infty}\widehat{\Re}\lambda_2(t) = \frac{1}{\sqrt{2}}-1<0.$$

iii) Similarly, with $\lambda_3 = -1-\frac{1}{2}i$ we also get

$$\Upsilon[e_{\lambda_3}(\cdot,0)] \leqslant \limsup_{t\to\infty}\widehat{\Re}\lambda_3(t) = \frac{1}{\sqrt{2}}-1<0.$$

Therefore, by Theorem 29, the above equation is exponentially asymptotically stable.

Make a note that the equation $x^\Delta(t) = -2x(t)$, $t\in\mathbb{T} = \cup_{k=0}^{\infty}[2k,2k+1]$ is exponentially asymptotically stable, meanwhile $\limsup_{t\to\infty}\widehat{\Re}(-2)(t) = 0$. This indicates that, in general, the inverse of Theorem 31 is not true.

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