

**THE STRONG LAW OF LARGE NUMBERS FOR MULTIVARIATE
FUNCTIONS OF CONTINUOUS-STATE NONHOMOGENEOUS
MARKOV CHAINS**

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ABSTRACT: In this paper, we are going to study the strong law of large numbers for multivariate functions of continuous-state nonhomogeneous Markov chains. Firstly, we give a primary proof of equivalence of the ergodicities for continuous-state homogeneous Markov chains. Then, we establish some lemmas which are the basis of the main result. Finally, we study the strong law of large numbers for functions of $N + 1$ variables of continuous-state nonhomogeneous Markov chains.

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1. INTRODUCTION

For any $x \in R$, let $p(x, y)$ be a nonnegative Borel measurable function, and

$$\int_R p(x, y)dy = 1.$$

$p(x, y)$ will be called transition probability density. If $p(x_1, y) = p(x_2, y)$ for any $x_1, x_2 \in R$, $p(x, y)$ will be called constant transition density. If $p(x, y)$ is a constant transition density, then $p(x, y) = \pi(y)$ is a probability density.

Let $\{X_n, n \geq 0\}$ be a nonhomogeneous Markov chain taking values in R with the transition kernels $P_n(x, B)$ ($P_n(x, B) = \mathbf{P}(X_{n+1} \in B | X_n = x)$, where $B \in \mathcal{B}(R)$).

Let $p_n(x, y)$ be the transition probability densities. If

$$P_n(x, B) = \int_B p_n(x, y) dy,$$

$\{X_n, n \geq 0\}$ will be called a continuous-state nonhomogeneous Markov chain.

Let

$$p^{(n, n+2)}(x, y) = \int_R p_n(x, z) p_{n+1}(z, y) dz,$$

and

$$p^{(n, n+k)}(x, y) = \int_R p^{(n, n+k-1)}(x, z) p_{n+k-1}(z, y) dz,$$

then $p^{(n, n+k)}(x, y)$ is also a transition probability density. It is easy to see that for any $B \in \mathcal{B}(R)$,

$$\mathbf{P}(X_{n+k} \in B | X_n = x) = \int_B p^{(n, n+k)}(x, y) dy.$$

If the continuous-state Markov chain is homogeneous, $p_n(x, y)$ will be denoted simply by p and $p^{(n, n+k)}$ by $p^{(k)}$, and we have

$$p^{(n+k)}(x, y) = \int_R p^{(n)}(x, z) p^{(k)}(z, y) dz.$$

$p^{(n)}(x, y)$ is called n -step transition probability density of the homogeneous Markov chains.

Let $f(x)$ be a Borel measurable function defined on R , we define the norm $\|\cdot\|$ of $f(x)$ as follows:

$$\|f(x)\| = \int_R |f(x)| dx.$$

There have been some works on strong law of large numbers for nonhomogeneous Markov chains. Liu and Liu [3] have studied the strong law of large numbers for multivariate functions of countable nonhomogeneous Markov chains. Liu and Liu [4] have studied a class of strong law of large numbers for functionals of countable nonhomogeneous Markov chains. Liu and Yang [6] have studied the strong law of large numbers and Shannon-McMillan-Breiman theorem for finite nonhomogeneous Markov chains. Yang [11] has studied the convergence in the Cesàro sense and strong law of large numbers for bivariate functions of countable nonhomogeneous Markov chains under the condition $\lim_{n \rightarrow \infty} (1/n) \sum_{k=1}^n \|P_k - P\| = 0$ where P is periodic strongly ergodic. Yang [12] has also studied strong law of large numbers for countable nonhomogeneous Markov chains under the condition of uniform convergence in the

Cesàro sense which is different from the result of [11]. Dietz and Sethuraman [1] have studied large deviations for a class of finite nonhomogeneous Markov chains.

In this paper, we are going to study the strong law of large numbers for multivariate functions of continuous-state nonhomogeneous Markov chains. Firstly, we give a primary proof of equivalence of the ergodicities for continuous-state homogeneous Markov chains. Then, we establish some lemmas which are the basis of the main result. Finally, we study the strong law of large numbers for multivariate functions of continuous-state nonhomogeneous Markov chains. As corollaries, we give a strong law of large numbers for functions of two variables of continuous-state nonhomogeneous Markov chains.

