UPPER AND LOWER SOLUTIONS METHOD FOR IMPULSIVE DIFFERENTIAL INCLUSIONS WITH NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT: This paper discusses the antiperiodic and nonlinear boundary problem for first-order impulsive differential inclusions. We establish the existence results by using Martelli's fixed point theorem with upper and lower solutions method.

1. INTRODUCTION

In this paper, we consider the first order impulsive differential inclusions with antiperiodic and nonlinear boundary conditions

$$\begin{cases} x'(t) \in F(t, x(t)), \ t \in J', \\ \Delta x(t_k) = I_k(x(t_k)), \ k = 1, \ldots, m, \\ g(x(0), x(T)) = 0, \end{cases} \tag{1.1}$$

where $J' = J \backslash \{t_1, \ldots, t_m\}$, $J = [0, T]$, $T > 0$, $0 < t_1 < t_2 < \cdots < t_m < T$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon)$, $x(t_k^-) = \lim_{\varepsilon \to 0^+} x(t_k - \varepsilon)$, $F : J \times R \to P(R)$ is a multivalued map, $P(R)$ is the family of all nonempty subsets of R, $I_k \in C(R, R)$ $(k = 1, \ldots, m)$, and $g : R^2 \to R$ is a single valued map.

The theory of impulsive differential equations or inclusions has become an active area of investigation due to their applications in the fields of mechanics, electrical

engineering, medicine biology, ecology, and so on, see [5, 6, 12, 18]. Critical point theory plays a major role in discussing the existence of solutions for boundary problem for impulsive differential inclusions, see [9, 10, 11, 16, 17]. There are many other methods such as in [1, 3, 7]. In [4], authors considered periodic boundary conditions $g(x, y) = x - y$, i.e., $x(0) = x(T)$. Those results are applicable in some important cases. However, they are not valid for antiperiodic boundary conditions, for example, $x(0) = -x(T)$, which corresponds to $g(x, y) = x + y$. Note that in this case g is nondecreasing in the second variable, and hence the results are not applicable. To the author's best knowledge, there is no paper discussing such boundary problem for impulsive differential inclusions.

Motivated by the above mentioned works, the aim of this paper is to study the existence of solutions for antiperiodic and nonlinear boundary problem (1.1) by Martelli's fixed point theorem with upper and lower solutions method. The rest of the paper is organized as follows. In Section 2, we introduce briefly some notations and necessary preliminaries. In Section 3, we prove the existence results of solutions for system (1.1).

2. PRELIMINARIES

In the paper, we introduce some notations, definitions and preliminary facts, which are useful for the development of this paper.

Let X be a Banach space, Z be a subset of X. We denote $P(X) = \{Z \subset X \mid Z \neq \emptyset\}$, $P_{cv}(X) = \{Z \subset P(X) \mid Z \text{ is convex}\}$, $P_{cp}(X) = \{Z \subset P(X) \mid Z \text{ is compact}\}$, $P_{cv,cp}(X) = P_{cv}(X) \cap P_{cp}(X)$, and so forth. For example, let $X = R$, we have notations $P(R)$, $P_{cv}(R)$, $P_{cp}(R)$ and $P_{cv,cp}(R)$. Let

$$L^1(J, R) = \{x : J \to R \mid |x| : J \to [0, +\infty) \text{ is Lebesgue integrable}\},$$

then $L^1(J, R)$ is a Banach space with the norm $\|x\|_{L^1} = \int_0^T |x(t)| \, dt$.

$$PC(J, R) = \left\{ x : J \to R \mid x(t) \text{ is continuous everywhere except for some } t_k, \text{ at which} \right.$$

$$\left. x(t_k^-), \ x(t_k^+) \text{ exist and } x(t_k^-) = x(t_k), \ k = 1, \ldots, m \right\},$$

which is a Banach space with the norm $\|x\|_{PC} = \sup\{|x(t)| : t \in J\}$.

$AC(J, R)$ is the space of all absolutely continuous functions $x : J \to R$.

