Dynamic Systems and Applications, **27**, No. 4 (2018), 927-942 ISSN: 1056-2176

STABILIZATION OF VECTOR THIRD-ORDER SYSTEMS VIA A STATIC OUTPUT-FEEDBACK CONTROL WITH STABILIZATION DELAYS

YALI DONG[1] AND JING HAO[2]

[1,2]School of Science
Tianjin Polytechnic University
Tianjin, 300387, P.R. CHINA

ABSTRACT: This paper addresses the problem of stabilization of the vector third-order systems via multi-delay output feedback control. Two main approaches are suggested for the Lyapunov stability analysis via simple Lyapunov-Krasovskii functionals. The first one is based on the application of Writinger's inequality to establish the delay-dependent sufficient conditions of asymptotically stabilization for a class of vector third-order systems. Another approach is the neutral type model transformation of the system to lead to the stabilization criteria for a class of vector third-order systems. Both have obtained the sufficient conditions for stabilization vector third-order systems. A numerical example is provided to demonstrate the effectiveness of the proposed methods.

AMS Subject Classification: 93D15, 93D20, 34D20

Key Words: stabilization by using delay, time-delay systems, linear matrix inequalities, Lyapunov-Krasovskii method

Received: June 30, 2018; **Accepted:** October 13, 2018;
Published: December 20, 2018 **doi:** 10.12732/dsa.v27i4.14
Dynamic Publishers, Inc., Acad. Publishers, Ltd. https://acadsol.eu/dsa

1. INTRODUCTION

Time delays are often encountered in various practical systems such as chemical processes, neural networks and long transmission lines in pneumatic systems [1-2]. It has been shown that the existence of time delays may lead to oscillation, divergence, instability, greatly increasing the difficulty of stability analysis and control design. Due to their success in practical applications and importance in theory development, time-delay systems received more and more attention during the past decades. Refs. [1,5,8,10,12] investigated the problem of stability for systems with time delay. Several papers address the problem of control design and the stabilization for systems with time-delays [3,4,6,7,9,11]. For some systems, the presence of delay has a stabilizing effect [3-5]. The Lyapunov-Krasovskii functional method was used to study the problems of stability and stabilization for some important classes of systems in recent years. Zhang et al. [8] considered the robust stability for a class of linear systems with interval time-varying delay and nonlinear perturbations. In [9], the exponential stabilization and L_2-gain for uncertain switched nonlinear systems with interval time-varying delay were investigated. In [10], the problem of robust exponential stability for neutral systems with interval time-varying delays and nonlinear perturbations was investigated. Based on Lyapunov stability theory, some new exponential delay-dependent stability conditions were derived. Borne et al. [11] proposed a model transformation-based method for stabilization by using constant artificial delay of the scalar second-order system that models inverted pendulum, and the stability conditions were given regarding inequalities on the system coefficients.

It is also well known that static output feedback controllers have advantages over observer-based controllers in the presence of uncertainties in the system matrices and/or uncertain input/output delays, where the observer-based design becomes complicated. Some important classes of systems can not be stabilized by a static output-feedback, such as inverted pendulums, oscillactors, double integrators, three integrators. However, these systems can be stabilized by inserting artificial multiple delays in the feedback. Fridman et al. [5] gave an example to show that the system which cannot be stabilized by static output feedback control, can be stabilized by static output feedback control with time delay. In [6], the static output feedback sliding mode control was studied via an artificial stabilizing delay. In [7], the idea of using artificial delay was applied for delay-induced consensus in multi-agent systems.

In this paper, the problem of delay-dependent asymptotical stabilization for a class of vector third-order systems is investigated by a static output-feedback control with time delays. Based on Lyapunov-Krasovskii functional method and linear matrix inequality technique (LMI), novel sufficient conditions, which guarantee that the class of vector third-order systems are asymptotically stabilizable, are established and expressed in terms of linear matrix inequality (LMI). The controllers gains may be found from the gains of the corresponding state feedback controllers. A numerical example is given to show the effectiveness of the proposed method.

This paper is organized as follows. In Section 2, the system description and some preliminaries are given. By Wirtinger's inequality-based approach, the delay-induced stability conditions are given in Section 3. Using Model transformation-based approach, the delay-induced stability conditions are established in Section 4. Section

5 gives an example to show the performances of our method. Finally, Section 6 concludes the paper.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following vector third-order system

$$\dddot{y} = A_1 y(t) + A_2 \dot{y}(t) + A_3 \ddot{y}(t) + Bu(t - h_1), \tag{1}$$

where $y(t) \in R^n$ is the measurement, $u(t) \in R^k$ $(k \leq n)$ is the control input, $A_1, A_2, A_3 \in R^{n\times n}$ are system matrices, $h_1 \geq 0$ is the input delay.