This paper is organized as follows: In Section 2, we give a primary proof of equivalence of the ergodicities for continuous-state homogeneous Markov chains. In Section 3, we study some lemmas which are the basis of the main result, then we obtain the strong law of large numbers for multivariate functions of continuous-state nonhomogeneous Markov chains.

2. A PRIMARY PROOF OF ERGODICITIES FOR CONTINUOUS-STATE HOMOGENEOUS MARKOV CHAINS

In this section, we introduce the definitions of geometric strongly ergodic, strongly ergodic and weakly ergodic for continuous-state homogeneous Markov chains, then we give a primary proof of equivalence of above three ergodicities for continuous-state homogeneous Markov chains.

Definition 2.1. ([9], Definition 2) Let $p(x, y)$ be the transition probability density. Define Dobrushin coefficient of p as follows:

$$C(p) = \frac{1}{2} \sup_{x, y} \int_R |p(x, z) - p(y, z)| dz. \quad (2.1)$$

Definition 2.2. Let $\{X_n, n \geq 0\}$ be a continuous-state homogeneous Markov chain with the transition probability density $p(x, y)$. Let $p^{(n)}(x, y)$ be the n -step transition probability density, and let $\pi(y)$ be a probability density on R .

(i) If there exist two constants $c > 0$, $0 < r < 1$ and a positive integer n_0 such that as $n \geq n_0$, for any x ,

$$\int_R |p^{(n)}(x, y) - \pi(y)| dy \leq cr^n, \quad (2.2)$$

$\{X_n, n \geq 0\}$ will be called geometric strongly ergodic. We also call that $p(x, y)$ is geometric strongly ergodic with probability density $\pi(y)$. (see [7]).

(ii) If

$$\sup_x \int_R |p^{(n)}(x, y) - \pi(y)| dy \rightarrow 0 (n \rightarrow \infty), \tag{2.3}$$

$\{X_n, n \geq 0\}$ will be called strongly ergodic. We also call that $p(x, y)$ is strongly ergodic with probability density $\pi(y)$. (see [9]).

(iii) $\{X_n, n \geq 0\}$ will be called weakly ergodic if $C(p^{(n)}) \rightarrow 0$ as $n \rightarrow \infty$. (see [9]).

Definition 2.3. Let $\{X_n, n \geq 0\}$ be a continuous-state homogeneous Markov chain with the transition probability density $p(x, y)$. Let $\pi(y)$ be a probability density on R . If

$$\pi(y) = \int_R \pi(x)p(x, y)dx, \tag{2.4}$$

$\pi(y)$ will be called stationary distribution of this Markov chain. We also called $\pi(y)$ is the stationary distribution determined by p .

If $\{X_n, n \geq 0\}$ is strongly ergodic with the probability density $\pi(y)$, it is easy to see that $\pi(y)$ is the stationary distribution of $\{X_n, n \geq 0\}$.

In the remark of Definition 5 of [9], the author of this paper pointed out that $C(p) = 0$ if and only if p is a constant transition density. It is easy to see that this conclusion is not correct. Here we give a revised lemma as follows.

Lemma 2.1. *Let $p(x, y)$ be a transition probability density. $C(p) = 0$ if and only if there exists a constant transition density $\bar{p}(y)$ such that*

$$\sup_x \int_R |p(x, y) - \bar{p}(y)| dy = 0. \tag{2.5}$$

Proof. Sufficiency is obvious. We only need to prove necessity. Let $\bar{p}(y) = p(0, y)$, it is easy to see that $\bar{p}(y)$ is a constant transition density. Since $C(p) = 0$, we have $\frac{1}{2} \sup_{x_1, x_2} \int_R |p(x_1, y) - p(x_2, y)| dy = 0$, and (2.5) holds.

From above lemma, if $C(p) = 0$, we can regard p as a constant transition density or a probability density.

Lemma 2.2. ([9], Lemma 6) *Let $p(x, y)$ be a transition probability density and let $r(x)$ be a Borel function such that $\|r\| < \infty$ and $\int_R r(x)dx = 0$. Let*

$$rp(y) = \int_R r(x)p(x, y)dx, \tag{2.6}$$

then

$$\|rp\| \leq \|r\|C(p). \tag{2.7}$$

Remark. In Lemma 6 of [9], the author has just given this conclusion but not provided a detailed proof. Here we give a completed proof as follows.