Definition 1. Throughout this paper, the multivalued map $F : J \times R \to P(R)$ is said to be L^1-Carathéodory if

(i) $t \to F(t, x)$ is measurable for each $x \in R$,

(ii) $x \to F(t, x)$ is upper semicontinuous on R for almost all $t \in J$,

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J, [0, +\infty))$ such that

$$\|F(t, x)\|_{P(R)} = \sup\{|v| : v \in F(t, x)\} \leq \varphi_\rho(t), \ \forall \ |x| \leq \rho, \ \text{a.e. } t \in J.$$

Definition 2.   The function $\alpha, \ \beta \in PC(J, R) \cap AC(J', R)$ are said to be related lower and upper solutions for the antiperiodic problem (1.1) if there exist $v_1, \ v_2 \in L^1(J, R)$ such that

$$\left\{ \begin{array}{l} v_1(t) \in F(t, \alpha(t)), a.e. \ t \in J, \\ \alpha'(t) \leq v_1(t), \ \text{a.e. } t \in J', \\ \Delta\alpha(t_k) \leq I_k(\alpha(t_k)), \ k = 1, \ldots, m, \\ g(\alpha(0), \beta(T)) \leq 0, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} v_2(t) \in F(t, \beta(t)), a.e. \ t \in J, \\ \beta'(t) \geq v_2(t), \ \text{a.e. } t \in J', \\ \Delta\beta(t_k) \geq I_k(\beta(t_k)), \ k = 1, \ldots, m, \\ g(\beta(0), \alpha(T)) \geq 0. \end{array} \right.$$

Definition 3.   A function $x \in PC(J, R) \cap AC(J', R)$ is said to be a solution of (1.1) if there exists a function $v \in L^1(J, R)$ such that $v(t) \in F(t, x(t))$ a.e. on J, $x'(t) = v(t)$ a.e. on J', $\Delta x(t_k) = I_k(x(t_k))$, $k = 1, \ldots, m$, and $g(x(0), x(T)) = 0$.

Lemma 4.   *(see [14]) Let X be a Banach space, and $F : J \times X \to P_{cv,cp}(X)$ be a L^1-Carathéodory multivalued map with*

$$S_{F,x} = \{f \in L^1(J, X) | \ f(t) \in F(t, x(t)) \text{ for a.e. } t \in J\} \neq \emptyset,$$

and let $\Gamma : L^1(J, X) \to C(J, X)$ be a linear continuous mapping, then the operator

$$\Gamma \circ S_F : C(J, X) \to P_{cv,cp}(C(J, X)), \ u \mapsto (\Gamma \circ S_F)(x) := \Gamma(S_{F,x})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 5.   *(Martelli's fixed point theorem [15]) Let X be a Banach space and $G : X \to P_{cv,cp}(X)$ be an upper semicontinuous and condensing map. If the set $\Re = \{x \in X : \lambda x \in G(x) \text{ for some } \lambda > 1\}$ is bounded, then G has a fixed point.*

Remark 6.   **(i)** If multivalued map F is completely continuous with nonempty compact values, then F is upper semicontinuous if and only if F has a closed graph (*i.e.*, $x_n \to x^*$, $y_n \to y^*$, $y_n \in F(x_n)$ imply $y^* \in F(x^*)$).

(ii) If multivalued map F is completely continuous, then F is condensing. For general information the reader can see [13].

Let $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, \ldots, m$, $t_{m+1} = T$.

Definition 7. (See [2]) A family of functions S is said to be quasiequicontinuous on J, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in S$, $k = 0, 1, \ldots, m$, then

$$\|x(t_1) - x(t_2)\| < \varepsilon, \ \forall \ t_1, \ t_2 \in J_k \text{ and } |t_1 - t_2| < \delta.$$

Lemma 8. *(Compactness Criterion, see [2]) The set $S \in PC(J, R^n)$ is relatively compact if and only if*
    (i) *S is bounded, i.e., $\|x\| < c$ for each $x \in S$ and some $c > 0$,*
    (ii) *S is quasiequicontinuous on J.*

Definition 9. Let X be a Banach space, a multivalued map F is said to be completely continuous if $F(U)$ is relatively compact for every bounded subset $U \subseteq X$.

