

VALUATION OF REVERSE MORTGAGE WITH DEPENDENT JOINT LIVES

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ABSTRACT: Utilizing the principle of balance between expected gain and expected payment, this work obtains the analytic valuation formula for reverse mortgage. In particular, we provide the formulas for the lump sum payment, joint annuity, increasing (decreasing) annuity, and level annuity of reverse mortgage. We also derive the valuation equation that the variable payment annuities satisfy. We then discuss the monotonicity of the lump sum, annuity, and annuity payment factors with respect to the parameters associated with the home price and the interest rate model. Finally, we analyze the sensitivity of the joint annuity with respect to the parameters associated with the home price, interest rate, and lifetime model. The numerical results show that the average return of home price exerts a dominating influence on the joint annuity, followed by the mean reversion level of interest rate, and both of them have stronger impact on the annuities of younger applicants than those of older applicants. Meanwhile, the initial age of male and that of female produce asymmetrical effect on the joint annuity. Remarkably, the dependence of joint-lifetime significantly affect the joint annuity value. In case that the male and female initial ages are both less than 80 years old, annuity values on average increase approximately 4.5%, and the greatest increment of annuity value approaches 9%.

Key Words: reverse mortgage, valuation, joint annuity, jump-diffusion, Vasicek model, lifetime model

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1. INTRODUCTION

Reverse Mortgage is an inviting financial lending product offered to any senior citizen who owns a house. It is normally categorized by law into two categories, namely *collateral reverse mortgage* and *ownership conversion reserve mortgage*, (Ohgaki, 2003). The collateral reverse mortgage is redeemable while the ownership conversion reverse mortgage is not. Home Equity Conversion Mortgage System is a typical collateral reverse mortgage in USA. In a collateral reverse mortgage, the elderly householder borrows annuity like periodical installment mortgage on his/her residential house. With the collateral reverse mortgage, the borrower is able to redeem the reverse mortgage by repaying the loan principal and accumulated interests through property sale at any time from the mortgage's effective date to due date. *Rente Viager* is a typical ownership conversion reverse mortgage in France. In the case of the ownership conversion reverse mortgage, the borrower enters into a contract with a lending institution to obtain an annuity until his/her death, and at death the pledged property ownership is transferred to the lender.

Since the advent of reverse mortgage, many scholars and practitioners have engaged in research related to this area, mainly on the basic principles, operation modes, feasibility, effectiveness, policies, laws, risks, and valuation. Compared to the valuation problem, other aspects of reverse mortgage are well studied, and we note here that the literature on valuation is not as rich as those on other aspects. We focus, in this article, our attention to the valuation problem. The valuation of reverse mortgage mainly includes three aspects: (1) determining a lump sum and annuity payments that the lender can pay before signing the reverse mortgage contract, (2) pricing the redemption right of the collateral reverse mortgage before signing the reverse mortgage contract, and (3) finding the value of reverse mortgage at any time t after signing the RM contract. The main idea behind the first aspect is to employ the principle of expected balance between gain and payment under the assumption of perfect competition market, which makes the discounted present value of payment of the lender be equal to a certain proportion of discounted present value of the mortgaged property, (see, DiVenti and Herzog (1990), Szymanoski(1994), Tse(1995), Mitchell and Piggott (2004)). The main valuation idea of the last two aspects is to apply the option pricing idea, which regards the mortgaged property as the underlying asset and, the loan principal and accumulated interests as the strike price of underlying asset. (Here, the pledged property is usually assumed to follow a stochastic process or stochastic series.) When the contract expires, the lender or its successor determines whether to execute the option (i.e., redeem the pledged property) according to the difference between the price of pledged property and the loan principle and accumulated interests, see Li et al. (2010) and Tsay et al. (2014).

The main risks involved with reverse mortgage, as pointed out by Szymanoski (1994), include property-value risk, interest-rate risk, and longevity risk. In order to rationally price reverse mortgage, one must build an appropriate model that takes into account the above mentioned risks. In general, the risk of house price is modeled in two ways. One is to assume directly that the dynamics of house-price is driven by a forward stochastic differential equation, as in Bardhan *et al.* (2006), Wang *et al.* (2008), Mizrach (2012), Huang *et al.* (2011), Chen *et al.* (2010a), Lee *et al.* (2012), and Tsay *et al.* (2014). The other is to fit the time series model based on the historical data of the house price, as discussed by Nothaft *et al.* (1995), Chinloy *et al.* (1997), Chen *et al.* (2010b), and Li *et al.* (2010). The literature on classical interest rate model mainly includes: the Dothan (1978) model, Vasicek (1977) model, Cox, Ingersoll and Ross (1985) model, Exponential Vasicek model, Hull and White (1990) model, Black and Karasinski (1991) model, Mercurio and Moraleda (2000) model, the CIR++ model, and the Extended Exponential Vasicek model (Brigo and Mercurio, 2006). There are usually several ways to describe the longevity risk, such as a life table and the force of mortality model. For the classical force of mortality model, refer to de Moivre (1724), Gompertz (1825), Makeham (1860, 1867), Weibull (1951), Heligman and Pollard (1980), and Lee-Carter (1992).

As pointed out earlier, this work focuses on the valuation problem of reverse mortgage. We provide analytic valuation formulas for the lump sum and three different annuity aspects of reverse mortgage. We also derive the valuation equation that the variable payment annuities satisfy. For our analysis, we appeal to the principle of balance between expected gain and expected payment. Taking into account of the influence of the parameters associated with the home-price and interest-rate models, we discuss the monotonicity of the lump sum, annuity, and annuity payment factors. Finally, we analyze the sensitivity of the joint annuity with respect to the parameters associated with the home price, interest rate and lifetime model. Our numerical results show that the average return of home price exerts a dominating influence on the joint annuity, followed by the mean reversion level of interest rate. We observe here that both of them have more impact on the annuities of young applicants than those of old applicants. It is interesting to note that the initial age of male and that of female produce asymmetrical effect on the joint annuity. Remarkably, the dependence of joint-lifetime significantly affects the joint annuity value.

The rest of this article is organized as follows. Section 2 presents the models of risk factors. In Section 3, we first design the reverse mortgage predicated on the ownership conversion with fixed yearly payment until death, and then derive the valuation model for the lump sum and annuity payments under the principle of balance between expected gain and expected payment. Section 4 analyzes the monotonicity of the lump sum, annuity payments, and annuity payment factors with respect to the

parameters involved in housing price and interest rate models. We provide in Section 5 some numerical results to examine how the housing price risk, interest rate risk, and longevity risk impact the lump sum, annuity payment, and the annuity payment factors. The final section draws conclusions about our findings.

2. RISK FACTORS

Our goal is to determine the lump sum and annuity of the reverse mortgage without redemption-right applied by a joint lives. In order to build a suitable model to analyze these factors, we must first explore how to describe the risk factors that the reverse mortgage without redemption-right involves. TOWARD THIS: (1) we employ the jump-diffusion model to mimic the dynamics of home price, (2) the Vasicek model to drive the instantaneous interest rate, and (3) a bivariate distribution function to describe the dependent longevity risk of a joint-life (i.e. a couple). All our random elements are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t \geq 0})$, where $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space and $\{\mathcal{F}_t\}_{t \geq 0}$ is a right continuous increasing family of sub σ -algebras of \mathcal{F} with all the null events in \mathcal{F}_0 .

2.1. HOUSE PRICE

We shall use a jump diffusion to model the risk caused by the house price. We assume that the *house price* $h(t)$ ($t \geq 0$) follows the exponential Lévy process

$$h(t) = h(0) \exp \left[\int_0^t \mu_h(s) ds - \left(\frac{1}{2} \sigma_h^2 + \lambda_h k_h \right) t + \sigma_h W_h(t) + \sum_{i=1}^{N(t)} J_i \right], \quad (1)$$

$$h(0) = h_0.$$

This is a generalization by Lee et al., 2012, of the Merton jump diffusion model (Merton, 1976). HERE:

- the \mathcal{P} -standard Brownian motion process $\{W_h(t), t \geq 0\}$ captures the unanticipated instantaneous change of house price.
- the Wiener process $\{W_h(t)\}$ will not capture the abnormal shocks caused by sudden rise or drop in the house price. We model the rise/drop in the house price by independent Gaussian random jumps $\{J_i, i \geq 0\}$ with mean μ_J and variance σ_J^2 , and we count the number of price jumps during the time interval $(0, t]$ using a Poisson process $\{N(t), t \geq 0\}$ with intensity λ_h .
- We assume that the processes $\{W_h(t), t \geq 0\}$, $\{N(t), t \geq 0\}$, and $\{J_i, i \geq 0\}$ are

independent.

- The drift coefficient $\mu_h(t)$ denotes the average rate of return.
- The diffusion coefficient $\sigma_h (> 0)$ represents the volatility of the house price.
- The parameter k_h is one less than the mean of $\exp(J_i)$, i.e., $k_h = \exp(\mu_J + \frac{1}{2}\sigma_J^2) - 1$.

2.2. INTEREST RATE

We next assume that the instantaneous short-rate dynamics evolves as Vasicek model (Vasicek, 1977). Specifically, the *interest rate process* $\{r(t), t \geq 0\}$ is governed by the following stochastic differential equation

$$dr(t) = \alpha_r(\mu_r - r(t))dt + \sigma_r dW_r(t), \quad r(0) = r_0, \tag{2}$$

where $\{W_r(t), t \geq 0\}$ is a \mathcal{P} -standard Brownian motion with $Cov(dW_r(t), dW_h(t)) = \rho_{hr}dt$. We assume that $r_0, \alpha_r, \mu_r, \sigma_r$ are positive constants.

Applying Itô’s formula to $e^{\alpha_r u}r(u)$ and integrating from 0 to t , we obtain

$$r(t) = e^{-\alpha_r t}r(0) + \mu_r(1 - e^{-\alpha_r t}) + \sigma_r \int_0^t e^{-\alpha_r(t-u)} dW_r(u), \quad t \geq 0. \tag{3}$$

The *discount factor* at time t , denoted by $d(t)$, is defined as

$$d(t) := \exp\left(-\int_0^t r(s)ds\right). \tag{4}$$

After some some trivial computations, we have

$$E[d(t)] = \exp\left\{\left(\frac{\sigma_r^2}{2\alpha_r^2} - \mu_r\right)t + \frac{1}{\alpha_r}(\mu_r - r_0)(1 - e^{-\alpha_r t}) + \frac{\sigma_r^2}{4\alpha_r^3}[1 - (2 - e^{-\alpha_r t})^2]\right\}, \tag{5}$$

see also Norberg (2004).