Proof. Since $r(x) = r^+(x) - r^-(x)$, $|r(x)| = r^+(x) + r^-(x)$, where $r^+(x) = \max\{r(x), 0\}$, $r^-(x) = \max\{-r(x), 0\}$. Let $\int_R r(x)dx = 0$, we have

$$\int_R r^+(x)dx = \int_R r^-(x)dx, \text{ and } \int_R |r(x)|dx = 2 \int_R r^+(x)dx.$$

It is easy to see that $\int_R(\int_R r(x)p(x, y)dx)dy = 0$. Similarly, we have

$$C(p) = \sup_{x,y} \int_R (p(x, z) - p(y, z))^+ dz.$$

Let

$$E = \{y : \int_R r(x)p(x, y)dx > 0\}.$$

Since $\int_R |r(x)|dx \int_E p(x, y)dy < \infty$, by Fubini theorem, we have

$$\begin{aligned} \|rp\| &= \int_R \left| \int_R r(x)p(x, y)dx \right| dy \\ &= 2 \int_R \left(\int_R r(x)p(x, y)dx \right)^+ dy \\ &= 2 \int_E \left(\int_R r(x)p(x, y)dx \right) dy \\ &= 2 \int_R r(x)dx \int_E p(x, y)dy \\ &= 2 \int_R (r^+(x) - r^-(x))dx \int_E p(x, y)dy \\ &\leq 2 \left(\int_R r^+(x)dx \sup_{x_1} \int_E p(x_1, y)dy - \int_R r^-(x)dx \inf_{x_2} \int_E p(x_2, y)dy \right) \\ &= 2 \int_R r^+(x)dx \sup_{x_1, x_2} \int_E (p(x_1, y) - p(x_2, y))dy \\ &\leq 2 \int_R r^+(x)dx \sup_{x_1, x_2} \int_R (p(x_1, y) - p(x_2, y))^+ dy \\ &= \|r\|C(p). \end{aligned} \tag{2.8}$$

The proof of this lemma is completed.

Corollary 2.1. (Dobrushin inequality) *Let $p(x, y)$ be a transition probability density of a continuous-state homogeneous Markov chain, and let $\rho_1(x)$ and $\rho_2(x)$ be two probability densities. Then*

$$\left\| \int_R \rho_1(x)p(x, *)dx - \int_R \rho_2(x)p(x, *)dx \right\| \leq C(p)\|\rho_1 - \rho_2\|. \tag{2.9}$$

Proof. Let $r(x) = \rho_1(x) - \rho_2(x)$ in Lemma 2.2, then $\int_{\mathbb{R}}(\rho_1(x) - \rho_2(x))dx = 0$. By Lemma 2.2, we have

$$\begin{aligned} \left\| \int_{\mathbb{R}} \rho_1(x)p(x, *)dx - \int_{\mathbb{R}} \rho_2(x)p(x, *)dx \right\| &= \|(\rho_1 - \rho_2)p\| \\ &\leq C(p)\|\rho_1 - \rho_2\|. \end{aligned} \tag{2.10}$$

That is (2.9) holds.

Lemma 2.3. ([8], Lemma 2) *Let p and q be two transition probability densities. Then*

$$C(pq) \leq C(p)C(q). \tag{2.11}$$

Lemma 2.4. *Let $\{X_n, n \geq 0\}$ be a continuous-state homogeneous Markov chain with the transition probability density $p(x, y)$. Then $\{X_n, n \geq 0\}$ is weakly ergodic if and only if there exists a positive integer number n_0 such that $C(p^{(n_0)}) < 1$.*

Proof. The necessity is obvious. We only need to prove sufficiency. Since $p^{(n+1)}(x, y) = \int_{\mathbb{R}} p(x, z)p^{(n)}(z, y)dz$, by (2.11), we have $C(p^{(n+1)}) \leq C(p)C(p^{(n)}) \leq C(p^{(n)})$. Hence $\{C(p^{(n)})\}$ is a nondecreasing sequence. Let $C(p^{(n_0)}) = \delta < 1$, we have $C(p^{(kn_0)}) \leq \delta^k \rightarrow 0$ as $k \rightarrow \infty$. Since $\{C(p^{(kn_0)})\}$ is a subsequence of $\{C(p^{(n)})\}$, hence we have $C(p^{(n)}) \rightarrow 0$.

The following theorem is the main result in this section.