For system (1), we consider the following static output-feedback controller

$$u(t) = K_1 y(t) + K_2 y(t-h) + K_3 y(t-\bar{h}), \tag{2}$$

where $K_1, K_2, K_3 \in R^{n\times n}$ are controller gains. $h > 0$ and $\bar{h} > 0$ are stabilizing delays respectively. Let $h_2 = h_1 + h > 0, h_3 = h_1 + \bar{h} > 0$.

Denoting $x(t) = (x_1^T(t), x_2^T(t), x_3^T(t))^T$, $x_1(t) = y(t)$, $\;x_2(t) = \dot{y}(t)$, $\;x_3(t) = \ddot{y}(t)$, we present the closed-loop system (1), (2) as

$$\dot{x}(t) = \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_n \\ A_1 & A_2 & A_3 \end{bmatrix} x(t) + \sum_{i=1}^{3} \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix} K_i x_1(t-h_i). \tag{3}$$

For the delay h_1, we will consider two case:

Case 1: h_1 is constant and known;

Case 2: $h_1 = h_1(t)$ is piecewise-continuous in time and bounded $h_1(t) \in [0, h_{1M}]$, $h_2(t) = h + h_1(t) \in [h, h + h_{1M}]$, $h_3(t) = \bar{h} + h_1(t) \in [\bar{h}, \bar{h} + h_{1M}]$, where h_{1M} is known.

Let

$$\bar{A} = \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_n \\ A_1 & A_2 & A_3 \end{bmatrix}, Y = \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}.$$

Throughout the paper we given the following assumption.

Assumption 1. The pair $\left(\bar{A}, Y\right)$ is stabilizable.

In Case 1, under Assumption 1, there exist $k \times n$ gains $\bar{K}_1$, $h_2\bar{K}_2$ and $\frac{h_3^2}{2}\bar{K}_3$ such that the following matrix is Hurwitz:

$$\bar{D}_1 = \bar{D}_1(h_2, h_3) = \bar{A} + Y \begin{bmatrix} \bar{K}_1 & h_2\bar{K}_2 & \frac{h_3^2}{2}\bar{K}_3 \end{bmatrix}. \tag{4}$$

Assuming $h_1 = O(h_2^2) = O(h_3^3)$ and noticing

$$\begin{aligned} x_1(t) &= x_1(t-h_1) + O(h_1), \\ h_2 x_2(t) &= x_1(t-h_1) - x_1(t-h_2) + O(h_2^2), \\ \tfrac{h_3^2}{2} x_3(t) &= \tfrac{h_3 - h_2}{h_2} x_1(t-h_1) + x_1(t-h_3) - \tfrac{h_3}{h_2} x_1(t-h_2) + O(h_3^3), \end{aligned}$$

the system $\dot{x}(t) = \bar{D}_1 x(t)$ can be written as

$$\dot{x}(t) = \bar{A}x(t) + \sum_{i=1}^{3} YK_i x_1(t-h_i) + O(h_3^3),$$

where

$$K_1 = \bar{K}_1 + \bar{K}_2 + \tfrac{h_3-h_2}{h_2}\bar{K}_3, \quad K_2 = -\bar{K}_2 - \tfrac{h_3}{h_2}\bar{K}_3, \quad K_3 = \bar{K}_3. \tag{5}$$

Hence, under Assuption 1, the system (3) is as asymptotically stable for small enough h_2 and h_3.

From (5), we have $\bar{K}_3 = K_3, \bar{K}_1 = K_1 + K_2 + K_3, \bar{K}_2 = -K_2 - \frac{h_3}{h_2}K_3$. Substituting the latter into $\bar{D}_1$, we get the following Hurwitz matrix

$$\bar{D}_1 = \bar{D}_1(h_2, h_3) = \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_n \\ A_1 + B\sum\limits_{i=1}^{3} K_i & \Upsilon & A_3 + \frac{h_3^2}{2}BK_3 \end{bmatrix}, \tag{6}$$

where $\Upsilon = A_2 - h_2BK_2 - h_3BK_3$.

In this paper, for Case 1 given $K_i, h_i (i = 1, 2, 3)$, we assume that $\bar{D}_1$ defined by (6) is Hurwitz. In Case 2, K_1, K_2 and K_3 may be found from (5), where $\bar{K}_1, \bar{K}_2$ and $\bar{K}_3$ are such that matrix $\bar{D}_1$ defined by (4) is Hurwitz. In both cases, we will derive sufficient stability conditions for the system (3).

We present below some useful lemmas.