2.3. JOINT LIVES

By the initial time $t = 0$ we mean the time at which the reverse mortgage (without redemption-right involving a joint-life) is signed. Let x_0 and y_0 represent the age of the husband and wife at time 0, respectively. Let X and Y be the age-at-death of the husband and wife, respectively. By $F(x, y) := P(X \leq x, Y \leq y)$ we denote the joint distribution function of random vector (X, Y) , with $F_1(x)$ and $F_2(y)$ denoting the respective marginal distributions, (i.e., $F_1(x) = F(x, +\infty)$ and $F_2(y) = F(+\infty, y)$).

The bivariate distribution function can be specified by a copula function and two marginal distributions; that is,

$$F(x, y) = C(F_1(x), F_2(y)),$$

where C is a real-valued copula function that provides a link between the marginal distributions and the corresponding bivariate distribution. The copula function can be expressed as, (see Frees et al. (1996)),

$$C(u, v) = \frac{1}{\alpha} \ln \left[1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^\alpha - 1} \right],$$

with the two marginal distributions following the Gompertz distribution:

$$\begin{aligned} F_1(x) &= 1 - \exp \left[e^{-\frac{m_1}{\sigma_1}} (1 - e^{\frac{x}{\sigma_1}}) \right], \\ F_2(y) &= 1 - \exp \left[e^{-\frac{m_2}{\sigma_2}} (1 - e^{\frac{y}{\sigma_2}}) \right], \end{aligned}$$

(see Carrière (1994)). It is easy to see that the marginal density distributions for X and Y , respectively denoted by $f_1(x)$ and $f_2(y)$, are given by

$$\begin{aligned} f_1(x) &= \frac{1}{\sigma_1} e^{\frac{x-m_1}{\sigma_1}} \exp \left[e^{-\frac{m_1}{\sigma_1}} (1 - e^{\frac{x}{\sigma_1}}) \right], \\ f_2(y) &= \frac{1}{\sigma_2} e^{\frac{y-m_2}{\sigma_2}} \exp \left[e^{-\frac{m_2}{\sigma_2}} (1 - e^{\frac{y}{\sigma_2}}) \right]. \end{aligned}$$

We consider a bivariate residual lifetime random vector $(X - x_0, Y - y_0)$, where $X - x_0$ and $Y - y_0$ represents *the time-until-death* of the husband and that of the wife, respectively. The joint distribution function for $(X - x_0, Y - y_0)$ is expressed as

$$F_c(x, y) = \frac{1}{p_0} [F(x_0 + x, y_0 + y) - F(x_0 + x, y_0) - F(x_0, y_0 + y) + F(x_0, y_0)], \quad (6)$$

where

$$p_0 = 1 - F_1(x_0) - F_2(y_0) + F(x_0, y_0).$$

Let

$$T_1 = \min\{X - x_0, Y - y_0\}, \quad T_2 = \max\{X - x_0, Y - y_0\}, \quad (7)$$

then the density function $f_T(t)$ for T_2 is

$$f_T(t) = \frac{1}{p_0} \left[\frac{dF(x_0 + t, y_0 + t)}{dt} - \frac{dF(x_0 + t, y_0)}{dt} - \frac{dF(x_0, y_0 + t)}{dt} \right]. \quad (8)$$

Abbreviating $\frac{dF(x_0+t, y_0+t)}{dt}$ as $\frac{dF}{dt}$, we have

$$\frac{dF}{dt} = \frac{f_1(x_0 + t)e^{\alpha F_1(x_0+t)}(e^{\alpha F_2(y_0+t)} - 1) + f_2(y_0 + t)e^{\alpha F_2(y_0+t)}(e^{\alpha F_1(x_0+t)} - 1)}{e^\alpha - 1 + (e^{\alpha F_1(x_0+t)} - 1)(e^{\alpha F_2(y_0+t)} - 1)},$$

$$\frac{dF(x_0 + t, y_0)}{dt} = \frac{(e^{\alpha F_2(y_0)} - 1)e^{\alpha F_1(x_0+t)} f_1(x_0 + t)}{e^\alpha - 1 + (e^{\alpha F_1(x_0+t)} - 1)(e^{\alpha F_2(y_0)} - 1)},$$

$$\frac{dF(x_0, y_0 + t)}{dt} = \frac{(e^{\alpha F_1(x_0)} - 1)e^{\alpha F_2(y_0+t)} f_2(y_0 + t)}{e^\alpha - 1 + (e^{\alpha F_1(x_0)} - 1)(e^{\alpha F_2(y_0+t)} - 1)}.$$

3. VALUATION OF REVERSE MORTGAGE

In this section, we will first introduce a reverse mortgage with the joint and γ annuities applied by the dependent joint lives. Then, the valuation models are built based on the principle of balance between expected gain and expected payment. Under the two-dimensional Gauss distribution and independence assumptions, we obtain the analytic valuation formulas for the lump sum, joint and γ annuities, increasing (decreasing) annuities, and level annuities of reverse mortgage without redemption right, and derive the valuation equation that the variable payment annuities satisfy.

3.1. REVERSE MORTGAGE WITH JOINT AND γ ANNUITIES

In this subsection, we will design a reverse mortgage with the joint and γ annuities. The joint-life contract offers a yearly annuity payment until the last annuitant dies. In this article, the *joint-life* means a couple. The product that we design has the following basic features:

- (I) The lender starts the payments of annuity to the joint annuitants at the beginning of signing the contract. The annuity payment is terminated upon the death of the last annuitant. While both annuitants are alive, the lender pays annuity amount A at the beginning of each year, and γA while only one annuitant is alive.
- (II) When the last applicant dies, the lender will take over the annuitant's pledged property, sell it in the market, and keep all of the proceeds from the sale of the property.

The essence of reverse mortgage with the joint and γ annuities is to exchange the profit from selling the mortgaged house with the joint-life's annuities until the last annuitant's death. Upon the passing of the last annuitant, the lender will take over the homeowner's mortgaged property and sell it. The cash that is thus acquired is used to repay loan (including annuities and accumulated interests) that the joint annuitants owe to the lender. Reverse mortgage possesses the non-recourse clauses, which is, that the lender may not reclaim the loan against the annuitants' other assets or cash income except for their pledged property. So, the lender will suffer a loss when the

cash of selling the mortgaged property is less than those annuities and accumulated interests, otherwise the lender will make a profit.

Next, we will demonstrate how the reverse mortgage runs with the joint and γ annuities. Assume that the age-at-death of the husband and the wife be $X = 65.7$ and $Y = 67.9$ years, respectively. Let the initial age be respectively $x_0 = 65$ and $y_0 = 64$ years. Then the loan tenure is $T_2 = \max\{X - x_0, Y - y_0\} = 3.9$ years. This implies that the couple claims once cash payment A at the beginning of the first year of the contract. The wife as the last annuitant claims three times cash payments γA at the beginning of the second, third and fourth year of the contract, respectively. When the wife, the last survivor, dies at age of 67.9 years, the lender will take over the pledged house and sell it in the market. Most of the time, it is impossible to sell out the pledged house as soon as the lender takes over it. Thus the time of selling out the pledged house usually lags behind that of taking over the pledged house for a time. In the following valuation models, we will take the delay time into consideration.

3.2. VALUATION OF REVERSE MORTGAGE WITH JOINT AND γ ANNUITIES

We assume that we are in the perfectly competitive market. We price the reverse mortgage with the joint and γ annuity by the principle of balance between expected gain and expected payment. Recall that the terminology *principle of balanced expected gain and payment* means that the expected discounted present value of future sale of the pledged property is the same as the expected discounted present value of annuities that the lender pays during the whole loan period.

At time T_2 , the lender takes over the annuitants' mortgaged property, and sells it at time $T_2 + t_0$ where $t_0 \geq 0$ is the delay time between the lender taking over the pledged property and the sale of that property. We assume that t_0 is fixed (not a random variable). Then the expectation of discounted present value of the sale price of the property (i.e., the lender's expected gain) is

$$E [h(T_2 + t_0)d(T_2 + t_0)], \quad (9)$$

where $h(t)$ is the value of the mortgaged property at time t given by the Lévy Equation (1), and $d(t)$ is the discount factor at time t given by the Equation (4).

The expectation of discounted present value of the joint and γ annuities during the whole loan period (i.e., the lender's expected payment) is

$$E \left[\sum_{k=0}^{\lfloor T_1 \rfloor} Ad(k) + \sum_{k=\lfloor T_1 \rfloor+1}^{\lfloor T_2 \rfloor} \gamma Ad(k) \right], \quad (10)$$

where $\lfloor x \rfloor$ is the floor function (the largest integer not greater than x). Then, the principle of balance between expected gain and expected payment yields

$$E [h(T_2 + t_0)d(T_2 + t_0)] = E \left[\sum_{k=0}^{\lfloor T_1 \rfloor} Ad(k) + \sum_{k=\lfloor T_1 \rfloor+1}^{\lfloor T_2 \rfloor} \gamma Ad(k) \right]. \tag{11}$$

In general, the explicit formula of annuity payment is difficult to obtain from the Equation (11). But we can obtain the analytic annuity formula under the two-dimensional Gaussian distribution and independence assumption. The following Proposition 1 presents the analytic formula for the expected discounted present value of the mortgaged property at any time t .

Proposition 1: *Define*

$$Y(t) := \int_0^t \left[\sigma_r e^{-\alpha_r s} \int_0^s e^{\alpha_r u} dW_r(u) \right] ds. \tag{12}$$

Assume that (a) the dynamics of home price follows the exponential Lévy process given by the Equation (1), (b) the instantaneous short interest rate is governed by the Equation (2), (c) the joint distribution of $(W_h(t), Y(t))$ follows the two dimensional Gaussian distribution, and that (d) $\sigma_h W_h(t) - Y(t)$ is independent of $\sum_{i=1}^{N(t)} J_i$. Then, the expectation of discounted present value of the mortgaged property at time- t is given by

$$E [h(t)d(t)] = G(t)D(t), \tag{13}$$

where

$$G(t) = h_0 \exp \left\{ \int_0^t \mu(s) ds - \sigma_h \sigma_r \rho_{hr} \frac{1}{\alpha_r} \left(t + \frac{1}{\alpha_r} e^{-\alpha_r t} - \frac{1}{\alpha_r} \right) \right\}, \tag{14}$$

$$D(t) = \exp \left\{ \left(\frac{\sigma_r^2}{2\alpha_r^2} - \mu_r \right) t + \frac{1}{\alpha_r} (\mu_r - r_0) (1 - e^{-\alpha_r t}) + \frac{\sigma_r^2}{4\alpha_r^3} [1 - (2 - e^{-\alpha_r t})^2] \right\}. \tag{15}$$

Proof: see Appendix A.