Theorem 2.1. *Let $\{X_n, n \geq 0\}$ be a continuous-state homogeneous Markov chain with the transition probability density $p(x, y)$. The following propositions are equivalent:*

- (i) $\{X_n, n \geq 0\}$ is geometric strongly ergodic;
- (ii) $\{X_n, n \geq 0\}$ is strongly ergodic;
- (iii) $\{X_n, n \geq 0\}$ is weakly ergodic.

Remark. The results of this theorem are known (see [2], [7] and [10, p.384]). But here, we give a primary proof of this theorem.

Proof. (i) \Rightarrow (ii) is obvious. Next we prove (ii) \Rightarrow (iii). Since

$$\begin{aligned} C(p^{(n)}) &= \frac{1}{2} \sup_{x_1, x_2} \int_{\mathbb{R}} |p^{(n)}(x_1, y) - p^{(n)}(x_2, y)|dy \\ &\leq \frac{1}{2} \sup_{x_1} \int_{\mathbb{R}} |p^{(n)}(x_1, y) - \pi(y)|dy + \frac{1}{2} \sup_{x_2} \int_{\mathbb{R}} |p^{(n)}(x_2, y) - \pi(y)|dy, \end{aligned} \tag{2.12}$$

the weak ergodicity follows from the strong ergodicity of $\{X_n, n \geq 0\}$.

Finally, we prove (iii) \Rightarrow (i). Let $x_0 \in R$, by Dobrushin inequality, for any n, k , we have

$$\begin{aligned} \int_R |p^{(n+k)}(x_0, y) - p^{(n+1)}(x_0, y)|dy &= \int_R \left| \int_R (p^{(k)}(x_0, z) - p(x_0, z))p^{(n)}(z, y)dz \right| dy \\ &\leq \| (p^{(k)}(x_0, *) - p(x_0, *)) \| C(p^{(n)}) \\ &\leq 2C(p^{(n)}) \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned} \tag{2.13}$$

Thus there exists a sequence such that

$$\int_R p^{(n_1)}(x_0, y)dy + \sum_{k=1}^{\infty} \int_R |p^{(n_{k+1})}(x_0, y) - p^{(n_k)}(x_0, y)|dy < \infty. \tag{2.14}$$

It follows from (2.14) that

$$p^{(n_1)}(x_0, y) + \sum_{k=1}^{\infty} (p^{(n_{k+1})}(x_0, y) - p^{(n_k)}(x_0, y)) \text{ converges a.e..} \tag{2.15}$$

On convergence points of (2.15), let $\pi(x_0, y) = p^{(n_1)}(x_0, y) + \sum_{k=1}^{\infty} (p^{(n_{k+1})}(x_0, y) - p^{(n_k)}(x_0, y))$, otherwise, let $\pi(x_0, y) = 0$. It is easy to see that

$$\lim_{k \rightarrow \infty} p^{(n_k)}(x_0, y) = \pi(x_0, y) \text{ a.e..} \tag{2.16}$$

By (2.13), (2.16) and Fatou lemma, we have

$$\int_R |p^{(n+1)}(x_0, y) - \pi(x_0, y)|dy \leq 2C(p^{(n)}). \tag{2.17}$$

By (2.17), we conclude that $\pi(x_0, y)$ is a transition probability density and $C(\pi) = 0$, by Lemma 2.1, $\pi(x_0, y)$ is a constant transition density ($\pi(x_0, y) = \pi(y)$). Hence, for any x , we have

$$\int_R |p^{(n+1)}(x, y) - \pi(y)|dy \leq 2C(p^{(n)}). \tag{2.18}$$

Since $C(p^{(n)}) \rightarrow 0$, let $C(p^{(n_0)}) = r < 1$ and $n = ln_0 + k$ ($k = 0, 1, \dots, n_0 - 1$), by (2.18), for any x , when $n \geq n_0$, we have

$$\begin{aligned} \int_R |p^{(n+1)}(x, y) - \pi(y)|dy &\leq 2C(p^{(n)}) \leq 2C(p^{(n_0)}) \\ &\leq 2r^l = 2(r^{\frac{1}{n+1}})^{n+1} \leq 2(r^{\frac{1}{2n_0}})^{n+1} = 2\delta^{n+1}, \end{aligned}$$

where $0 < \delta < 1$. We obtain that $\{X_n, n \geq 0\}$ is geometric strongly ergodic. Thus we have completed the proof of this theorem.