Lemma 1 [12]. Let $z(t) : (a, b) \to R^n$ be absolutely continuous with $\dot{z} \in L_2(a, b)$ and $z(a) = 0$ or $z(b) = 0$. Then for any $n \times n$ matrix $W > 0$ the following inequality holds:

$$\int_a^b z^T(\xi)Wz(\xi)d\xi \le \frac{4(b-a)^2}{\pi^2}\int_a^b \dot{z}^T(\xi)W\dot{z}(\xi)d\xi. \tag{7}$$

Lemma 2 [13]. Suppose that $x(t) \in R^n$ and $\eta \in R$. For any positive definite matrix P the following inequalities hold:

$$-\eta\int_{t-\eta}^{t} x^T(s)Px(s)ds \le -\left(\int_{t-\eta}^{t} x^T(s)ds\right) P \left(\int_{t-\eta}^{t} x(s)ds\right),$$

$$\begin{aligned} -\tfrac{\eta^2}{2}\int_{-\eta}^{0}\int_{t+\beta}^{\beta} x^T(s)Px(s)dsd\beta \le \; & -\left(\int_{-\eta}^{0}\int_{t+\beta}^{\beta} x^T(s)dsd\beta\right) P \\ & \times \left(\int_{-\eta}^{0}\int_{t+\beta}^{\beta} x(s)dsd\beta\right), \end{aligned}$$

$$-\frac{\eta^3}{6}\int_{-\eta}^{0}\int_{\beta}^{0}\int_{t+\lambda}^{t} x^T(s)Px(s)dsd\beta d\lambda \le -\vartheta^T P\vartheta,$$

where $\vartheta = \int_{-\eta}^{0}\int_{\beta}^{0}\int_{t+\lambda}^{t} x(s)dsd\beta d\lambda$.

Lemma 3 [14]. Denote $G = \int_b^a f(s)x(s)ds$, where $a \leq b, f : [a, b] \to [0, +\infty), x(s) \in R^n$ and the integration concerned is well defined. Then for any $n \times n$ matrix $R > 0$ the following inequality holds:

$$G^T RG \leq \int_a^b f(\theta)d\theta \int_a^b f(s)x^T(s)Rx(s)ds. \tag{8}$$

Lemma 4 [15]. Consider the integral equation

$$z(t) = \sum_{i=0}^{m} \int_{-h_i}^{0} F_i(s)z(t+s)ds, \tag{9}$$

where $z(t) \in R^n, F_i(s) \in R^{n\times n}$ is integrable. If there exists a continuous functional $V(\varphi)$ such that $V(z_t)$ is differentiable in $t \geq 0$ and the following conditions hold

$$\alpha_1 \sum_{i=1}^{m} \int_{-h_i}^{0} |\varphi(s)|^2 ds \leq V(\varphi) \leq \alpha_2 \sum_{i=1}^{m} \int_{-h_i}^{0} |\varphi(s)|^2 ds, \tag{10}$$

$$\dot{V}(z_t) \leq -\beta \sum_{i=1}^{m} \int_{-h_i}^{0} |z(t+s)|^2 ds, \tag{11}$$

with some positive constants $\alpha_1 \leq \alpha_2$ and β, then (9) is exponentially stable.

3. STABILITY ANALYSIS VIA WIRTINGER'S INEQUALITY

In this section we consider (3) in Case 2. We have

$$\begin{aligned} x_1(t-h_2) &= x_1(t) - h_2x_2(t) + \delta_2(t), \\ \delta_2(t) &= \textstyle\int_{t-h_2}^{t} [x_2(t) - x_2(s)]ds. \end{aligned} \tag{12}$$

Since $\dot{x}_1(t) = x_2(t)$, the term $x_1(t-h_1)$ can be presented as

$$\begin{aligned} x_1(t-h_1) &= x_1(t) + \delta_1(t), \\ \delta_1(t) &= -\textstyle\int_{t-h_1}^{t} x_2(s)ds, \end{aligned} \tag{13}$$

and

$$\begin{aligned} x_1(t-h_3) &= x_1(t) - h_3x_2(t) + \tfrac{h_3^2}{2}x_3(t) + \delta_3(t), \\ \delta_3(t) &= -\tfrac{h_3^2}{2}x_3(t) + h_3x_2(t) - [x_1(t) - x_1(t-h_3)]. \end{aligned} \tag{14}$$

Substituting (12), (13) and (14) into (3), the following system can be obtained:

$$\dot{x}(t) = \bar{D}_1(h_2, h_3)x(t) + \sum_{i=1}^{3} YK_i\delta_i(t). \tag{15}$$