The Proposition 2 presented below provides the analytic valuation formula for the expected lump sum that the householder can borrow in average at time 0, and the analytic valuation formula for the joint annuity. In this, the house price $h(t)$, the interest rate $r(t)$, the discount factor $d(t)$, and the survival periods T_1 and T_2 are respectively given by the Equations (1), (3), (4), and (7). Also recall that the *lump sum* is the total cash that the annuitants can obtain in average at the time of signing the reverse mortgage contract.

Proposition 2: *Assume that (a) $h(t)d(t)$, ($t \geq 0$), is independent of T_2 , (b) $r(t)$ is independent of (T_1, T_2) , and (c) the pledged property is sold at time $T_2 + t_0$. Then:*

(1) The expectation of the lump sum, denoted by \tilde{G} , is

$$\tilde{G} = \int_0^{+\infty} G(x + t_0)D(x + t_0)f_T(x)dx, \tag{16}$$

where $G(x + t_0)$, $D(x + t_0)$ and $f_T(x)$ are given by Equations (14), (15) and (8), respectively.

(2) For the joint and γ annuity, the fixed amount A of annuity is given by

$$A = \frac{\tilde{G}}{\tilde{F}_1}, \tag{17}$$

where

$$\tilde{F}_1 = \sum_{i=0}^{+\infty} D(i) [1 - (1 - \gamma)F_c(i, +\infty) - (1 - \gamma)F_c(+\infty, i) + (1 - 2\gamma)F_c(i, i)], \tag{18}$$

and $F_c(x, y)$ is given by the Equation (6).

Proof: Since the lender’s only gain is from the proceeds of selling the homeowner’s pledged house, with the principle of balance between expected gain and expected payment, the lump sum that the applicants can borrow at time $t = 0$ of signing the reverse mortgage contract is $h(T_2 + t_0)d(T_2 + t_0)$, which is a random variable. Further, noting that $h(t)d(t)$ ($t \geq 0$) is independent of T_2 , we get

$$\begin{aligned} \tilde{G} &= E [h(T_2 + t_0)d(T_2 + t_0)] \\ &= \int_0^{+\infty} E [h(x + t_0)d(x + t_0)] f_T(x)dx \\ &= \int_0^{+\infty} G(x + t_0)D(x + t_0)f_T(x)dx, \end{aligned}$$

where $G(x + t_0)$ and $D(x + t_0)$ can be characterized by replacing t with $x + t_0$ in Equations (14) and (15), respectively, and $f_T(x)$ denotes the probability density function for T_2 given by the Equation (8).

From the independence of $r(t)$ and (T_1, T_2) , we have

$$\begin{aligned} &E \left[\sum_{k=0}^{\lfloor T_1 \rfloor} Ad(k) + \sum_{k=\lfloor T_1 \rfloor+1}^{\lfloor T_2 \rfloor} \gamma Ad(k) \right] \\ &= E \left[\sum_{i=0}^{+\infty} \sum_{k=0}^i Ad(k) \mathbf{1}_{\{\lfloor T_1 \rfloor=i\}} \right] + E \left[\sum_{i=0}^{+\infty} \sum_{j=i+1}^{+\infty} \sum_{k=i+1}^j \gamma Ad(k) \mathbf{1}_{\{\lfloor T_1 \rfloor=i, \lfloor T_2 \rfloor=j\}} \right] \\ &= A \sum_{i=0}^{+\infty} \sum_{k=0}^i E[d(k)]P(\lfloor T_1 \rfloor = i) + \gamma A \sum_{i=0}^{+\infty} \sum_{j=i+1}^{+\infty} \sum_{k=i+1}^j E[d(k)]P(\lfloor T_1 \rfloor = i, \lfloor T_2 \rfloor = j) \end{aligned}$$

$$\begin{aligned}
 &= A \sum_{i=0}^{+\infty} D(i)P(T_1 \geq i) + \gamma A \sum_{i=0}^{+\infty} D(i)P(T_1 < i, T_2 \geq i) \\
 &= A \cdot \tilde{F}_1
 \end{aligned}$$

where $D(k)$ is characterized as in (15). Recalling that the probability density function for T_2 is given by the Relation (8), we get the Equation (17). The proof is complete.

3.3. VALUATION OF REVERSE MORTGAGE WITH VARIABLE PAYMENT ANNUITIES

By the terminology phrase *reverse mortgage with variable payment annuity* one means that the lender starts the payments of annuity to the joint lives at the beginning of signing the contract until the death of the last survivor, and that the annuity payment amount at year k , ($k \geq 1$), is A_k . Here, if the last annuitant passes away at k -th year, then a total amount $\sum_{i=1}^k A_i$ of annuity payments has been paid. The *increasing* or *decreasing* annuity is a special case of the variable payment annuity with $A_k \equiv A_0 + d \cdot k$ ($k = 1, 2, \dots$). At the beginning of k -th period, the annuity payment is $A_0 + d \cdot k$, as long as at least one of the annuitants is alive. In the following, we call A_0 the *basic annuity*, and d the *annuity increment*. The *level annuity* is a special case of the joint and γ annuity with $\gamma = 1$, and it is also a special case of the variable payment annuity with A_k being the same constant for all $k \geq 1$. For the level annuity, the fixed amount A^* of annuity is paid until the death of the last survivor. Because the analysis needed in this subsection parallels that of the previous subsection we omit the proof of Proposition 3 below.

Proposition 3: *Assume that (a) $h(t)d(t)$ ($t \geq 0$) is independent of T_2 , (b) $r(t)$ is also independent of T_2 , and (c) The pledged property is sold at time $T_2 + t_0$. Then:*

(1) *For the variable payment annuity, the annuity payments A_k ($k = 1, 2, \dots$) satisfy the following valuation equation*

$$\int_0^{+\infty} G(x + t_0)D(x + t_0)f_T(x)dx = \sum_{k=0}^{+\infty} A_{k+1}D(k)P(T_2 \geq k), \tag{19}$$

where

$$P(T_2 \geq k) = 1 - F_c(k, k), \tag{20}$$

and $D(k)$ is given by Equation (15).

(2) *For the increasing (decreasing) annuity, A_0 and d are determined by the simultaneous equations*

$$A_0 = \frac{\tilde{G} - d \cdot \tilde{F}_3}{\tilde{F}_2}, \quad d = \frac{\tilde{G} - A_0 \cdot \tilde{F}_2}{\tilde{F}_3}, \tag{21}$$

where

$$\tilde{F}_2 = \sum_{k=0}^{+\infty} D(k)P(T_2 \geq k), \quad \tilde{F}_3 = \sum_{k=0}^{+\infty} kD(k)P(T_2 \geq k), \quad (22)$$

and $D(k)$, \tilde{G} and $P(T_2 \geq k)$ are respectively defined by Equations (15), (16) and (20).

(3) For the level annuity, the fixed annuity amount A^* is given by

$$A^* = \frac{\tilde{G}}{\tilde{F}_2}, \quad (23)$$

where \tilde{F}_2 is given by Equation (22).

4. EFFECT OF PARAMETERS ON THE ANNUITY

We analyze in this and the following sections, how the parameters associated with the house price, interest rate, and the delay duration in selling the pledged house would affect the various annuity payments, (particularly, annuity payments, lump sum payments, and annuity payment factors).

4.1. MONOTONICITY W.R.T THE PARAMETERS OF HOUSE PRICE

In the following, we assume that the rate of return $\mu_h(t) \equiv \mu_h$ of the house price (that is the drift coefficient) is held constant. The next Proposition 4 analyzes the monotonicity of the annuity payment, lump sum, and annuity payment factors w.r.t the parameters related with the house price model, including the average return rate μ_h , the volatility σ_h , the initial house price h_0 , the correlation coefficient between the Brownian motion driving the house price and those driving the interest rate ρ_{hr} , and the delay time in selling the pledged house t_0 . For the descriptions of the parameters μ_h , σ_h , ρ_{hr} , and h_0 connected to the house price, we refer to Section 2.

Proposition 4: *With respect to the parameters μ_h , σ_h , ρ_{hr} , and h_0 of the house price, the joint and γ annuity payment A , the basic annuity A_0 , the fixed annuity increment d of the increasing (decreasing) annuity, the level annuity payment A^* , the lump sum \tilde{G} , and the annuity payment factors \tilde{F}_i ($i = 1, 2, 3$) (given by Propositions 2 and 3) have the following properties:*

4.1. PARAMETER μ_h : (a) *The annuity payment factors \tilde{F}_i ($i = 1, 2, 3$) are independent of μ_h ;* (b) *The lump sum \tilde{G} is an increasing function of μ_h ;* and (c) *The A , A_0 , d and A^* are all increasing functions of μ_h .*

4.2. PARAMETER σ_h : (a) The annuity payment factors \tilde{F}_i ($i = 1, 2, 3$) are independent of the volatility σ_h of the house price. (b) If $\rho_{hr} > 0$, $\sigma_r > 0$ and $\alpha_r \neq 0$, then the lump sum \tilde{G} is a decreasing function of σ_h . (c) If $\rho_{hr} < 0$, $\sigma_r > 0$ and $\alpha_r \neq 0$, then the lump sum \tilde{G} is an increasing function of σ_h . (d) If $\rho_{hr} > 0$, $\sigma_r > 0$ and $\alpha_r \neq 0$, then A , A_0 , d and A^* all are decreasing functions of σ_h . (e) If $\rho_{hr} < 0$, $\sigma_r > 0$ and $\alpha_r \neq 0$, then A , A_0 , d and A^* all are increasing functions of σ_h .

4.3. PARAMETER ρ_{hr} : (a) The annuity payment factors \tilde{F}_i , $i = 1, 2, 3$, are independent of ρ_{hr} . (b) If $\sigma_h > 0$, $\sigma_r > 0$ and $\alpha_r \neq 0$, then the lump sum \tilde{G} is a decreasing function of ρ_{hr} . (c) If $\sigma_h > 0$, $\sigma_r > 0$ and $\alpha_r \neq 0$, then A , A_0 , d and A^* are all decreasing functions of ρ_{hr} .

4.4. PARAMETER h_0 : (a) The annuity payment factors \tilde{F}_i , $i = 1, 2, 3$, are independent of the initial house price h_0 . (b) The lump sum \tilde{G} is an increasing function of h_0 . (c) The A , A_0 , d and A^* are all increasing functions of h_0 .

Proof: From the definitions of the annuity payment factors \tilde{F}_i ($i = 1, 2, 3$) (see Relations (18) and (22)), we note that these two annuity payment factors are independent of μ_h .