3. STRONG LAW OF LARGE NUMBERS

In this section, we will establish the strong law of large numbers for functions of $N + 1$ variables of continuous-state nonhomogeneous Markov chains. Before proving the main result, we need to prove the following lemmas.

Lemma 3.1. *Let $\{X_n, n \geq 0\}$ be a continuous-state nonhomogeneous Markov chain with the transition probability densities $\{p_n(x, y), n \geq 1\}$. Let $p(x, y)$ be another transition probability density. If*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sup_x \|p_k(x, *) - p(x, *)\| = 0, \tag{3.1}$$

then for any positive integer numbers m and l

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sup_x \|p^{(m+k, m+k+l)}(x, *) - p^{(l)}(x, *)\| = 0. \tag{3.2}$$

Proof. Now

$$\begin{aligned} & \sup_x \|p^{(m+k, m+k+2)}(x, *) - p^{(2)}(x, *)\| \\ &= \sup_x \int_R |p^{(m+k, m+k+2)}(x, y) - p^{(2)}(x, y)| dy \\ &= \sup_x \int_R \left| \int_R p_{m+k}(x, z) p_{m+k+1}(z, y) dz - \int_R p(x, z) p(z, y) dz \right| dy \\ &\leq \sup_x \int_R \left| \int_R p_{m+k}(x, z) p_{m+k+1}(z, y) dz - \int_R p_{m+k}(x, z) p(z, y) dz \right| dy \\ &\quad + \sup_x \int_R \left| \int_R (p_{m+k}(x, z) p(z, y) dz - \int_R p(x, z) p(z, y) dz) \right| dy \\ &\leq \sup_z \|p_{m+k+1}(z, *) - p(z, *)\| + \sup_z \|p_{m+k}(z, *) - p(z, *)\|. \end{aligned} \tag{3.3}$$

It follows from (3.1) that for any m

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sup_x \|p^{(m+k, m+k+2)}(x, *) - p^{(2)}(x, *)\| = 0.$$

By induction, (3.2) holds.

Lemma 3.2. *Let $\{X_n, \mathcal{F}_n, n \geq 0\}$ be a stochastic sequence on the probability space (Ω, \mathcal{F}, P) , and let $\mathcal{F}_{-n} = \{\Omega, \phi\}$, $n \geq 1$. Let $\{\varphi_n, n \geq 1\}$ be a sequence of nonnegative, even functions defined on R such that as $|x|$ increase*

$$\frac{\varphi_n(x)}{|x|} \uparrow, \quad \frac{\varphi_n(x)}{x^2} \downarrow. \tag{3.4}$$

If

$$\sum_{n=1}^{\infty} \frac{E[\varphi_n(X_n)]}{\varphi_n(n)} < \infty, \tag{3.5}$$

then for any $k \geq 1$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n \{X_m - E[X_{m+k} | \mathcal{F}_m]\} = 0 \text{ a.e.} \tag{3.6}$$

Proof. By (3.5) and Corollary 1 of [5], for any $k \geq 1$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n \{X_m - E[X_m | \mathcal{F}_{m-k}]\} = 0 \text{ a.e.} \tag{3.7}$$

By using similar proof of Lemma 3 of [12]. By (3.5), we can obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n+k} E[X_{n+k} | \mathcal{F}_n] = 0 \text{ a.e.}, \tag{3.8}$$

(3.6) follows from (3.7) and (3.8).

The following theorem is the main result in this paper.

Theorem 3.1. *Let $\{X_n, n \geq 0\}$ be a continuous-state nonhomogeneous Markov chain with the transition probability densities $\{p_n(x, y), n \geq 1\}$. Let $\{f_n, n \geq 1\}$ be a sequence of Borel measurable functions defined on $\mathcal{B}(R^{N+1})$, and $\{\varphi_n, n \geq 1\}$ be the same as in Lemma 3.2. Let*

$$g_n(x) = \mathbf{E}[f_n(X_{n-N}, \dots, X_n) | X_{n-N} = x], \tag{3.9}$$

where $\sup_x |g_n(x)| < \infty$. Let $g(x)$ be another Borel function defined on R such that $\sup_x |g(x)| < \infty$, and $p(x, y)$ be another transition probability density. Assume that $p(x, y)$ is strongly ergodic with probability density $\pi(y)$. If

$$\sum_{n=1}^{\infty} \frac{E[\varphi_n(f_n(X_{n-N}, \dots, X_n))]}{\varphi_n(n)} < \infty, \tag{3.10}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n \sup_x \|p_m(x, *) - p(x, *)\| = 0, \tag{3.11}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n \sup_x |g_m(x) - g(x)| = 0, \tag{3.12}$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n f_m(X_{m-N}, \dots, X_m) = \int_R g(x)\pi(x)dx \text{ a.e.} \tag{3.13}$$