3. MAIN RESULT

Theorem 10. *Assume the following conditions hold.*
(H1) *$F : J \times R \to P_{cv,cp}(R)$ is an L^1-Carathéodory multivalued map.*
(H2) *Functions $\alpha, \beta \in PC(J, R) \cap AC(J', R)$ are related lower and upper solutions of problem (1.1) such that $\alpha(t) \leq \beta(t)$, $t \in J$.*
(H3) *$I_k \in C(R, R)$, $k = 1, \ldots, m$.*
(H4) *g is a continuous single-valued map in $(x, y) \in [\alpha(0), \beta(0)] \times [\alpha(T), \beta(T)]$, and nondecreasing in $y \in [\alpha(T), \beta(T)]$.*
Then the system (1.1) has at least one solution x such that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in J$.

Proof. We transform (1.1) into a fixed point problem. Consider the modified problem

$$\begin{cases} x'(t) + x(t) \in F_1(t, x(t)), \ t \in J', \\ \Delta x(t_k) = I_k(\tau(t_k, x(t_k))), \ k = 1, \ldots, m, \\ x(0) = \tau(0, x(0) - g(\tau(0, x), \tau(T, x))), \end{cases} \tag{3.1}$$

where $F_1(t, x) = F(t, \tau(t, x)) + \tau(t, x)$, $\tau : C(J, R) \to C(J, R)$ is defined by

$$\tau(t, x) = \begin{cases} \beta(t), & x(t) > \beta(t), \\ x(t), & \alpha(t) \leq x(t) \leq \beta(t), \\ \alpha(t), & x(t) < \alpha(t). \end{cases}$$

Evidently, if x is a solution of (3.1), $\alpha(t) \leq x(t) \leq \beta(t)$, and $\alpha(0) \leq x(0) - g(\tau(0, x), \tau(T, x)) \leq \beta(0)$, then x is a solution of (1.1).

By straightforward calculation, we have that a solution of (3.1) is a fixed point of the operator $N : PC(J, R) \to P(PC(J, R))$ defined by

$$N(x) = \left\{ h \in PC(J, R) : h(t) = x(0) + \int_0^t [v(s) + \tau(s, x) - x(s)]ds \right.$$

$$\left. + \sum_{0 < t_k < t} I_k(\tau(t_k, x(t_k))), \ v \in \widetilde{S}_{F,x} \right\},$$

where

$$\widetilde{S}_{F,x} = \{v \in S_{F,\tau(t,x)} : v(t) \geq v_1(t) \ a.e. \ on \ A_1, \ v(t) \leq v_2(t) \ a.e. \ on \ A_2\},$$

$$S_{F,\tau(t,x)} = \{v \in L^1(J, R) : v(t) \in F(t, \tau(t, x)) \ for \ a.e. \ t \in J\},$$

$$A_1 = \{t \in J : x(t) < \alpha(t) \leq \beta(t)\}, \ A_2 = \{t \in J : \alpha(t) \leq \beta(t) < x(t)\}.$$

Note that for each $x \in C(J, R)$, $S_{F,x}$ is nonempty (see [14]), so $\widetilde{S}_{F,x}$ is nonempty.

Next we will show that N has a fixed point by applying Lemma 5. The proof will be given in several steps. We first show that N is a completely continuous multivalued map, upper semicontinuous with convex closed values.

Step 1. $N(x)$ is convex for each $x \in PC(J, R)$.

