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# MEASURE EXPANSIVENESS FOR $C^1$ GENERIC DIFFEOMORPHISMS

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**ABSTRACT:** Let $M$ be a closed smooth Reimannian manifold, and let $f : M \to M$ be a diffeomorphism. In the paper, we show that $C^1$ generically, a diffeomorphism $f$ is measure expansive then it is Axiom A without cycles.

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**Key Words:** expansive, measure expansive, partially hyperbolic, Axiom A

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## 1. INTRODUCTION

Let $M$ be a closed smooth Reimannian manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of $M$ endowed with the $C^1$ topology. Denote by $d$ the distance on $M$ induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle $TM$. Let $f \in \mathrm{Diff}(M)$. We say that $f$ is *expansive* if there is $e > 0$ such that for any $x, y \in M$ if $d(f^i(x), f^i(y)) < e$ for all $i \in \mathbb{Z}$ then $x = y$.

In differentiable dynamical systems, a main goal is the stability theory, and many peoples have studied the topic (see [3, 4, 12, 17, 18]). Among the various properties, expansiveness has been used to investigate the stability theory. For instance, Mañé [17] proved that a diffeomorphism $f$ belongs to the $C^1$-interior of the set of expansive diffeomorphisms coincides with the set of quasi-Anosov diffeomorphisms. Here $f$ is *quasi-Anosov* if for all $v \in TM \setminus \{0\}$, the set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded. Lee [14] proved that a diffeomorphism $f$ belongs to the $C^1$-interior of the set of measure expansive diffeomorphisms coincides with the set of quasi-Anosov diffeomorphisms. Sakai *et al* [20] proved that a diffeomorphism $f$ belongs to the $C^1$-interior of the set of measure expansive diffeomorphisms coincides with the set of quasi-Anosov diffeomorphisms. Sakai [19] roved that a diffeomorphism $f$ belongs to the $C^1$-interior of the set of continuum-wise expansive diffeomorphisms coincides with the set of quasi-Anosov diffeomorphisms.

From now, we can find a general expansivities - N-expansive [15], measure expansive [16], countably expansive [16], continuum-wise expansive [8], etc. We say that $f$ is *N-expansive* if there is $e > 0$ such that for any $x \in M$, the number of elements of the set $\Gamma_e(x) = \{y \in M : d(f^i(x), f^i(y)) < e$ for all $i \in \mathbb{Z}\}$ is less than $N$. We say that $f$ is *countably expansive* if there is $e > 0$ such that for $x \in M$, the number of elements of the set $\Gamma_e(x)$ is countable, where $e$ is an expansive constant for $f$.

Note that if a diffeomorphism $f$ is expansive then $\Gamma_e(x) = \{x\}$ for $x \in M$. Thus if a diffeomorphism $f$ is expansive then $f$ is countably expansive, but the converse is not true (see [16]).

The following notion was introduced by Morales and Sirvent [16]. For a Borel probability measure $\mu$ on $M$, we say that $f$ is $\mu$-expansive if there is $\delta > 0$ such that $\mu(\Gamma_e(x)) = 0$ for all $x \in M$.

**Definition 1.1.** We say that $f$ is *measure expansive* if it is $\mu$-expansive for every non-atomic Borel probability measure $\mu$ on $M$.

Let $\Lambda$ be a closed $f$-invariant set. We say that $\Lambda$ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a $Df$-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \le C\lambda^n \ \text{ and } \ \|D_x f^{-n}|_{E^u_x}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. If $\Lambda = M$ then $f$ is said to be Anosov.

It is well know that if a diffeomorphism $f$ is Anosov then it is quasi-Anosov, but the converse is not true (see [6]). Note that if a diffeomorphism $f$ is Anosov then $f$ is expansive, N-expansive, measure expansive, countably expansive and continuum-wise expansive. In the paper, we consider measure expansivity.

A subset $\mathcal{G} \subset \mathrm{Diff}(M)$ is called *residual* if it contains a countable intersection of open and dense subsets of $\mathrm{Diff}(M)$. A dynamic property is called $C^1$*generic* if it holds in a residual subset of $\mathrm{Diff}(M)$. Arbieto [2] proved that $C^1$ generically, if a diffeomorphism $f$ is expansive then it is Axiom A without cycles. Lee [14] proved that $C^1$ generically, if a diffeomorphism $f$ is N-expansive then it is Axiom A without cycles. Yang and Gan [21] proved that $C^1$ generically, if a homoclinic class is expansive then it is hyperbolic. For measure expansive diffeomorphisms, Koo *et al* [9] proved that $C^1$ generically, if a locally maximal homoclinic class is measure expansive then it is hyperbolic. Lee and Lee [10] proved that if the homoclinic class is $C^1$ robust way then it is hyperbolic which is a general result of [11]. From the result of [9], Lee

[13] studied in a general condition and showed that if a homoclinic class is measure expansive then it is hyperbolic.

For these results, we consider measure expansiveness for $C^1$ generic diffeomorphsims.