Proof. Let X_{-n} be a constant as $n \geq 1$, and let $\mathcal{F}_n = \sigma(X_n, X_{n-1}, \dots)$. Then $\{f_n(X_{n-N}, \dots, X_n), \mathcal{F}_n, n \geq 0\}$ is a stochastic sequence. By (3.10) and Lemma 3.2, we have for any $k \geq 1$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n \{f_m(X_{m-N}, \dots, X_m) - E[f_{m+k+N}(X_{m+k}, \dots, X_{m+k+N}) | \mathcal{F}_m]\} = 0 \text{ a.e.} \quad (3.14)$$

By Markov property and the properties of conditional expectation

$$\begin{aligned} & \frac{1}{n} \sum_{m=1}^n \mathbf{E}[f_{m+k+N}(X_{m+k}, \dots, X_{m+k+N}) | \mathcal{F}_m] \\ &= \frac{1}{n} \sum_{m=1}^n \mathbf{E}[\mathbf{E}[f_{m+k+N}(X_{m+k}, \dots, X_{m+k+N}) | \mathcal{F}_{m+k}] | \mathcal{F}_m] \\ &= \frac{1}{n} \sum_{m=1}^n \mathbf{E}[\mathbf{E}[f_{m+k+N}(X_{m+k}, \dots, X_{m+k+N}) | X_{m+k}] | \mathcal{F}_m] \\ &= \frac{1}{n} \sum_{m=1}^n \mathbf{E}[g_{m+k+N}(X_{m+k}) | X_m] \\ &= \frac{1}{n} \sum_{m=1}^n \int_R g_{m+k+N}(x) p^{(m, m+k)}(X_m, x) dx. \end{aligned} \quad (3.15)$$

By (3.15), we have

$$\begin{aligned} & \left| \frac{1}{n} \sum_{m=1}^n E[f_{m+k+N}(X_{m+k}, \dots, X_{m+k+N}) | \mathcal{F}_m] - \frac{1}{n} \sum_{m=1}^n \int_R g(x) p^{(k)}(X_m, x) dx \right| \\ &= \left| \frac{1}{n} \sum_{m=1}^n \int_R g_{m+k+N}(x) p^{(m, m+k)}(X_m, x) dx - \frac{1}{n} \sum_{m=1}^n \int_R g(x) p^{(k)}(X_m, x) dx \right| \\ &\leq \frac{1}{n} \sum_{m=1}^n \left| \int_R g_{m+k+N}(x) p^{(m, m+k)}(X_m, x) dx - \int_R g(x) p^{(m, m+k)}(X_m, x) dx \right| \\ &\quad + \frac{1}{n} \sum_{m=1}^n \left| \int_R g(x) p^{(m, m+k)}(X_m, x) dx - \int_R g(x) p^{(k)}(X_m, x) dx \right| \\ &= \frac{1}{n} \sum_{m=1}^n \left| \int_R p^{(m, m+k)}(X_m, x) (g_{m+k+N}(x) - g(x)) dx \right| \\ &\quad + \frac{1}{n} \sum_{m=1}^n \left| \int_R g(x) (p^{(m, m+k)}(X_m, x) - p^{(k)}(X_m, x)) dx \right| \\ &\leq \frac{1}{n} \sum_{m=1}^n \sup_x |g_{m+k+N}(x) - g(x)| \end{aligned}$$

$$+ \frac{1}{n} \sup_x |g(x)| \sum_{m=1}^n \sup_x \|p^{(m,m+k)}(x, *) - p^{(k)}(x, *)\|. \tag{3.16}$$

By (3.12), the first term of right side of (3.16) converges to zero as $n \rightarrow \infty$. By (3.11) and Lemma 3.1, the second term of right side of (3.16) also converges to zero as $n \rightarrow \infty$. By (3.14) and (3.16), we have for any $k \geq 1$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n \{f_m(X_{m-N}, \dots, X_m) - \int_R g(x)p^{(k)}(X_m, x)dx\} = 0 \text{ a.e.} \tag{3.17}$$