Noting that $D(x + t_0)$ and $f_T(x)$ do not depend on μ_h , the partial derivative of the integrand in the definition of \tilde{G} (see Relation (16)), is

$$\frac{\partial [G(x + t_0)D(x + t_0)f_T(x)]}{\partial \mu_h} = (x + t_0)G(x + t_0)D(x + t_0)f_T(x).$$

Since $G(x + t_0) > 0$, $D(x + t_0) > 0$, $f_T(x) \geq 0$ and $x + t_0 \geq 0$, this implies that the lump sum \tilde{G} is an increasing function of μ_h .

Furthermore, from the Equations (17), (21) and (23), we note that A , A_0 , d and A^* are increasing functions of μ_h . Thus, Part 4.1 of the proposition is proved.

From the Equations (18) and (22), defining the annuity payment factors \tilde{F}_i ($i = 1, 2, 3$) we see that these annuity payment factors are independent of σ_h .

Define

$$g_1(z) := \frac{1}{\alpha_r} \left[-z + \frac{1}{\alpha_r} (1 - e^{-\alpha_r z}) \right]. \tag{24}$$

Noting that $D(x + t_0)$ and $f_T(x)$ do not depend on σ_h , the partial derivative of the integrand in the definition of \tilde{G} is

$$\frac{\partial [G(x + t_0)D(x + t_0)f_T(x)]}{\partial \sigma_h} = \sigma_r \rho_{hr} G(x + t_0)D(x + t_0)f_T(x)g_1(x + t_0),$$

where $g_1(x + t_0) = \frac{1}{\alpha_r} \left[-(x + t_0) + \frac{1}{\alpha_r} (1 - e^{-\alpha_r(x+t_0)}) \right]$.

In case that $\alpha_r \neq 0$ and $z \geq 0$, we have $g_1(z) \leq 0$. Thus, Part 4.2 of the proposition is proved.

Since $D(x + t_0)$ and $f_T(x)$ are free from ρ_{hr} , the partial derivative w.r.t. ρ_{hr} of the integrand in the definition of \tilde{G} is

$$\frac{\partial [G(x + t_0)D(x + t_0)f_T(x)]}{\partial \rho_{hr}} = \sigma_h \sigma_r G(x + t_0)D(x + t_0)f_T(x)g_1(x + t_0).$$

Noting that $g_1(z) \leq 0$ in case of $z \geq 0, \alpha_r \neq 0$, Part 4.3 follows.

Since the Part 4.4 is obvious, we omit its proof.

Proposition 5: *With respect to the delay time t_0 , the factors $A, A_0, d, A^*, \tilde{G}$, and \tilde{F}_i ($i = 1, 2, 3$) have the following properties:*

- (a) *The annuity payment factors \tilde{F}_i ($i = 1, 2, 3$) do not depend on t_0 .*
- (b) *Now set*

$$\Delta = \left(\mu_r - r_0 + \frac{\sigma_h \sigma_r \rho_{hr}}{\alpha_r} \right)^2 + \frac{2\sigma_r^2(r_0 - \mu_h)}{\alpha_r^2}, \tag{25}$$

$$z_1 = -\frac{\alpha_r^2}{\sigma_r^2} \left[\mu_r - r_0 - \frac{\sigma_r^2}{\alpha_r^2} + \frac{\sigma_h \sigma_r \rho_{hr}}{\alpha_r} + \sqrt{\Delta} \right], \tag{26}$$

and

$$z_2 = -\frac{\alpha_r^2}{\sigma_r^2} \left[\mu_r - r_0 - \frac{\sigma_r^2}{\alpha_r^2} + \frac{\sigma_h \sigma_r \rho_{hr}}{\alpha_r} - \sqrt{\Delta} \right]. \tag{27}$$

(b-1) *If any one of the following conditions is satisfied*

$$\begin{aligned} \Delta &\leq 0, \\ \Delta &\geq 0, \alpha_r > 0, z_1 \geq 1, \\ \Delta &\geq 0, \alpha_r > 0, z_2 \leq 0, \end{aligned}$$

then the lump sum \tilde{G} is an increasing function of t_0 . Also, the quantities A, A_0, d and A^* are increasing functions of t_0 .

(b-2) *If*

$$\Delta \geq 0, \alpha_r > 0, z_1 \leq 0, z_2 \geq 1, \tag{28}$$

holds, then the lump sum \tilde{G} is a decreasing function of t_0 . The quantities A, A_0, d and A^* are also decreasing functions of t_0 .

Proof: Define

$$g_2(z) := \frac{\sigma_r^2}{2\alpha_r^2} z^2 + \beta_1 z + \beta_0, \quad (-\infty < z < +\infty),$$

where

$$\beta_0 = \mu_h - \mu_r - \frac{\sigma_h \sigma_r \rho_{hr}}{\alpha_r} + \frac{\sigma_r^2}{2\alpha_r^2},$$

$$\beta_1 = \mu_r - r_0 + \frac{\sigma_h \sigma_r \rho_{hr}}{\alpha_r} - \frac{\sigma_r^2}{\alpha_r^2}.$$

The partial derivative of the integrand in the definition of \tilde{G} is

$$\frac{\partial [G(x + t_0)D(x + t_0)f_T(x)]}{\partial t_0} = G(x + t_0)D(x + t_0)f_T(x)g_2(e^{-\alpha_r(x+t_0)}),$$

where

$$g_2(e^{-\alpha_r(x+t_0)}) = \frac{\sigma_r^2}{2\alpha_r^2}e^{-2\alpha_r(x+t_0)} + \beta_1 e^{-\alpha_r(x+t_0)} + \beta_0.$$

The minimum of $g_2(z)$ is $-\frac{\alpha_r^2}{2\sigma_r^2}\Delta$, (Δ given by Equation (25)). If the condition $\Delta \leq 0$ holds, we then have $g_2(z) \geq 0$, and thus \tilde{G} is an increasing function of t_0 .

Recall the definitions of z_1 and z_2 given above by the Relations (26) and (27), respectively. Now, if the condition $\Delta \geq 0$ holds, then $g_2(z_i) = 0$, $i = 1, 2$. Moreover, it is obvious that $0 < \exp(-\alpha_r(x + t_0)) \leq 1$ whenever $\alpha_r > 0$ and $x + t_0 \geq 0$. Thus the lump sum \tilde{G} is a decreasing function of t_0 if the Condition (28) holds. One similarly obtains the rest of the properties in Part (b-1).

4.2. MONOTONICITY W.R.T PARAMETERS OF INTEREST RATE

The following Proposition 6 analyzes how the annuity payment, lump sum payment, and the annuity payment factors vary with the parameters involved in the interest rate model, including the initial interest rate r_0 , the mean reversion level μ_r , and the volatility σ_r .

Proposition 6: *With respect to the parameters of the interest rate model, the factors A , A_0 , d , A^* , \tilde{G} , \tilde{F}_i ($i = 1, 2, 3$) have the following properties:*

6.1. PARAMETER r_0 : *If $\alpha_r \neq 0$, then \tilde{G} , \tilde{F}_i , ($i = 1, 2, 3$), are decreasing functions of r_0 .*

6.2. PARAMETER μ_r : *If $\alpha_r > 0$, then \tilde{G} , \tilde{F}_i , ($i = 1, 2, 3$), are decreasing functions of μ_r . If the opposite case $\alpha_r < 0$ holds, then \tilde{G} , \tilde{F}_i , ($i = 1, 2, 3$), are increasing functions of μ_r .*

6.3. PARAMETER σ_r : **(a)** *If $\alpha_r > 0, \sigma_h > 0$ and $\rho_{hr} \geq 0$, then \tilde{G} is a decreasing function of σ_r in the interval $\sigma_r \in (0, \sigma_h \rho_{hr} \alpha_r]$. If $\alpha_r > 0, \sigma_h > 0$ and $\rho_{hr} \leq 0$, then \tilde{G} is an increasing functions of σ_r .*

(b) *In case of $\alpha_r \neq 0, \sigma_r > 0$, then \tilde{F}_i , ($i = 1, 2, 3$), are increasing functions of σ_r .*

(c) If $\alpha_r > 0, \sigma_h > 0$ and $\rho_{hr} \geq 0$, then A, A_0, d and A^* are decreasing functions of σ_r in the interval $\sigma_r \in (0, \sigma_h \rho_{hr} \alpha_r]$.

Proof: Noting that both $G(x + t_0)$ and $f_T(x)$ are independent of r_0 , the partial derivative w.r.t r_0 of the integrand in the definition of \tilde{G} is

$$\frac{\partial [G(x + t_0)D(x + t_0)f_T(x)]}{\partial r_0} = -\frac{1}{\alpha_r} \left[1 - e^{-\alpha_r(x+t_0)} \right] G(x + t_0)D(x + t_0)f_T(x).$$

The partial derivatives w.r.t r_0 of the annuity payment factors $\tilde{F}_i, (i = 1, 2, 3)$, are

$$\begin{aligned} \frac{\partial \tilde{F}_1}{\partial r_0} &= -\frac{1}{\alpha_r} \sum_{k=1}^{+\infty} (1 - e^{-\alpha_r k}) D(k) [P(T_1 \geq k) + \gamma P(T_1 < k, T_2 \geq k)], \\ \frac{\partial \tilde{F}_2}{\partial r_0} &= -\frac{1}{\alpha_r} \sum_{k=1}^{+\infty} (1 - e^{-\alpha_r k}) D(k) P(T_2 \geq k), \\ \frac{\partial \tilde{F}_3}{\partial r_0} &= -\frac{1}{\alpha_r} \sum_{k=1}^{+\infty} k(1 - e^{-\alpha_r k}) D(k) P(T_2 \geq k). \end{aligned}$$

Since $\frac{1}{\alpha_r}(1 - e^{-\alpha_r z}) \geq 0$ whenever $\alpha_r \neq 0$ and $z \geq 0$, we obtain Part 6.1 of the Proposition.

Define

$$g_3(z) := -z + \frac{1}{\alpha_r} (1 - e^{-\alpha_r z}). \tag{29}$$

It is obvious that

$$\frac{\partial D(x)}{\partial \mu_r} = D(x)g_3(x).$$

Since $G(x + t_0)$ and $f_T(x)$ are free from μ_r , the partial derivative w.r.t. μ_r of the integrand in the definition of \tilde{G} is

$$\frac{\partial [G(x + t_0)D(x + t_0)f_T(x)]}{\partial \mu_r} = G(x + t_0)D(x + t_0)f_T(x)g_3(x + t_0),$$

where $g_3(x + t_0) = -(x + t_0) + \frac{1}{\alpha_r} (1 - e^{-\alpha_r(x+t_0)})$.