Theorem 1. *For given $K_i \in R^{k\times n} (i = 1, 2, 3), h_{1M} \geq 0$ and $h > 0, \bar{h} > 0$, if there exist positive definite $k \times k$-matrices W, R, N and $3n \times 3n$-matrix $P > 0$ such that the following LMIs hold:*

$$\Xi(h, \bar{h}) < 0, \quad \Xi(h, h_{3M}) < 0, \quad \Xi(h_{2M}, \bar{h}) < 0, \quad \Xi(h_{2M}, h_{3M}) < 0, \tag{16}$$

where $h_{2M} = h + h_{1M}, \quad h_{3M} = \bar{h} + h_{1M}$, and

$$\begin{aligned} \Xi(h_2, h_3) &= \begin{bmatrix} X & PY & PY & PY + D^T NZ & D^T N \\ * & -R & 0 & -Y^T NZ & Y^T N \\ * & * & -\frac{\pi^2}{4} W & -Y^T NZ & Y^T N \\ * & * & * & -2Z^T NY & Z^T N + Y^T N \\ * & * & * & * & -2N \end{bmatrix}, \\ X &= \bar{D}_1(h_2, h_3)^T P + P\bar{D}_1(h_2, h_3) \\ &\quad + diag\{0, h_{1M}^2 K_1^T RK_1, h_{2M}^4 K_2^T WK_2\}, \\ Y &= (0 \ \ 0 \ \ B^T)^T, \quad Z = (0 \ \ 0 \ \ I_n)^T, \end{aligned}$$

and the matrix $\bar{D}_1(h_2, h_3)$ is given by (6), then the system (3) is asymptotically stable for all $h_1(t) \in [0, h_{1M}], h_2(t) \in [h, h_{2M}], h_3(t) \in [\bar{h}, h_{3M}]$.

In case of constant $h_1 \equiv h_{1M}$, the system (3) is asymptotically stable if $\Xi(h_{2M}, h_{3M}) < 0$.

Proof. The Lyapunov-Krasovskii functional is constructed as follows:

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)), \tag{17}$$

where

$$\begin{aligned} &V_1(x(t)) = x^T(t)Px(t), \qquad P > 0, \\ &V_2(x(t)) = h_{1M} \int_{t-h_{1M}}^{t} (s - t + h_{1M}) x_2^T(s)\hat{R}x_2(s)ds, \quad \hat{R} = K_1^T RK_1, \\ &V_3(x(t)) = h_{2M}^3 \int_{t-h_{2M}}^{t} (s - t + h_{2M}) x_3^T(s)\hat{W}x_3(s)ds, \hat{W} = K_2^T WK_2. \end{aligned}$$

Time derivatives of $V_i(x(t)), i = 1, 2, 3$, along the trajectories of (15) are as follows:

$$\begin{aligned} \dot{V}_1(x(t)) &= 2x^T(t)P[\bar{D}_1 x(t) + \sum_{i=1}^{3} YK_i\delta_i(t)], \\ \dot{V}_2 &\leq h_{1M}^2 x_2^T(t)\hat{R}x_2(t) - h_{1M} \int_{t-h_1}^{t} x_2^T(s)\hat{R}x_2(s)ds, \\ \dot{V}_3 &\leq h_{2M}^4 x_3^T(t)\hat{W}x_3(t) - h_{2M}^3 \int_{t-h_2}^{t} x_3^T(s)\hat{W}x_3(s)ds. \end{aligned} \tag{18}$$

Then by Lemma 2, we have

$$\dot{V}_2 \leq h_{1M}^2 x_2^T(t)\hat{R}x_2(t) - \delta_1^T(t)\hat{R}\delta_1(t). \tag{19}$$

By Lemma 1, we get

$$\begin{aligned} &-h_{2M}^3 \int_{t-h_2}^{t} x_3^T(s)\hat{W}x_3(s)ds \\ \leq \; &-\frac{\pi^2}{4} h_{2M} \int_{t-h_2}^{t} [x_2(t) - x_2(s)]^T \hat{W}[x_2(t) - x_2(s)]ds. \end{aligned}$$

Using $\int_{t-h_2}^{t}[x_2(t)-x_2(s)]ds = \delta_2(t)$, we have

$$\dot{V}_3 \le h_{2M}^4 x_3^T(t)\hat{W}x_3(t) - \frac{\pi^2}{4}\delta_2^T(t)\hat{W}\delta_2(t). \tag{20}$$

Furthermore, for any matrix $N > 0$ with appropriate dimension, the following equation holds:

$$2(-K_3\delta_3(t)Z + \dot{x}(t))^T N[\bar{D}_1(h_2,h_3)x(t) + \sum_{i=1}^{3} YK_i\delta_i(t) - \dot{x}(t)] = 0. \tag{21}$$