Indeed, if h_1, h_2 belong to $N(x)$, then there exist $\overline{v}_1$, $\overline{v}_2 \in \widetilde{S}_{F,x}$ such that

$$h_i(t) = x(0) + \int_0^t [\overline{v}_i(s) + \tau(s, x) - x(s)]ds + \sum_{0 < t_k < t} I_k(\tau(t_k, x(t_k))),$$

$i = 1, 2$. Let $0 \leq l \leq 1$, then for each $t \in J$, we have

$$[lh_1 + (1 - l)h_2](t) = x(0) + \sum_{0 < t_k < t} I_k(\tau(t_k, x(t_k)))$$

$$+ \int_0^t [l\overline{v}_1(s) + (1 - l)\overline{v}_2(s) + \tau(s, x) - x(s)]ds.$$

Since $\widetilde{S}_{F,x}$ is convex (because F has convex values in (H1)), then $lh_1 + (1 - l)h_2 \in N(x)$, so $N(x)$ is convex.

Step 2. N is completely continuous.

Firstly, we show that N maps bounded sets into bounded sets in $PC(J, R)$. Let q be a positive constant, $B_q = \{x \in PC(J, R) : \|x\|_{PC} < q\}$ be a bounded set, and $x \in B_q$. Then for each $h \in N(x)$, there exists $v \in \widetilde{S}_{F,x}$ such that

$$h(t) = x(0) + \int_0^t [v(s) + \tau(s, x) - x(s)]ds + \sum_{0 < t_k < t} I_k(\tau(t_k, x(t_k))). \tag{3.2}$$

Note the boundary condition of (3.1) and the definition of τ, we have

$$\alpha(0) \leq x(0) \leq \beta(0), \tag{3.3}$$

$$\alpha(t) \leq \tau(t, x) \leq \beta(t). \tag{3.4}$$

Let $\rho_1 = \max(q, \sup_{t \in J} |\alpha(t)|, \sup_{t \in J} |\beta(t)|)$, then $|\tau(t, x)| \leq \rho_1$. From (H1), there exists $\varphi_{\rho_1} \in L^1(J, [0, +\infty))$ such that

$$\sup\{|v| : v \in F(t, \tau(t, x))\} \leq \varphi_{\rho_1}(t). \tag{3.5}$$

If $x \in B_q$, then there exist $c_k > 0$, $k = 1, \ldots, m$ such that

$$|I_k(\tau(t_k, x(t_k)))| \leq c_k,$$

since I_k are continuous in (H3) and (3.4). So, with (3.3) and (3.5), we have

$$|h(t)| \leq |x(0)| + \int_0^T [|v(s)| + |\tau(s, x)| + |x(s)|] ds + \sum_{0 < t_k < t} |I_k(\tau(t_k, x(t_k)))|$$

$$\leq \max(|\alpha(0)|, |\beta(0)|) + \|\varphi_{\rho_1}\|_{L^1} + T(\rho_1 + q) + \sum_{k=1}^m c_k := K,$$

then $\|N(x)\|_{PC} \leq K$.

Secondly, we prove that N maps bounded sets into quasiequicontinuous sets of $PC(J, R)$. Let $u_1, u_2 \in J_k$, $k = 0, 1, \ldots, m$, $u_1 < u_2$, $x \in B_q$ and $h \in N(x)$. Then,

$$|h(u_2) - h(u_1)| \leq \int_{u_1}^{u_2} \varphi_{\rho_1}(s) ds + (u_2 - u_1)(\rho_1 + q) + \sum_{u_1 < t_k < u_2} c_k.$$

As $u_2 \to u_1$, the right-hand side of the above inequality tends to zero. This proves that $N(B_q)$ is quasiequicontinuous. By Lemma 8, N is completely continuous, and therefore a condensing map.

Step 3. N has a closed graph.