Since $g_3(z) \leq 0$ in case of $\alpha_r > 0, z \geq 0$, and $g_3(z) \geq 0$ in case of $\alpha_r < 0, z \geq 0$, we obtain Part 6.2.

For $-\infty < z < +\infty$, define

$$\begin{aligned} g_4(z) &= -\frac{\sigma_r}{2\alpha_r^3} e^{-2\alpha_r z} + \left(\frac{2\sigma_r}{\alpha_r^3} - \frac{\sigma_h \rho_{hr}}{\alpha_r^2} \right) e^{-\alpha_r z} \\ &\quad + \left(\frac{\sigma_r}{\alpha_r^2} - \frac{\sigma_h \rho_{hr}}{\alpha_r} \right) z + \left(\frac{\sigma_h \rho_{hr}}{\alpha_r^2} - \frac{3\sigma_r}{2\alpha_r^3} \right), \\ g_5(y) &= \frac{1}{\alpha_r} \left[\frac{\sigma_r}{\alpha_r} y^2 + \left(\sigma_h \rho_{hr} - \frac{2\sigma_r}{\alpha_r} \right) y + \left(\frac{\sigma_r}{\alpha_r} - \sigma_h \rho_{hr} \right) \right]. \end{aligned}$$

We have $g_5(y)$ has two zero point $y_1 = 1 - \frac{\sigma_h \rho_{hr} \alpha_r}{\sigma_r}$ and $y_2 = 1$ in case of $\sigma_h \rho_{hr} \geq 0$; $g_5(y)$ has two zero point $y_1 = 1$ and $y_2 = 1 - \frac{\sigma_h \rho_{hr} \alpha_r}{\sigma_r}$ in case of $\sigma_h \rho_{hr} \leq 0$.

The derivation of $g_4(z)$ w.r.t z is

$$\frac{dg_4(z)}{dz} = \frac{1}{\alpha_r} \left[\frac{\sigma_r}{\alpha_r} e^{-2\alpha_r z} + \left(\sigma_h \rho_{hr} - \frac{2\sigma_r}{\alpha_r} \right) e^{-\alpha_r z} + \left(\frac{\sigma_r}{\alpha_r} - \sigma_h \rho_{hr} \right) \right] = g_5(e^{-\alpha_r z}),$$

Note that $0 < e^{-\alpha_r z} \leq 1$ in case of $z \in [0, +\infty), \alpha_r > 0$. In case of $z \in [0, +\infty), \alpha_r > 0, \sigma_h > 0$ and $\rho_{hr} \geq 0$, we have $g_5(e^{-\alpha_r z}) \leq 0$ in the interval $\sigma_r \in (0, \sigma_h \rho_{hr} \alpha_r]$, then $g_4(z) \leq g_4(0) = 0$ in the interval $\sigma_r \in (0, \sigma_h \rho_{hr} \alpha_r]$. In case of $z \in [0, +\infty), \alpha_r > 0, \sigma_h > 0$ and $\rho_{hr} \leq 0$, we have $g_5(e^{-\alpha_r z}) \geq 0$, then $g_4(z)$ is an increasing function of z and $g_4(z) \geq g_4(0) = 0$.

Noting that $G(x + t_0)$ and $f_T(x)$ do not depend on σ_r , the partial derivative w.r.t σ_r of the integrand in the definition of \tilde{G} is

$$\frac{\partial [G(x + t_0)D(x + t_0)f_T(x)]}{\partial \sigma_r} = G(x + t_0)D(x + t_0)f_T(x)g_4(x + t_0).$$

Define

$$g_6(z) := z + [1 - (2 - e^{-\alpha_r z})^2] \frac{1}{2\alpha_r}.$$

The partial derivatives w.r.t σ_r of the annuity payment factors are

$$\begin{aligned} \frac{\partial \tilde{F}_1}{\partial \sigma_r} &= \frac{\sigma_r}{\alpha_r^2} \sum_{k=1}^{+\infty} D(k)g_6(k)[P(T_1 \geq k) + \gamma P(T_1 < k, T_2 \geq k)], \\ \frac{\partial \tilde{F}_2}{\partial \sigma_r} &= \frac{\sigma_r}{\alpha_r^2} \sum_{k=1}^{+\infty} D(k)g_6(k)P(T_2 \geq k), \\ \frac{\partial \tilde{F}_3}{\partial \sigma_r} &= \frac{\sigma_r}{\alpha_r^2} \sum_{k=1}^{+\infty} kD(k)g_6(k)P(T_2 \geq k). \end{aligned}$$

Noting that $g_6(z) \geq 0$ in case of $z \in [0, +\infty)$, we get Part 6.3. This completes the proof.

5. NUMERICAL EXPERIMENT

In this section, we illustrate the impact of risks involved in the house price, the interest rate, and the longevity on the annuity payment on the valuation.of reverse mortgage Table 1 below provides, as the standard case, the parametric values involved in the models of house price, interest rate, and lifetime. The values of the parameters $(m_1, m_2, \sigma_1, \sigma_2, \alpha)$ come from the bivariate distribution function of the joint lifetimes, (see Frees et al., 1996). For concreteness of exposition, we assume that the initial age of the male is two years greater than that of the female, that is, $x_0 = y_0 + 2$.

Table 1: Parameters of the standard case

<i>Para</i>	μ_h	σ_h	ρ_{hr}	h_0	t_0	r_0	μ_r	σ_r
<i>Value</i>	0.04	0.08	0.3	\$100	0	0.04	0.06	0.01
<i>Para</i>	α_r	y_0	m_1	m_2	σ_1	σ_2	α	γ
<i>Value</i>	0.5	$x_0 - 2$	85.82	89.40	9.98	8.12	-3.367	0.5

5.1. EFFECT OF HOUSE PRICE ON ANNUITY VALUES

Table 2: Effect of house price on annuity values

x_0	50	55	60	65	70	75	80	85	90	95	100
$\mu_h = 0.01$	1.072	1.429	1.913	2.578	3.497	4.793	6.684	9.586	14.333	22.413	35.177
$\mu_h = 0.025$	1.833	2.281	2.865	3.632	4.659	6.067	8.072	11.087	15.933	24.065	36.790
$\mu_h = 0.04$	3.193	3.708	4.356	5.187	6.276	7.745	9.809	12.877	17.757	25.876	38.505
$\mu_h = 0.055$	5.658	6.124	6.722	7.504	8.545	9.969	11.993	15.02	19.844	27.866	40.329
$\mu_h = 0.07$	10.191	10.271	10.52	10.989	11.755	12.939	14.753	17.594	22.235	30.055	42.273
$\sigma_h = 0.001$	3.247	3.762	4.411	5.242	6.331	7.798	9.861	12.927	17.803	25.916	38.535
$\sigma_h = 0.1$	3.179	3.694	4.342	5.173	6.262	7.731	9.796	12.865	17.746	25.866	38.497
$\sigma_h = 0.2$	3.112	3.626	4.274	5.105	6.194	7.664	9.73	12.802	17.687	25.816	38.459
$\sigma_h = 0.3$	3.046	3.559	4.207	5.038	6.127	7.597	9.665	12.739	17.629	25.767	38.42
$\sigma_h = 0.4$	2.982	3.494	4.141	4.971	6.061	7.531	9.6	12.677	17.572	25.717	38.382
$\rho_{hr} = -1$	3.438	3.955	4.603	5.433	6.519	7.984	10.041	13.099	17.961	26.050	38.638
$\rho_{hr} = -0.5$	3.342	3.858	4.506	5.337	6.425	7.891	9.951	13.013	17.882	25.983	38.587
$\rho_{hr} = 0$	3.248	3.763	4.412	5.243	6.331	7.799	9.862	12.928	17.804	25.916	38.535
$\rho_{hr} = 0.5$	3.156	3.671	4.319	5.151	6.240	7.709	9.774	12.844	17.726	25.850	38.484
$\rho_{hr} = 1$	3.068	3.582	4.229	5.060	6.150	7.619	9.686	12.760	17.649	25.783	38.433
$h_0 = 100$	3.193	3.708	4.356	5.187	6.276	7.745	9.809	12.877	17.757	25.876	38.505
$h_0 = 200$	6.385	7.415	8.712	10.375	12.552	15.489	19.618	25.755	35.514	51.752	77.009
$h_0 = 300$	9.578	11.123	13.068	15.562	18.828	23.234	29.426	38.632	53.272	77.628	115.514
$h_0 = 400$	12.771	14.830	17.424	20.749	25.105	30.979	39.235	51.509	71.029	103.505	154.019
$h_0 = 500$	15.963	18.538	21.780	25.936	31.381	38.723	49.044	64.387	88.786	129.381	192.523

We start the numerical analysis of how the parameters of the house price model and initial age impact the joint annuity and γ annuity, while we fix the other parametric values. Table 2 supports the following analysis.

(a) **Parameter μ_h , the average return of house price:** (1) As the initial age is fixed, the annuity increases significantly with the increase of the average return rate of house price μ_h ; this agrees with the theory established by Proposition 4. Indeed, this is reasonable. After all, the higher average return rate of house price implies the higher average gains that can be obtained by the lender when selling the mortgaged property in future. With the fair valuation principle, the lender is bound to pay enhanced annuities to the annuitants. (2) Compared to the applicant with higher initial age, the mean return has a stronger impact on the annuity of the annuitant with lower initial age. As the initial age increases, the annuity under different average return rates are stabilizing. While the average return rate of house price μ_h remains

unchanged, the annuity increases with the increase of male initial age x_0 .

(b) Parameter σ_h , the volatility in house price: **(1)** When the initial age is fixed, the annuity decreases with the increase of σ_h , (as supported by Proposition 4). This also is reasonable. After all, higher the volatility of house price, greater the market risk. In order to avoid the higher market risk, the lender will have to reduce the amount of annuity. **(2)** As the volatility of house price σ_h remains unchanged, the annuity increases with the increase of the initial age; that is, the older applicant will get better annuity.

(c) Parameter ρ_{hr} , the correlation coefficient between Brown motions: Here, the parameter ρ_{hr} denotes the correlation coefficient between the Brownian motion driving house price and that driving interest rate. **(1)** As the initial age is fixed, the annuity decreases with the increase of the correlation coefficient ρ_{hr} , as proved in Proposition 4. When the Brownian motion driving the house price and the Brownian motion driving the interest rate are completely negatively correlated ($\rho_{hr} = -1$), the annuity reaches the maximum. If they are completely positively correlated ($\rho_{hr} = 1$), the annuity reaches the minimum. In addition, the annuity values under the different correlation coefficients are very close, which implies that the influence of the correlation coefficient ρ_{hr} on the annuity is very weak. As an example, consider a male annuitant with initial age of 50. As the correlation coefficient increases from -1 to 1, the annuity reduces from \$3.438 to \$3.068, the average change rate of annuity w.r.t ρ_{hr} is only 0.185. **(2)** Compared with the older applicants, the annuity of the younger applicant is more susceptible to the correlation coefficient. When the correlation coefficient ρ_{hr} is fixed, the annuity increases with the increase of the initial age, that is, the older applicant will receive a larger annuity.