From (17)-(21), it follows that$\dot{V}(x(t)) \le \xi^T(t)\Xi(h_2,h_3)\xi(t)$, where

$$\xi(t) = \begin{pmatrix} x^T(t) & [K_1\delta_1(t)]^T & [K_2\delta_2(t)]^T & [K_3\delta_3(t)]^T & \dot{x}^T(t) \end{pmatrix}^T,$$

$$\Xi(h_2,h_3) = \begin{bmatrix} X & PY & PY & PY + D^TNZ & D^TN \\ * & -R & 0 & -Y^TNZ & Y^TN \\ * & * & -\frac{\pi^2}{4}W & -Y^TNZ & Y^TN \\ * & * & * & -2Z^TNY & Z^TN + Y^TN \\ * & * & * & * & -2N \end{bmatrix}. \tag{22}$$

We conclude that $\dot{V} \le -l\|x(t)\|^2$ for some $l > 0$, if $\Xi(h_2,h_3) < 0$. Since $\bar{D}_1(h_2,h_3)$ is affine in h_2 and h_3, the matrix $\Xi(h_2,h_3)$ is affine in h_2 and h_3. Therefore, the feasibility of the LMIs (16) yields the inequality $\Xi(h_2,h_3) < 0$ for all $h_2 = h_2(t) \in [h, h_{2M}], h_3 = h_3(t) \in [\bar{h}, h_{3M}]$. The latter inequality implies the asymptotic stability of (15) and thus of (3).

For constant delay $h_1 = h_{1M}, h_2 = h_{2M}$ and $h_3 = h_{3M}$ the result follows from (22). This completes the proof of the Theorem 1. □

4. STABILITY ANALYSIS VIA MODEL TRANSFORMATION

In this section, we consider the case of constant and known delays.

4.1. NEUTRAL TYPE MODEL TRANSFORMATION OF THE SYSTEM

The ideal of the transformation that we use below is to represent the system (3) in the form of a neutral type system without delays in the right-hand side of the equation. The term with stabilizing delay $x_1(t-h_2)$ can be presented as

$$x_1(t-h_2) = x_1(t) - h_2x_2(t) + \dot{G}_2(x_{2t}), \tag{23}$$

where

$$G_2(x_{2t}) = \int_{t-h_i}^{t} (s - t + h_i)x_2(s)ds. \tag{24}$$

Indeed, since $\dot{x}_1(t) = x_2(t)$ we have

$$\dot{G}_2(x_{2t}) = h_i x_2(t) - [x_1(t) - x_1(t - h_i)].$$

The term $x_1(t - h_1)$ can be presented either as

$$x_1(t - h_1) = x_1(t) + \dot{G}_1(x_{1t}), \tag{25}$$

where

$$G_1(x_{1t}) = -\int_{t-h_1}^{t} x_1(s)ds. \tag{26}$$

And the term $x_1(t - h_i)$ can be presented either as

$$x_1(t - h_i) = x_1(t) - h_i x_2(t) + \frac{h_i^2}{2} x_3(t) + \dot{G}_i(x_{3t}), \tag{27}$$

where $i = 1, 2, 3$, and

$$G_i(x_{3t}) = -\int_0^{h_i} \int_{t-\theta}^{t} (s - t + \theta)x_3(s)dsd\theta. \tag{28}$$

Different ways of the presentation for $x_1(t - h_1)$ and $x_1(t - h_2)$ lead to different neutral type system and to different (complementary) stability conditions.

From (23), (25) and (27), we represent the system (3) in the form of a neutral type system

$$\dot{z}(t) = \bar{D}_1 x(t), \qquad x(t), z(t) \in R^{3n}, \tag{29}$$

where $\bar{D}_1 = \bar{D}_1(h_2, h_3)$ is given by (6) and $x(t) = (x_1^T(t), x_2^T(t), x_3^T(t))^T$,

$$z(t) = x(t) - \sum_{i=1}^{3} Y K_i G_i(x_{it}). \tag{30}$$

Using G_1, G_2 and G_3 of (28) we represent the system (3) in the form

$$\dot{z} = D_2 x(t), \qquad x(t), z(t) \in R^{3n}, \tag{31}$$

where

$$x(t) = (x_1^T(t), x_2^T(t), x_3(t))^T,$$

$$z(t) = x(t) - \sum_{i=1}^{3} Y K_i G_i(x_{3t}),$$

$$D_2 = \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_n \\ A_1 + \sum\limits_{i=1}^{3} BK_i & A_2 - \sum\limits_{i=1}^{3} BK_i h_i & A_3 + \sum\limits_{i=1}^{3} \frac{h_i^2}{2} BK_i \end{bmatrix}. \tag{32}$$

Note that if the matrix D_1 is Hurwitz, then for small enough $h_1 = O(h_2^2)$, the matrix D_2 is Hurwitz too.