Let $x_n \to x^*$, $h_n \in N(x_n)$, and $h_n \to h^*$. We will prove that $h^* \in N(x^*)$. $h_n \in N(x_n)$ means that there exists $v_n \in \widetilde{S}_{F, x_n}$ such that

$$h_n(t) = x_n(0) + \int_0^t [v_n(s) + \tau(s, x_n) - x_n(s)] ds + \sum_{0 < t_k < t} I_k(\tau(t_k, x_n(t_k))).$$

Next we need prove that there exists $v^* \in \widetilde{S}_{F, x^*}$ such that for each $t \in J$,

$$h^*(t) = x^*(0) + \int_0^t [v^*(s) + \tau(s, x^*) - x^*(s)] ds + \sum_{0 < t_k < t} I_k(\tau(t_k, x^*(t_k))).$$

Since $x_n \to x^*$, $h_n \to h^*$, τ and I_k $(k = 1, 2, \ldots, m)$ are continuous, we have

$$\left\| \left[h_n(t) - x_n(0) - \int_0^t [\tau(s, x_n) - x_n(s)] ds - \sum_{0 < t_k < t} I_k(\tau(t_k, x_n(t_k))) - \right. \right.$$

$$\left. \left[h^*(t) - x^*(0) - \int_0^t [\tau(s, x^*) - x^*(s)] ds - \sum_{0 < t_k < t} I_k(\tau(t_k, x^*(t_k))) \right] \right\|_{PC} \tag{3.6}$$

$$\to 0$$

as $n \to \infty$. Consider the linear and continuous operator $\Gamma : L^1(J, R) \to C(J, R)$,

$$v \mapsto \Gamma(v)(t) = \int_0^t v(s)ds.$$

From Lemma 4, $\Gamma \circ \widetilde{S}_F$ is a closed graph operator. Moreover,

$$h_n(t) - x_n(0) - \int_0^t [\tau(s, x_n) - x_n(s)]ds - \sum_{0 < t_k < t} I_k(\tau(t_k, x_n(t_k))) \in \Gamma(\widetilde{S}_{F,x_n}). \quad (3.7)$$

Since $x_n \to x^*$, (3.6) and (3.7), there exists $v^* \in \widetilde{S}_{F,x^*}$ satisfying

$$h^*(t) - x^*(0) - \int_0^t [\tau(s, x^*) - x^*(s)]ds - \sum_{0 < t_k < t} I_k(\tau(t_k, x^*(t_k))) = \int_0^t v^*(s)ds.$$

As a consequence of Steps 1 to 3, N is a completely continuous multivalued map, upper semicontinuous with convex closed values.

Step 4. The set $\Re = \{x \in PC(J, R) : \lambda x \in N(x) \text{ for some } \lambda > 1\}$ is bounded.

Let $x \in \Re$, then $\lambda x \in N(x)$ for some $\lambda > 1$. Thus, for each $t \in J$,

$$x(t) = \lambda^{-1} \left[x(0) + \int_0^t [v(s) + \tau(s, x) - x(s)]ds + \sum_{0 < t_k < t} I_k(\tau(t_k, x(t_k))) \right],$$

for some $v \in \widetilde{S}_{F,x}$. Let $\rho_2 = \max(\sup_{t \in J} |\alpha(t)|, \sup_{t \in J} |\beta(t)|)$, it follows from (3.4) that $|\tau(t, x)| \le \rho_2$. From (H1), there exists $\varphi_{\rho_2} \in L^1(J, [0, +\infty))$ such that

$$\sup\{|v| : v \in F(t, \tau(t, x))\} \le \varphi_{\rho_2}(t). \quad (3.8)$$

Since I_k are continuous in (H3) and (3.4), there exist $c'_k > 0$, $k = 1, \dots, m$ such that $|I_k(\tau(t_k, x(t_k)))| \le c'_k$. So, with (3.3) and (3.8), for each $t \in J$ we have

$$\begin{aligned}
|x(t)| &\le |x(0)| + \int_0^t [|v(s)| + |\tau(s, x)| + |x(s)|]ds \\
&\quad + \sum_{0 < t_k < t} |I_k(\tau(t_k, x(t_k)))| \\
&\le \max(|\alpha(0)|, |\beta(0)|) + \|\varphi_{\rho_2}\|_{L^1} + T\rho_2 \\
&\quad + \int_0^t |x(s)|ds + \sum_{k=1}^m c'_k.
\end{aligned}$$