**Theorem A** *For $C^1$ generic $f \in \mathrm{Diff}(M)$, if $f$ is measure expansive then it is Axiom A without cycles.*

## 2. PROOF OF THEOREM A

Let $M$ be as before, and let $f \in \mathrm{Diff}(M)$. The following Franks' lemma [5] will play essential roles in our proofs.

**Lemma 2.1.** *Let $\mathcal{U}(f)$ be any given $C^1$-neighborhood of $f$. Then there exist $\epsilon > 0$ and a $C^1$-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of $f$ such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \cdots, x_N\}$, a neighborhood $U$ of $\{x_1, x_2, \ldots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \leq \epsilon$ for all $1 \leq i \leq N$, there exists $\widehat{g} \in \mathcal{U}(f)$ such that $\widehat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \cdots, x_N\} \cup (M \setminus U)$ and $D_{x_i}\widehat{g} = L_i$ for all $1 \leq i \leq N$.*

Denote by $P(f)$ the set of periodic points of $f$ and by $P_h(f)$ the set of hyperbolic periodic points of $f$.

**Lemma 2.2.** *If $f \in \mathrm{Diff}(M)$ has a non-hyperbolic periodic point, then for any neighborhood $\mathcal{U}(f)$ of $f$ and any $\eta > 0$, there are $g \in \mathcal{U}(f)$ and a curve $\gamma$ with the following properties:*

1. *$\gamma$ is $g$ periodic, that is, there is $n \in \mathbb{Z}$ such that $g^n(\gamma) = \gamma$;*
2. *the length of $g^i(\gamma)$ is less than $\eta$, for all $i \in \mathbb{Z}$;*
3. *the two end points of $\gamma$ are hyperbolic periodic points of $g$, and $\gamma$ is normally hyperbolic with respect to $g$.*

**Proof.** Let $\mathcal{U}(f)$ be a $C^1$ neighborhood of $f$. Suppose that $p \in P(f)$ is not a hyperbolic point of $f$. Let $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ be a $C^1$ neighborhood given by Lemma 2.1. Then there is $g \in \mathcal{U}_0(f)$ such that $D_p g^{\pi(p)}$ has an eigenvalue $\lambda$ with $|\lambda| = 1$. For simplicity, we may assume that $g^{\pi(p)}(p) = g(p) = p$. Denote by $E^c_p$ the eigenspace corresponding to $\lambda$. If $\lambda \in \mathbb{R}$ then $\dim E^c_p = 1$ and if $\lambda \in \mathbb{C}$ then $\dim E^c_p = 2$.

First, we consider $\dim E^c_p = 1$. Then we assume that $\lambda = 1$ (the other case is similar). By Lemma 2.1, there are $\epsilon > 0$ and $h \in \mathcal{U}(f)$ such that

- $h(p) = g(p) = p$,
- $h(x) = \exp_p \circ D_p g \circ \exp_p^{-1}(x)$ if $x \in B_\epsilon(p)$, and

· $h(x) = g(x)$ if $x \notin B_{4\epsilon}(p)$.

Since $\lambda = 1$, we can construct a closed small arc $\mathcal{I}_p \subset B_\epsilon(p) \cap \exp_p(E_p^c(\epsilon))$ with its center at $p$ such that

· $\mathrm{diam}\mathcal{I}_p = \epsilon/4$,

· $h(\mathcal{I}_p) = \mathcal{I}_p$, and

· $h|_{\mathcal{I}_p}$ is the identity map.

Here $E_p^c(\epsilon)$ is the $\epsilon$-ball in $E_p^c$ centered at the origin $\overrightarrow{o}_p$. From Franks' lemma, we can take $h_1$ $C^1$ close to $h$ ($C^1$ close to $f$), such that the two end points of $\mathcal{I}_p$ can make hyperbolic points for $h_1$, and $h_1$ satisfies the above construction of the small arc $\mathcal{I}_p$. Then $\mathcal{I}_p$ is normally hyperbolic with respect to $h_1$, and for any $\eta < \epsilon/4$, the length of $\mathcal{I}_p$ is less than $\eta$.

Finally, we consider $\mathrm{dim}E_p^c = 2$. For convenience, we assume that $g^{\pi(p)}(p) = g(p) = p$. By Lemma 2.1, there is $\epsilon > 0$ and $g_1 \in \mathcal{U}(f)$ such that

· $g_1(p) = g(p) = p$,

· $g_1(x) = \exp_p \circ D_p g \circ \exp_p^{-1}(x)$ if $x \in B_\epsilon(p)$, and

· $g_1(x) = g(x)$ if $x \notin B_{4\epsilon}(p)$.

For any $v \in E_p^c(\epsilon)$, there is $l > 0$ such that $D_p g^l(v) = v$. Take $u \in E_p^c(\epsilon)$ such that $\|u\| = \epsilon/2$. As in the first case, we can construct a closed connected small arc $\mathcal{J}_p \subset B_\epsilon(p) \cap \exp_p(E_p^c(\epsilon))$ such that

· $\mathrm{diam}\mathcal{J}_p = \epsilon/4$,

· $g_1^l(\mathcal{J}_p) = \mathcal{J}_p$, and

· $g_1^l|_{\mathcal{J}_p}$ is the identity map.