4.2. STABILITY OF THE INTEGRAL EQUATIONS

In order to use the Lyapunov-Krasorskii theorem for the stability of the neutral type systems, we first derive conditions for the exponential stability of the corresponding integral equations

$$z(t) = 0. \tag{33}$$

We will start with (33), where $z(t)$ is defined by (30), i.e. with the following system:

$$\begin{aligned} &x_1(t) = 0, \\ &x_2(t) = 0, \\ &x_3(t) - BK_1G_1(x_{1t}) - BK_2G_2(x_{2t}) - BK_3G_3(x_{3t}) = 0. \end{aligned}$$

It is clear that the latter system is exponentially stable if the equation $x_3(t) = BK_3G_3(x_{3t})$ is exponentially stable, i.e. if the following integral equation

$$x_3(t) = BK_3G_3(x_{3t}) = -\int_0^{h_3}\int_{t-\theta}^{t}(s-t+\theta)BK_3x_3(s)dsd\theta, \tag{34}$$

is exponentially stable.

Lemma 5. If there exist an matrix $S_3 > 0$ such that the following LMI holds:

$$h_3^3K_3^TB^TS_3BK_3 - 36h_3^{-3}S_3 < 0, \tag{35}$$

then the integral equation (34) (and, thus, (33) with (30)) is exponentially stable.

Proof. It is easy to see that the functional

$$\bar{V}(x_{3t}) = \int_0^{h_3}\int_{t-\theta}^{t}(s-t+\theta+\alpha)^2x_3^T(s)S_3x_3(s)dsd\theta, \tag{36}$$

where $\alpha > 0$ and $0 < S_3 \in R^{n\times n}$, satisfies the condition (10) of Lemma 4. Caculating the derivative of the functional (36), we obtain

$$\begin{aligned} \dot{\bar{V}}(x_{3t}) \le \quad & \tfrac{(h_3+\alpha)^3}{3}x_3^T(t)S_3x_3(t) - 2\alpha\textstyle\int_0^{h_3}\int_{t-\theta}^{t}x_3^T(s)S_3x_3(s)dsd\theta \\ & -2\textstyle\int_0^{h_3}\int_{t-\theta}^{t}(s-t+\theta)x_3^T(s)S_3x_3(t)dsd\theta. \end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned} &-2\textstyle\int_0^{h_3}\int_{t-\theta}^{t}(s-t+\theta)x_3^T(s)S_3x_3(t)dsd\theta \\ &\le -\tfrac{12}{h_3^3}(\textstyle\int_0^{h_3}\int_{t-\theta}^{t}\int_{t-s}^{\theta}x_3^T(s)dmdsd\theta)S_3(\int_0^{h_3}\int_{t-\theta}^{t}\int_{t-s}^{\theta}x_3(t)dmdsd\theta) \\ &= -\tfrac{12}{h_3^3}G_3^T(x_{3t})S_3G_3(x_{3t}). \end{aligned} \tag{37}$$

From (37), we have

$$\begin{aligned} \dot{\bar{V}}(x_{3t}) \le \quad & \tfrac{(h_3+\alpha)^3}{3}x_3^T(t)S_3x_3(t) - \tfrac{12}{h_3^3}G_3^T(x_{3t})S_3G_3(x_{3t}) \\ & -\beta\textstyle\int_0^{h_3}\int_{t-\theta}^{t}\|x_3^T(s)\|^2dsd\theta, \end{aligned} \tag{38}$$

where $\beta = 2\alpha\lambda_{min}(S_3) > 0$. Here, $\lambda_{min}(S_3)$ is the minimal eigenvalue of S_3.

Substituting $x_3(t) = BK_3G_3(x_{3t})$ into (38), we obtain

$$\begin{aligned}\dot{V}(x_{3t}) \leq \quad & G_3^T(x_{3t})[\tfrac{(h_3+\alpha)^3}{3}K_3^TB^TS_3BK_3 - \tfrac{12}{h_3^3}S_3]G_3(x_{3t}) \\ & -\beta\int_0^{h_3}\int_{t-\theta}^{t}\|x_3^T(s)\|^2 dsd\theta.\end{aligned}$$

So, if

$$\frac{(h_3+\alpha)^3}{3}K_3^TB^TS_3BK_3 - \frac{12}{h_3^3}S_3 < 0, \tag{39}$$

then the functional (36) satisfies also the condition (11) of Lemma 4 and therefore (34) is exponentially stable. It is easy to see that if (35) holds then (39) holds with a small enough $\alpha > 0$. This completes the proof of the Lemma 5. $\square$