(d) Parameter h_0 , the initial house price: **(1)** As the initial age is fixed, the annuity increases obviously with the increase of initial house price. The higher initial house price implies that the lender reaps greater benefits while selling the mortgaged property in the future. With the fair valuation, the lender will pay better annuity to the borrower. As is clear from Table 2 that these annuity values get closer to each other in the case of lower initial age, and while the initial age increases these annuity values gradually diverge. It means that the annuity for the older applicant is more affected by the initial house price than that of the younger applicant. **(2)** When the initial house price is fixed, the annuity increases with the increase of initial age; that is, the older applicant will be paid higher annuities every year as other factors, except for the initial age, are same.

Table 3: Effect of interest rate on annuity values

x_0	50	55	60	65	70	75	80	85	90	95	100
$r_0 = 0.01$	3.222	3.744	4.402	5.246	6.354	7.851	9.959	13.101	18.104	26.418	39.304
$r_0 = 0.04$	3.193	3.708	4.356	5.187	6.276	7.745	9.809	12.877	17.757	25.876	38.505
$r_0 = 0.07$	3.162	3.670	4.310	5.128	6.198	7.638	9.657	12.652	17.410	25.337	37.713
$r_0 = 0.1$	3.131	3.633	4.262	5.067	6.118	7.529	9.503	12.426	17.064	24.801	36.931
$r_0 = 0.13$	3.100	3.594	4.214	5.006	6.037	7.419	9.349	12.199	16.717	24.268	36.156
$\mu_r = 0.02$	8.821	8.894	9.139	9.602	10.355	11.515	13.285	16.045	20.550	28.178	40.236
$\mu_r = 0.04$	5.346	5.766	6.319	7.056	8.052	9.428	11.396	14.354	19.083	26.987	39.351
$\mu_r = 0.06$	3.193	3.708	4.356	5.187	6.276	7.745	9.809	12.877	17.757	25.876	38.505
$\mu_r = 0.08$	1.899	2.384	3.011	3.831	4.917	6.393	8.477	11.588	16.557	24.839	37.695
$\mu_r = 0.1$	1.136	1.544	2.099	2.852	3.881	5.309	7.360	10.462	15.470	23.870	36.919
$\sigma_r = 0.001$	3.226	3.741	4.390	5.222	6.311	7.779	9.843	12.910	17.788	25.903	38.526
$\sigma_r = 0.01$	3.193	3.708	4.356	5.187	6.276	7.745	9.809	12.877	17.757	25.876	38.505
$\sigma_r = 0.02$	3.187	3.701	4.349	5.179	6.267	7.734	9.797	12.865	17.744	25.863	38.492
$\sigma_r = 0.03$	3.215	3.727	4.373	5.201	6.287	7.753	9.814	12.879	17.754	25.867	38.491
$\sigma_r = 0.04$	3.276	3.786	4.430	5.256	6.339	7.801	9.858	12.918	17.786	25.888	38.502
$\alpha_r = 0.05$	4.426	4.897	5.504	6.299	7.358	8.804	10.848	13.892	18.719	26.728	39.186
$\alpha_r = 0.15$	3.360	3.895	4.568	5.430	6.557	8.073	10.191	13.318	18.244	26.369	38.942
$\alpha_r = 0.35$	3.204	3.722	4.375	5.213	6.310	7.791	9.872	12.963	17.869	26.009	38.641
$\alpha_r = 0.55$	3.192	3.706	4.353	5.184	6.271	7.737	9.797	12.860	17.733	25.845	38.470
$\alpha_r = 0.75$	3.191	3.704	4.350	5.177	6.260	7.720	9.771	12.820	17.671	25.757	38.364

5.2. EFFECT OF INTEREST RATE ON ANNUITY VALUES

This subsection provides the numerical analysis of how the interest rate influences the annuity A ; again, we keep the other parametric values fixed as the standard case. Table 3 portrays the following analysis.

(a) Parameter r_0 , the initial interest rate: Keeping the initial age fixed, the annuity decreases slightly with the increase of the initial interest rate r_0 , that is, the higher the initial interest rate the lower the annuity payment. The initial interest rate r_0 has a less influence on the annuity of younger applicants than those of older applicants. In general, the initial interest rate r_0 weakly affects the annuity payments. When the initial interest rate is fixed, the annuity increases with the increase of the initial age.

(b) Parameter μ_r , the average reversion level of interest rate: **(1)** Fixing the initial age, the annuity decreases with the increase of μ_r . The average reversion level μ_r impacts more the annuity of younger borrowers than that of older borrowers. Generally, μ_r has a significant effect on the annuity. **(2)** With fixed average reversion level μ_r , the annuity increases with the increase of initial age.

(c) Parameter σ_r , the volatility of interest rate: The volatility σ_r of interest rate in Table 2 takes five values: 0.001, 0.01, 0.02, 0.03, 0.04. The corresponding annuity values with different σ_r almost coincides with each other under fixed initial male age. This indicates that the volatility of interest rate has weak effect on the annuity while the volatility of interest rate is at a low level. It is known from the

original data that: while the initial age kept fixed, the annuity decreases slightly, with the increase of volatility rate σ_r in case that $\sigma_r \leq \sigma_h \rho_{hr} \alpha_r = 0.012$ (it is consistent with Proposition 6); and the annuity increases slightly with the increase of volatility rate σ_r in case that $\sigma_r \geq \sigma_h \rho_{hr} \alpha_r = 0.012$. Proposition 6 shows that the annuity amount decreases with increase of σ_r in case of $\sigma_r \leq \sigma_h \rho_{hr} \alpha_r$. It implies that the valuation models can be used to determine the annuity payments as long as the volatility of interest rate σ_r can be controlled by the quantity $\sigma_h \rho_{hr} \alpha_r$ (irrespective of the volatility rate).

(d) Parameter α_r , the reversion speed of interest rate: As the initial age is fixed, the annuity decreases slowly with the increase of the reversion speed α_r . While the reversion speed of interest rate is more than 0.75, the annuity is basically stable. To further validate this view, the α_r is relaxed to 2.05. For the applicant with the initial age of 50 years old, the annuity is \$3.192 with $\alpha_r = 0.85$, and the annuity is \$3.200 with $\alpha_r = 2.05$. The change of annuity is very small. The α_r has a greater impact on the annuity of younger applicants than that of older applicants. While α_r remains unchanged, the annuity increases with the increase of the initial age, that is, the older applicants will receive larger annuity.

5.3. EFFECT OF JOINT LIFETIME ON ANNUITY VALUES

Table 4: Effect of joint lifetime on annuity values

x_0	50	55	60	65	70	75	80	85	90	95	100
$m_1 = 69$	3.69	4.337	5.134	6.123	7.39	9.129	11.757	16.083	23.491	35.692	52.474
$m_1 = 79$	3.401	3.976	4.701	5.626	6.832	8.45	10.743	14.287	20.304	30.641	45.906
$m_1 = 89$	3.081	3.567	4.176	4.955	5.972	7.339	9.245	12.039	16.368	23.391	34.514
$m_1 = 99$	2.658	3.05	3.534	4.142	4.918	5.929	7.269	9.089	11.637	15.412	21.528
$m_1 = 109$	2.205	2.515	2.891	3.356	3.935	4.663	5.575	6.714	8.141	10.027	12.859
$\sigma_1 = 6$	3.138	3.66	4.342	5.259	6.529	8.339	11.001	15.037	21.627	33.609	51.684
$\sigma_1 = 8$	3.176	3.699	4.37	5.249	6.426	8.045	10.359	13.846	19.529	29.443	45.100
$\sigma_1 = 10$	3.193	3.708	4.356	5.186	6.275	7.742	9.804	12.868	17.741	25.843	38.443
$\sigma_1 = 12$	3.191	3.691	4.312	5.095	6.106	7.451	9.316	12.041	16.239	22.884	32.886
$\sigma_1 = 14$	3.175	3.658	4.251	4.991	5.935	7.179	8.882	11.327	14.974	20.478	28.458
$\alpha = -5$	3.223	3.736	4.378	5.195	6.260	7.690	9.689	12.640	17.335	25.322	38.180
$\alpha = -4$	3.205	3.720	4.365	5.190	6.267	7.717	9.751	12.768	17.573	25.645	38.376
$\alpha = -3$	3.185	3.700	4.350	5.186	6.283	7.765	9.851	12.953	17.879	26.021	38.581
$\alpha = -2$	3.160	3.677	4.334	5.184	6.310	7.842	10.006	13.224	18.286	26.464	38.797
$\alpha = -1$	3.134	3.652	4.316	5.185	6.350	7.954	10.236	13.615	18.832	26.987	39.023

In this subsection we discuss the impact on annuity value by joint lifetime, including the parameters m_1, σ_1 and α . Table 4 shows that:

(a) Parameter m_1 , the modal value of male lifetime: Since the parameters m_1 and m_2 in the respective Gompertz distributions have the same function, we consider only the parameter m_1 for illustration. The reader can draw the corresponding

analysis for m_2 . Part 1 of Table 4 shows that: **(1)** As the initial age is fixed, the annuity decreases with the increase of m_1 . The annuity for the older applicants is more sensitive to the change of m_1 than that for the younger applicants. **(2)** As m_1 is fixed, the annuity increases with the increase of the initial age, that is, older the applicant is, higher the annuity he will receive. **(3)** Smaller the parameter m_1 becomes, greater the impact on annuity the parameter m_1 will exert.

(b) Parameter σ_1 , the dispersion coefficient of male lifetime: The parameter σ_1 shares the same function with σ_2 . We shall treat only the parameter σ_1 for illustration purpose. With fixed initial age, the annuity shows two different trends with the change of σ_1 : **(1)** When the initial age is at a lower level (say, $x_0 = 50, 55, 60$), the annuity increases first and then decreases with the increase of σ_1 . **(2)** When the initial age is at a higher level (say, $x_0 = 70, 75, \dots, 100$), the annuity decreases with the increase of σ_1 . The annuity of an older applicant is more strongly affected by σ_1 . As σ_1 remains unchanged, the annuity increases with the increase of initial age x_0 .