Set

$$K_0 = \max(|\alpha(0)|, |\beta(0)|) + \|\varphi_{\rho_2}\|_{L^1} + T\rho_2 + \sum_{k=1}^m c'_k.$$

Using Gronwall's lemma (see [8], page 36), for each $t \in J$, we have

$$|x(t)| \le K_0 e^t.$$

So,

$$\|x\|_{PC} \le K_0 e^T.$$

This shows that the set $\Re$ is bounded. As a consequence of Lemma 5, we deduce that N has a fixed point, which is a solution of problem (3.1).

Step 5. The solution x of (3.1) satisfies

$$\alpha(t) \le x(t) \le \beta(t), \ t \in J, \tag{3.9}$$

and

$$\alpha(0) \le x(0) - g(\tau(0, x), \tau(T, x)) \le \beta(0). \tag{3.10}$$

We firstly prove (3.9). Let x be a solution of (3.1), we prove that $x(t) \le \beta(t)$, for all $t \in J$. Suppose that $x - \beta$ attains a positive maximum on J at s_0. As (3.3), we consider the only possible case $s_0 \in (0, T]$. Then there exists $s_1 \in (0, s_0)$ and $s_1 \ne t_k$ $(k = 1, 2, \ldots, m)$ such that

$$0 < x(t) - \beta(t) \le x(s_0) - \beta(s_0), \ t \in [s_1, s_0].$$

So, $\tau(t, x) = \beta(t)$ for $t \in [s_1, s_0]$, and there exists $v \in \widetilde{S}_{F,x}$ with $v(t) \le v_2(t)$ and $v(t) \in F(t, \beta(t))$ such that

$$\beta(s_0) - \beta(s_1) \le x(s_0) - x(s_1) = \int_{s_1}^{s_0} [v(s) + \beta(s) - x(s)]ds$$
$$< \int_{s_1}^{s_0} v(s)ds \le \int_{s_1}^{s_0} v_2(s)ds$$
$$\le \int_{s_1}^{s_0} \beta'(s)ds = \beta(s_0) - \beta(s_1).$$

This is a contradiction. Consequently, $x(t) \le \beta(t)$ for all $t \in J$.

Similarly, we can prove that $\alpha(t) \le x(t)$ on J. This shows that (3.9) holds.

Finally, we prove that the solution x of (3.1) satisfies (3.10). Suppose that

$$\alpha(0) > x(0) - g(\tau(0, x), \tau(T, x)). \tag{3.11}$$

Then with the boundary condition of (3.1) and the definition of τ, we have

$$x(0) = \alpha(0). \tag{3.12}$$

With (3.9) and the definition of τ, we have

$$\tau(0, x) = x(0), \ \tau(T, x) = x(T). \tag{3.13}$$

From (3.11) to (3.13), we get

$$g(\alpha(0), x(T)) = g(\tau(0, x), \tau(T, x)) > 0.$$

Since g is nondecreasing in the second variable in (H4) and $x(T) \leq \beta(T)$, we have

$$g(\alpha(0), \beta(T)) \geq g(\alpha(0), x(T)) > 0,$$

which contradicts $g(\alpha(0), \beta(T)) \leq 0$ in Definition 2. So, we have

$$\alpha(0) \leq x(0) - g(\tau(0, x), \tau(T, x)). \qquad (3.14)$$

Analogously, we can prove that

$$x(0) - g(\tau(0, x), \tau(T, x)) \leq \beta(0). \qquad (3.15)$$

(3.14) and (3.15) show that (3.10) holds.

According to Steps 1 to 5, the solution x of (3.1) is also a solution of (1.1). The proof is complete. $\qquad\square$

Remark 11. If $g(x(0), x(T)) = x(0) + x(T))$ in (1.1), i.e., $x(0) = -x(T)$, which satisfies (H4), then (1.1) becomes an antiperiodic boundary value problem for impulsive differential inclusions.

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