Consider next (33), where $z(t)$ is defined by (32), i.e. the following system :

$$\begin{aligned}& x_1(t) = 0, \\ & x_2(t) = 0, \\ & x_3(t) - BK_1G_1(x_{3t}) - BK_2G_2(x_{3t}) - BK_3G_3(x_{3t}) = 0.\end{aligned}$$

In this case, the stability of (33) is reduced to the stability of the integral equation with three delay

$$x_3(t) = -\sum_{i=1}^{3}\int_0^{h_i}\int_{t-\theta}^{t}(s-t+\theta)BK_ix_3(s)ds. \tag{40}$$

We immediately arrive at the following result.

Lemma 6. If there exist some positive definite $n\times n$-matrices S_1, S_2, S_3 such that the following LMIs holds:

$$\bar{B}^T(h_1^3S_1 + h_2^3S_2 + h_3^3S_3)\bar{B} - \bar{S} < 0, \tag{41}$$

where $\bar{B} = (BK_1, BK_2, BK_3), \bar{S} = diag\{36h_1^{-3}S_1, 36h_2^{-3}S_2, 36h_3^{-3}S_3\}$, then the integral equation (40)(and thus (33) with notation (32))is exponentially stable.

4.3. STABILITY OF SYSTEM (3): CONSTANT DELAYS

Consider the neutral type system (29), (30) where the matrix $\bar{D}_1$ is Hurwitz.

Theorem 2. *Given $K_i \in R^{k\times n}$ $(i = 1, 2, 3)$ and constant know delays $h_1 \geq 0, h_2 > 0$ and $h_3 > 0$ such that $\bar{D}_1$ defined by (6) is Hurwitz, if there exist positive definite matrices $S_3 \in R^{n\times n}, R_1, R_2, R_3 \in R^{k\times k}$ and $P \in R^{3n\times 3n}$ such that (35) and the* *following LMI hold:*

$$\Psi_1 = \begin{bmatrix} \Phi_1 & \Phi & \Phi & \Phi \\ * & -R_1 & 0 & 0 \\ * & * & -4R_2 & 0 \\ * & * & * & -12R_3 \end{bmatrix} < 0, \tag{42}$$

where

$$\begin{array}{c} \Phi = \bar{D}_1^T P \begin{pmatrix} 0 & 0 & B^T \end{pmatrix}^T, \\ \Phi_1 = \bar{D}_1^T P + P\bar{D}_1 + diag\{h_1^2 K_1^T R K_1, h_2^4 K_2^T R K_2, \frac{h_3^6}{3} K_3^T R K_3\}, \end{array} \tag{43}$$

then the system (3) is asymptotically stable.

Proof. Via the condition (35) the integral equation (34) is exponentially stable.

Differentiating $V_1(x_t) = z^T(t)Pz(t), P > 0$, along (29),(30) and using notion (43), we have

$$\dot{V}_1(x_t) = 2x^T(t)P\bar{D}_1 x(t) - 2\sum_{i=1}^{3} G_i^T(x_{it})K_i^T \Phi^T x(t). \tag{44}$$

In order to compensate G_i-terms in (44) consider

$$\begin{array}{rcl} V_2(x_t) & = & h_1 \int_{t-h_1}^{t} (s-t+h_1)x_1^T(s)\hat{R}_1 x_1(s)ds \\ & & +h_2^2 \int_{t-h_2}^{t} (s-t+h_2)^2 x_2^T(s)\hat{R}_2 x_2(s)ds \\ & & +h_3^3 \int_0^{h_3} \int_{t-\theta}^{t} (s-t+\theta)^2 x_3^T(s)\hat{R}_3 x_3(s)dsd\theta, \end{array} \tag{45}$$

where $\hat{R}_i = K_i^T R_i K_i, i = 1, 2, 3.$

We have

$$\begin{array}{l} \dot{V}_2(x_t) \le h_1^2 x_1^T(t)\hat{R}_1 x_1(t) - h_1 \int_{t-h_1}^{t} x_1^T(s)\hat{R}_1 x_1(s)ds \\ +h_2^4 x_2^T(t)\hat{R}_2 x_2(t) - 2h_2^2 \int_{t-h_2}^{t} (s-t+h_2)x_2^T(s)\hat{R}_2 x_2(s)ds \\ +\frac{h_3^6}{3} x_3^T(t)\hat{R}_3 x_3(t) - 2h_3^3 \int_0^{h_3} \int_{t-\theta}^{t} (s-t+\theta)x_3^T(s)\hat{R}_3 x_3(s)dsd\theta. \end{array} \tag{46}$$