(c) Parameter α , the dependence between male and female lifetime: As the initial age remains fixed, the change of annuity shows three different trends: **(1)** when the initial age is at a lower level (say $x_0 = 50, 55, 60$), the annuity decreases with the increase of α ; **(2)** when the initial age is at the middle level (say $x_0 = 65$), the annuity decreases first and then increases with the increase of α ; **(3)** as the initial age is at a higher level (say $x_0 = 70, 75, \dots, 100$), the annuity increases with the increase of α . **(4)** In general, the impact of α on older applicants' annuities is significantly stronger than that for young applicants' annuities. As α is fixed, the annuity increases with the increase of the applicant age, that is, the older the applicants are, the greater annuity they will be paid.

5.4. EFFECT OF INITIAL AGE ON ANNUITY VALUES

Table 5: Effect of joint lifetime on annuity values

(x_0, y_0)	(55,50)	(60,55)	(65,60)	(70,65)	(75,70)	(80,75)	(85,80)	(90,85)	(95,90)	(100,95)
Annuity	3.490	4.084	4.837	5.811	7.099	8.863	11.410	15.374	22.032	33.089
(x_0, y_0)	(50,55)	(55,60)	(60,65)	(65,70)	(70,75)	(75,80)	(80,85)	(85,90)	(90,95)	(95,100)
Annuity	3.590	4.206	4.995	6.028	7.418	9.358	12.184	16.502	23.417	34.329
(x_0, y_0)	(60,50)	(65,55)	(70,60)	(75,65)	(80,70)	(85,75)	(90,80)	(95,85)	(100,90)	
Annuity	3.653	4.293	5.103	6.142	7.505	9.369	12.110	16.572	24.357	
(x_0, y_0)	(50,60)	(55,65)	(60,70)	(65,75)	(70,80)	(75,85)	(80,90)	(85,95)	(90,100)	
Annuity	3.846	4.537	5.430	6.606	8.193	10.399	13.587	18.530	26.675	
(x_0, y_0)	(65,50)	(70,55)	(75,60)	(80,65)	(85,70)	(90,75)	(95,80)	(100,85)		
Annuity	3.811	4.492	5.342	6.417	7.816	9.756	12.723	17.726		
(x_0, y_0)	(50,65)	(55,70)	(60,75)	(65,80)	(70,85)	(75,90)	(80,95)	(85,100)		
Annuity	4.086	4.852	5.837	7.121	8.816	11.119	14.521	20.137		

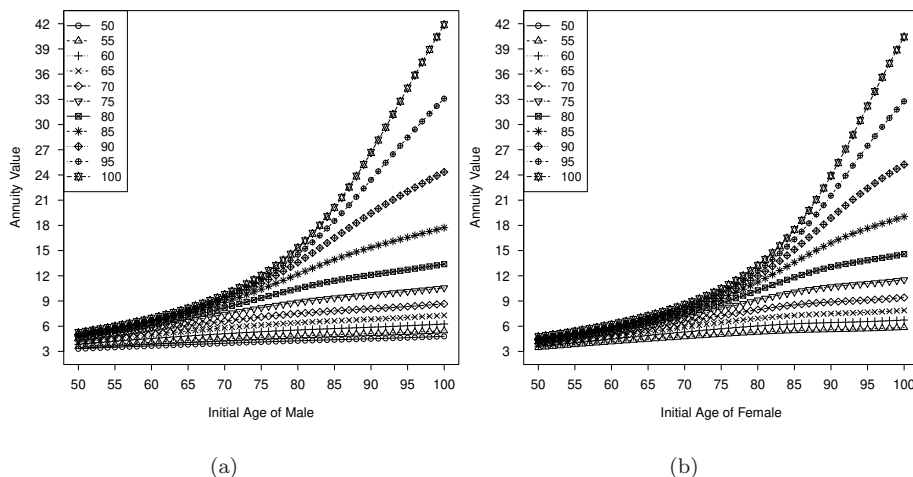


Figure 1: Effect of the initial age on annuity values with dependent assumption

In order to evaluate the effect of the initial ages x_0, y_0 on the annuity, Figure 1 presents the scatter plot of the joint and γ annuity. Figure 1(a) shows that: Keeping the initial age y_0 ($y_0 = 50, 55, \dots, 100$) of the female annuitant fixed, the annuity value increases with the increase of the male applicant’s initial age x_0 , (the initial age of male applicant is between 50 and 100). The larger the initial age of female is, the stronger effect it will exert on the annuity amount.

Figure 1(b) reveals that: When the initial age x_0 ($x_0 = 50, 55, \dots, 100$) of male applicant is fixed, the annuity amount increase with the increase of the female’s initial age y_0 , (the initial age of female is taken to be between 50 and 100). The larger the initial age of male is, the stronger influence it will produce on the the annuity value.

Comparing Figure 1(a) with Figure 1(b), it is not difficult to find that: the annuity value is approximately symmetrical, though not completely symmetrical in the initial x_0 and y_0 . When the age difference between the male and female annuitants is the same, different influences are made on the annuity value in the case with the initial age y_0 of female is greater than that of male and *vice versa*. Especially, when $y_0 - x_0 = d$ (d is a positive constant) the annuity value is greater than that when $x_0 - y_0 = d$. To validate this claim, we present the annuity values with the age difference of 5, 10 and 15 years (see Table 5).

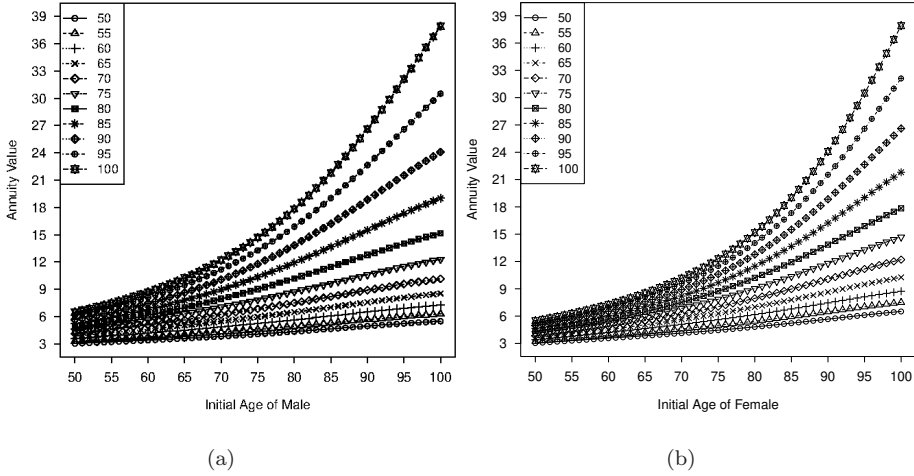


Figure 2: Effect of the initial age on annuity values with independent assumption.

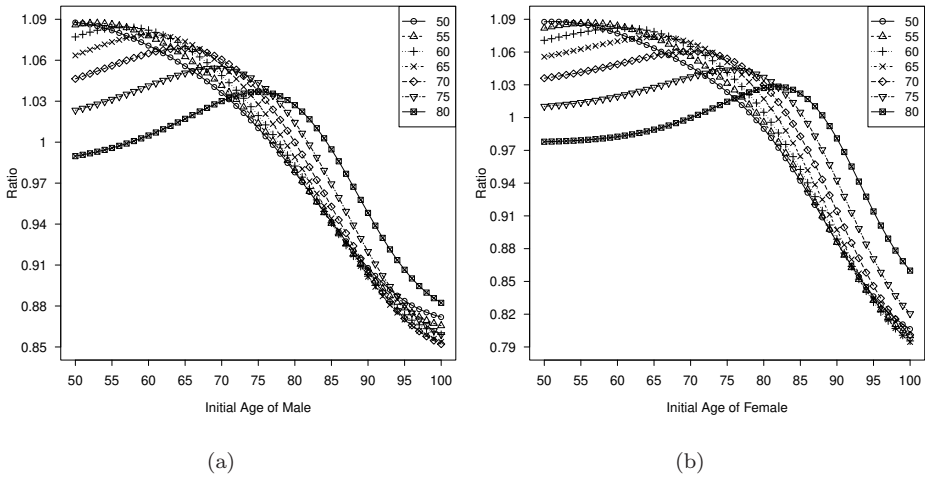


Figure 3: Effect of dependence on annuity values.

5.5. EFFECT OF DEPENDENCE ON ANNUITY VALUES

In order to evaluate the effect of the dependence of joint-lifetime on the annuity, we further compute the annuity value with the assumption of independence between the husband's lifetime and the wife's lifetime. Also, we assume that the husband is two

years older than his wife (i.e., $x_0 = y_0 + 2$), and the parameters of joint lifetime model take the following values: $m_1 = 86.38$, $m_2 = 92.17$, $\sigma_1 = 9.83$, $\sigma_2 = 8.11$ (see Frees et al., 1996). In Figure 2(a) (respectively, Figure 2(b)), the initial age of female (respectively, male) is fixed as 50, 55, ...,100. With these assumptions, Figure 2(a) (respectively, Figure 2(b)) shows how the annuity values vary with the initial age of male (respectively, female). Comparing Figure 2(a) (respectively, Figure 2(b)) with Figure 1(a) (respectively, Figure 1(b)), it is easy to note that the annuity values with the dependence assumption are not same with those with independence assumption. Figure 3(a) and 3(b) present the multiple scatter plot of the ratio of dependent to independent. Ratios that are greater than one indicate that annuity values calculated with dependence assumption of lives are larger than those calculated with independence assumption. Annuity values in average increase approximately 4.5%, and the greatest increment in annuity value approaches 9%.

5.6. EFFECT OF OTHER PARAMETERS ON ANNUITY VALUES

Table 6: Effect of other parameters on annuity values

x_0	50	55	60	65	70	75	80	85	90	95	100
$t_0 = 0$	3.193	3.708	4.356	5.187	6.276	7.745	9.809	12.877	17.757	25.876	38.505
$t_0 = 3$	3.004	3.489	4.099	4.881	5.907	7.290	9.236	12.133	16.752	24.475	36.573
$t_0 = 6$	2.827	3.283	3.857	4.593	5.558	6.860	8.692	11.420	15.772	23.057	34.487
$t_0 = 9$	2.660	3.089	3.629	4.322	5.230	6.455	8.179	10.746	14.843	21.702	32.466
$t_0 = 12$	2.503	2.907	3.415	4.067	4.922	6.074	7.696	10.112	13.967	20.422	30.553
$\gamma = 1/2$	3.193	3.708	4.356	5.187	6.276	.745	9.809	12.877	17.757	25.876	38.505
$\gamma = 2/3$	3.129	3.615	4.221	4.990	5.987	7.314	9.157	11.868	16.162	23.349	34.755
$\gamma = 3/4$	3.099	3.571	4.157	4.897	5.851	7.116	8.862	11.421	15.468	22.262	33.141
$\gamma = 4/5$	3.080	3.545	4.119	4.843	5.773	7.002	8.694	11.168	15.079	21.657	32.242
$\gamma = 1$	3.010	3.444	3.975	4.638	5.481	6.582	8.082	10.260	13.701	19.534	29.088

(a) **Parameter t_0 , the delay time in selling the pledged house:** Table 6 shows that: As the initial age is fixed, the annuity slowly decreases with the increase of the delay time of selling the pledged house, that is, the applicant will receive less annuities, while the delay time of selling house becomes longer. The impact of delay time on the annuity is complex. While the other parameters in the valuation model change, the annuity may also increase with the increase of delay time. The delay time more affects the annuity of the older borrowers than those of the younger borrowers. However, the delay time has a weak impact on the annuity.