Now

$$\begin{aligned} & \left| \frac{1}{n} \sum_{m=1}^n \int_R g(x)p^{(k)}(X_m, x)dx - \int_R g(x)\pi(x)dx \right| \\ & \leq \frac{1}{n} \sum_{m=1}^n \int_R |g(x)(p^{(k)}(X_m, x) - \pi(x))|dx \\ & \leq \frac{1}{n} \sum_{m=1}^n \sup_x |g(x)| \int_R |(p^{(k)}(X_m, x) - \pi(x))|dx \\ & \leq \sup_x |g(x)| \sup_y \int_R |p^{(k)}(y, x) - \pi(x)|dx. \end{aligned} \tag{3.18}$$

Since $p(y, x)$ is strongly ergodic with probability density $\pi(x)$, the right side of (3.18) is small provided k is large, equation (3.13) follows from (3.17) and (3.18) directly.

Corollary 3.1. *Let $\{X_n, n \geq 0\}$ be a continuous-state nonhomogeneous Markov chain with the transition probability densities $\{p_n(x, y), n \geq 1\}$. Let $\{f_n, n \geq 1\}$ be a sequence of Borel measurable functions defined on $\mathcal{B}(R^2)$, and $\{\varphi_n, n \geq 1\}$ be the same as in Lemma 3.2. Let*

$$g_n(x) = E[f_n(X_{n-1}, X_n) | X_{n-1} = x],$$

where $\sup_x |g_n(x)| < \infty$. Let $g(x)$ be another Borel function defined on R such that $\sup_x |g(x)| < \infty$, and $p(x, y)$ be another transition probability density. Assume that $p(x, y)$ is strongly ergodic with probability density $\pi(y)$. If (3.11) and (3.12) hold, and

$$\sum_{n=1}^{\infty} \frac{E[\varphi_n(f_n(X_{n-1}, X_n))]}{\varphi_n(n)} < \infty, \tag{3.19}$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n f_m(X_{m-1}, X_m) = \int_R g(x)\pi(x)dx \text{ a.e.} \tag{3.20}$$

Proof. Letting $N = 1$ in Theorem 3.1, this corollary follows.

Corollary 3.2. *Let $\{X_n, n \geq 0\}$ be a continuous-state nonhomogeneous Markov chain with the transition probability densities $\{p_n(x, y), n \geq 1\}$. Let $p(x, y)$ be another transition probability density. Assume that $p(x, y)$ is strongly ergodic with probability density $\pi(y)$. Let $S_n^B(\omega)$ ($B \in \mathcal{B}(R)$) be the times of visiting B of X_0, X_1, \dots, X_{n-1} , that is*

$$S_n^B(\omega) = \sum_{m=0}^{n-1} I_{\{X_m \in B\}}. \tag{3.21}$$

If (3.11) holds, then

$$\lim_{n \rightarrow \infty} \frac{S_n^B(\omega)}{n} = \int_B \pi(y) dy \quad a.e.. \tag{3.22}$$

Proof. Letting $f_n(x, y) = I_B(y)$ in Corollary 3.1, and $\varphi_n = x^2$, obviously (3.10) holds, and

$$g_n(x) = \int_R I_B(y) p_n(x, y) dy = \int_B p_n(x, y) dy.$$

Let

$$g(x) = \int_B p(x, y) dy,$$

by (3.11), we have (3.12) holds. Since $p(x, y)$ is strongly ergodic with probability density $\pi(y)$, then $\pi(y)$ is the stationary distribution determined by p , we have

$$\begin{aligned} \int_R g(x) \pi(x) dx &= \int_R \left(\int_B p(x, y) dy \right) \pi(x) dx \\ &= \int_B \left(\int_R p(x, y) \pi(x) dx \right) dy = \int_B \pi(y) dy. \end{aligned} \tag{3.23}$$

By (3.23) and Corollary 3.1, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n f_m(X_{m-1}, X_m) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n I_{\{X_m \in B\}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} S_n^B(\omega) = \int_R g(x) \pi(x) dx = \int_B \pi(y) dy \quad a.e.. \end{aligned} \tag{3.24}$$

Thus (3.22) holds.

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