As in the proof of the first case, there is $h$ $C^1$ close to $g_1$ such that $\mathcal{J}_p$ is normally hyperbolic with respect to $h$, and for any $\eta < \epsilon/4$, the length of $\mathcal{J}_p$ is less than $\eta$. □

**Remark 2.3.** (a) By the persistency of normally hyperbolic manifold we know that there is a neighborhood $\mathcal{U}(g)$ of $g$ such that for any $\tilde{g} \in \mathcal{U}(g)$ there is a curve $\tilde{\gamma}$ close to $\gamma$ such that all properties of $\gamma$ listed in the Lemma 2.2 is also satisfied for $\tilde{\gamma}$.

(b) For convenience, we call $\gamma$ is a $\varepsilon$-simply periodic curve for $g$ if $\gamma$ satisfies the items in the Lemma 2.2.

We say that $f$ is *Axiom A* if the non-wandering set $\Omega(f)$ is hyperbolic and it is the closure of $P(f)$. A point $x \in M$ is said to be *non-wandering* for $f$ if for any non-empty open set $U$ of $x$ there is $n \geq 0$ such that $f^n(U) \cap U \neq \emptyset$. Denote by $\Omega(f)$ the set of all non-wandering points of $f$. It is clear that $\overline{P(f)} \subset \Omega(f)$. A diffeomorphism $f$ is $\Omega$-*stable* if there is a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that for any $g \in \mathcal{U}(f)$ there is a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that $h \circ f = g \circ h$, where $\Omega(g)$ is the non-wandering set of $g$. For $f \in \mathrm{Diff}(M)$, we say that $f$ is the *star diffeomorphism* (or $f$ satisfies the *star condition*) if there is a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that all periodic points of $g \in \mathcal{U}(f)$ are hyperbolic. Denote by $\mathcal{F}(M)$ the set of all star diffeomorphisms. Aoki [1] and Hayashi [7] showed that for any dimension case, if $f \in \mathcal{F}(M)$ then $f$ is Axiom A without cycles.

**Lemma 2.4.** *There is a residual set $\mathcal{G} \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}$,*

- *either $f$ is a star diffeomorphism,*
- *or for any $\varepsilon > 0$ there is a periodic curve $\gamma$ such that the length of $f^n(\gamma)$ is less than $\varepsilon$, for any $n \in \mathbb{Z}$.*

**Proof.** Let $\mathcal{H}_n$ be the set of $C^1$ diffeomorphisms $f$ such that $f$ has a normally hyperbolic $\gamma$ which is $1/n$-simply periodic curve ($n \in \mathbb{N}$). By Remark 2.3, $\mathcal{H}_n$ is open. Let $\mathcal{N}_n = \mathrm{Diff}(M) \setminus \overline{\mathcal{H}_n}$. Then $\mathcal{H}_n(\eta) \cup \mathcal{N}_n(\eta)$ is open and dense in $\mathrm{Diff}(M)$. Let

$$\mathcal{G} = \bigcap_{n \in \mathbb{N}^+} (\mathcal{H}_n \cup \mathcal{N}_n).$$

Then $\mathcal{G}$ is $C^1$ residual in $\mathrm{Diff}(M)$. Let $f \in \mathcal{G}$ and assume $f$ is not the star diffeomorphism. Then we know that $f \in \overline{\mathcal{H}_n}$ for any $n \in \mathbb{N}^+$ by Lemma 2.2. Hence $f \notin \mathcal{N}_n$ and $f \in \mathcal{H}_n$ for any $n \in \mathbb{N}$. Thus we know that $f$ has a normally hyperbolic $\gamma$ which is $\varepsilon$-simply periodic curve, for any $\varepsilon > 0$. □

**Proof of Theorem A.** Let $f \in \mathcal{G}$ be measure expansive. Suppose by contradiction that $f \notin \mathcal{F}(M)$. From Lemma 2.4, for any $\varepsilon > 0$ there is a periodic curve $\gamma$ such that the length of $f^i(\gamma)$ is less than $\varepsilon$, for any $i \in \mathbb{Z}$. Let $\mu_\gamma$ be a normalized Lebesgue measure on $\gamma$, and define a non-atomic Borel probability measure $\mu$ by $\mu(C) = \mu_\gamma(C \cap \gamma)$, for any Borel set $C$ of $M$. For $x \in M$, let $\Gamma_\epsilon(x) = \{y \in M : d(f^i(x), f^i(y)) \leq \epsilon \text{ for all } i \in \mathbb{Z}\}$. Since $f^n(\gamma) = \gamma$ for some $n \in \mathbb{Z}$, for $x \in \gamma$, we define $\Delta_\epsilon(x) = \{y \in \gamma : d(f^{in}(x), f^{in}(y)) \leq \epsilon \text{ for all } i \in \mathbb{Z}\}$. Clearly $\Delta_\epsilon(x) \subset \Gamma_\epsilon(x)$. Then we have

$$0 < \mu(\Delta_\epsilon(x)) \leq \mu(\Gamma_\epsilon(x)).$$

This is a contradiction. Thus if $f \in \mathcal{G}$ is measure expansive then it is Axiom A without cycles. □

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