While the delay time of selling house remains unchanged, the annuity increases with the increase of the initial age. Facing the different delay time, the applicant with different initial age may get the approximately same annuity. For example, with no delay, the applicant with 53 years old obtains annuity amount \$3.488 every year; and

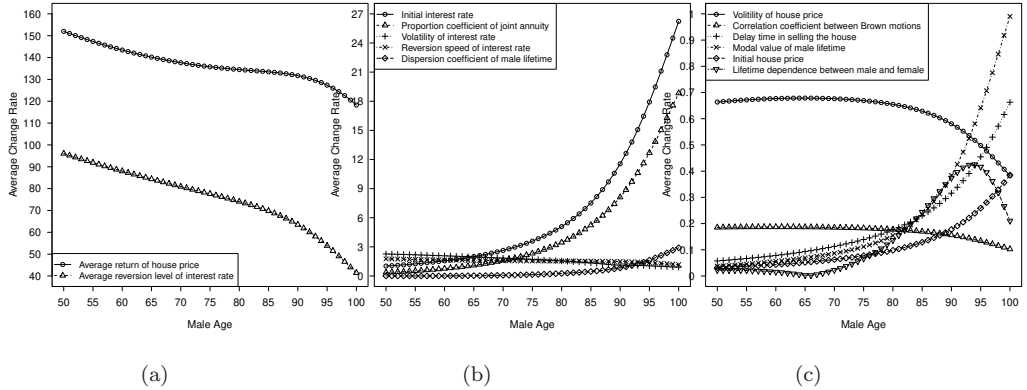


Figure 4: Average change rate for all parameters.

with three years delay, the applicant with 55 years old gets annuity amount \$3,489. These two annuity amounts are almost same.

(b) Parameter γ , the proportion coefficient of joint annuity: In order to evaluate the effect of the γ on the annuity, Table 6 presents the joint and γ annuity values. It shows that: As the initial age is fixed, the annuity payment decreases slowly with the increase of γ . The effect of γ on the annuity for older applicants is stronger than that of younger applicants. As γ remains unchanged, the annuity increases with the increase of initial age.

5.7. COMPARISON OF ALL PARAMETERS

The average return rate of house price μ_h exerts a dominating impact on the annuity. The rates of μ_h are between 118 and 152, (see Figure 4(a)), much much larger than that of other parameters. The μ_r has the second strongest impact on the annuity, and the average change rate of μ_r is between 41 and 97 (see Figure 4(a)), followed by r_0 , γ , σ_r , α_r and σ_1 (see Figure 4(b)). The average change rates of other parameters are between 0 and 1 (see Figure 4(c)), which implies that the other parameters have very weak influence on the annuity value. The parameters μ_h, μ_r, σ_r and α_r affect the annuity of younger borrower more than that of older borrower. The parameters $r_0, \gamma, \sigma, t_0, m_1$ and h_0 have much greater influence on the annuity of an older applicant than that of a young applicant. The parameters σ_h and ρ_{hr} exert almost same effect on the annuity of younger annuitant with that of older annuitant in case that the initial age of male is between 50 and 100.

6. CONCLUSION

Based on the valuation principle of balance between expected gain and expected payment, this paper builds a valuation model for reverse mortgage applied to a couple-annuitants. We obtain the analytic valuation formula for the lump sum, joint annuity, increasing (decreasing) annuity, level annuity of reverse mortgage, and derive the valuation equation that the variable payment annuities satisfy. Then, we discuss the monotonicity of the lump sum, annuity, and annuity payment factors with respect to the parameters associated with the home price and the interest rate model. Finally, we present a sensitivity analysis for the joint and γ annuity to the parameters associated with the home price, interest rate and mortality model. Corresponding to the average change rate, we compare the impact of the said parameters on the joint and γ annuity. The numerical results show that the average return of home price μ_h exerts a dominating influence on the joint and γ annuity, followed by the mean reversion level of interest rate μ_r . Next to μ_h and μ_r , the initial interest rate r_0 , the proportion coefficient of joint annuity γ , the volatility of interest rate σ_r , the reversion speed of interest rate α_r and the dispersion coefficient of male lifetime σ_1 also impact on the joint annuity value. The rest parameters $\sigma_h, \rho_{hr}, t_0, m_1, h_0$ and α have very weak influence on the annuity value. Meanwhile, the initial age of male x_0 and that of female y_0 produce asymmetrical effect on the joint and γ annuity. Remarkably, the dependence of joint-lifetime significantly affect the joint annuity value. Annuity values in average increase approximately 4.5%, and the greatest increment of annuity value approaches 9% in case that the initial age of male and female are both less than 80.

However, it should be noted that the average change rates depend on the ranges of the parameters. Once the ranges of parameters change, it may change the evaluation results for the parameters. Thus, the appropriate range should be chosen in order to make objective evaluations of the parameters' effects. Moreover, the model selection of the house price, interest rate and lifetime will directly affect the final valuation results. Therefore, it is suggested to collect the data of house price, interest rate and population in the area that the reverse mortgage product covers, and model the special house price, interest rate and lifetime model based on the collected historical data before employing the valuation model. This will be one of our future research works. It will offer reasonable valuation of reverse mortgage, and help the lender of reverse mortgage to better manage and control risk.

APPENDIX A: DERIVATION OF PROPOSITION 1

This appendix gives the derivations of Proposition 1.

Proof. It is easy to get that $Y(t)$ follows the normal distribution with the mean 0 and variance

$$\sigma_y^2(t) = \frac{\sigma_r^2}{\alpha_r^2}t + \frac{\sigma_r^2}{2\alpha_r^3} [1 - (2 - e^{-\alpha_r t})^2]. \tag{A.1}$$

Noting that $Cov(dW_h(t), dW_r(t)) = \rho_{hr}dt$, we can obtain the covariance between $W_h(t)$ and $Y(t)$, that is

$$Cov(W_h(t), Y(t)) = \sigma_r \rho_{hr} \frac{1}{\alpha_r} \left(t + \frac{1}{\alpha_r} e^{-\alpha_r t} - \frac{1}{\alpha_r} \right).$$

Thus the correlation coefficient between $W_h(t)$ and $Y(t)$ denoted by $\rho(t)$ is

$$\rho(t) = \frac{\sigma_r \rho_{hr}}{\alpha_r \sigma_y(t) \sqrt{t}} \left(t + \frac{1}{\alpha_r} e^{-\alpha_r t} - \frac{1}{\alpha_r} \right). \tag{A.2}$$

Since the joint distribution of $(W_h(t), Y(t))$ follows the two dimension normal distribution with the correlation coefficient $\rho(t)$ given by Eq. (A.2), we have with Eq. (A.1)

$$E \{ \exp [\sigma_h W_h(t) - Y(t)] \} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(\sigma_h x - y) f(x, y) dx dy,$$

where

$$f(x, y) = \frac{1}{2\pi\sigma_y(t)\sqrt{t(1-\rho^2(t))}} \exp \left\{ -\frac{1}{2(1-\rho^2(t))} S_{xy} \right\},$$

and

$$S_{xy} = \left(\frac{x}{\sqrt{t}} \right)^2 - 2\rho(t) \frac{x}{\sqrt{t}} \frac{y}{\sigma_y(t)} + \left(\frac{y}{\sigma_y(t)} \right)^2.$$

Making the transform $u = \frac{x}{\sqrt{t}}$ and $v = y - \rho(t)\sigma_y(t)u$, then we obtain

$$\begin{aligned} E \{ \exp [\sigma_h W_h(t) - Y(t)] \} &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left\{ \left[\sigma_h \sqrt{t} - \rho(t)\sigma_y(t) \right] u - v \right\} g(u, v) du dv \\ &= \exp \left[\frac{1}{2} \sigma_h^2 t - \rho(t)\sigma_y(t)\sigma_h \sqrt{t} + \frac{1}{2} \sigma_y^2(t) \right], \end{aligned} \tag{A.3}$$

where

$$g(u, v) = \frac{1}{2\pi\sigma_y(t)\sqrt{1-\rho^2(t)}} \exp \left\{ -\frac{1}{2} \left[u^2 + \frac{v^2}{\sigma_y^2(t)(1-\rho^2(t))} \right] \right\}.$$

Noting $\{N(t), t \geq 0\}$ and $\{J_i, i \geq 1\}$ are independent, and J_i follows the normal distribution with mean μ_J and variance σ_J^2 , we obtain

$$E \left[e^{\sum_{i=1}^{N(t)} J_i} \right] = \exp(k_h \lambda_h t), \tag{A.4}$$

where $k_h := \exp(\mu_J + \frac{1}{2}\sigma_J^2) - 1$ as defined in the interest rate model.

Let

$$m_1 := \int_0^t \mu_h(s) ds - \left(\frac{1}{2} \sigma_h^2 + \lambda_h k_h \right) t,$$

$$m_2 := \mu_r t + \frac{1}{\alpha_r} (\mu_r - r_0) (e^{-\alpha_r t} - 1).$$

It is easy to get

$$h(t) = h_0 \exp \left[m_1 + \sigma_h W_h(t) + \sum_{i=1}^{N(t)} J_i \right], \quad (\text{A.5})$$

$$\int_0^t r(u) du = m_2 + Y(t). \quad (\text{A.6})$$

Noting that $\sigma_h W_h(t) - Y(t)$ is independent of $\sum_{i=1}^{N(t)} J_i$, with Eq. (A.3)-(A.6), we have

$$\begin{aligned} E[h(t)d(t)] &= h_0 e^{m_1 - m_2} E \left[e^{\sigma_h W_h(t) - Y(t)} \right] E \left[e^{\sum_{i=1}^{N(t)} J_i} \right] \\ &= G(t) D(t), \end{aligned}$$

where $G(t)$ and $D(t)$ is respectively defined by Eq. (14) and (15). We obtain the Proposition 1.

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