Taking into account the representations (26), (24), (28) for $G_1(x_{1t})$, $G_2(x_{2t}), G_3(x_{3t})$ and applying Lemma 2, similarly to (37), we have

$$-2h_3^3 \int_0^{h_3} \int_{t-\theta}^{t} (s-t+\theta)x_3^T(s)\hat{R}_3 x_3(s)dsd\theta \le -12G_3^T(x_{3t})\hat{R}_3 G_3(x_{3t}), \tag{47}$$

$$-2h_2^2 \int_{t-h_2}^{t} (s-t+h_2)x_2^T(s)\hat{R}_2 x_2(s)ds \le -4G_2^T(x_{2t})\hat{R}_2 G_2(x_{2t}). \tag{48}$$

From (46)-(48), it follows that

$$\begin{array}{c} \dot{V}_2(x_t) \le h_1^2 x_1^T(t)\hat{R}_1 x_1(t) + h_2^4 x_2^T(t)\hat{R}_2 x_2(t) + \frac{h_3^6}{3} x_3^T(t)\hat{R}_3 x_3(t) \\ -G_1^T(x_{1t})\hat{R}_1 G_1(x_{1t}) - 4G_2^T(x_{2t})\hat{R}_2 G_2(x_{2t}) - 12G_3^T(x_{3t})\hat{R}_3 G_3(x_{3t}). \end{array} \tag{49}$$

Denote $\eta(t) = [x^T(t),\ -(K_1G_1(t,x_{1t}))^T,\ -(K_2G_2(t,x_{2t}))^T,$
$-(K_3G_3(t,x_{3t}))^T]^T$. Then for the Lyapunov-Krasovskii functional

$$V(x_t) = V_1(x_t) + V_2(x_t)$$

from (42), (44) and (49), we obtain

$$\begin{array}{l}\dot V(x_t) \leq 2x^T(t)P\bar D_1 x(t) - 2\sum\limits_{i=1}^{3} x^T(t)\Phi K_i G_i(x_{it}) \\ +h_2^4 x_2^T(t)\hat R_2 x_2(t) + \frac{h_3^6}{3} x_3^T(t)\hat R_3 x_3(t) - G_1^T(x_{1t})\hat R_1 G_1(x_{1t}) \\ -4G_2^T(x_{2t})\hat R_2 G_2(x_{2t}) - 12G_3^T(x_{3t})\hat R_3 G_1(x_{3t}) + h_1^2 x_1^T(t)\hat R_1 x_1(t) \\ = \eta^T(t)\Psi_1\eta(t) \\ \leq -c\|x(t)\|^2,\end{array} \tag{50}$$

for some $c > 0$. The latter inequality guarantees asymptotic stability of the neutral type system (29), (30)(and, thus, of (3))with the asymptotically stable integral equation. This completes the proof of the Theorem 2. □

Theorem 3. *Given $K_i \in R^{k\times n}, (i = 1,2,3)$ and constant known delays $h_1 \geq 0$ and $h_2 > 0, h_3 > 0$ such that D_2 defined by (32) is Hurwitz. The system (3) is asymptotically stable, if there exist positive definite matrices $S_1, S_2, S_3 \in R^{n\times n}, R_1, R_2, R_3 \in R^{k\times k}$ and $P \in R^{3n\times 3n}$ such that (41)and the following LMI hold:*

$$\Psi_2 = \begin{bmatrix} \Phi_2 & \bar\Phi & \bar\Phi & \bar\Phi \\ * & -12R_1 & 0 & 0 \\ * & * & -12R_2 & 0 \\ * & * & * & -12R_3 \end{bmatrix} < 0, \tag{51}$$

where

$$\begin{array}{c}\bar\Phi = D_2^T P \begin{pmatrix} 0 & 0 & B^T \end{pmatrix}^T, \\ \Phi_2 = D_2^T P + PD_2 + diag\{0, 0, \sum\limits_{i=1}^{3} \frac{h_3^6}{3} K_i^T R_i K_i\}.\end{array} \tag{52}$$

Proof. Consider the following Lyapunov functional:

$$\begin{array}{rl} V(x_t) = & \sum\limits_{i=1}^{3} h_i^3 \int_0^{h_i} \int_{t-\theta}^{t} (s-t+\theta)^2 x_3^T(s) K_i^T R_i K_i x_3(t) ds d\theta \\ & +z^T(t)Pz(t), \end{array} \tag{53}$$

where $0 < P \in R^{3n\times 3n},\ 0 < R_i \in R^{k\times k} (i = 1,2,3)$. Along the trajectories of system (31), the time derivative of $V(x_t)$ can be obtained

$$\begin{array}{rl} \dot V(x_t) \leq & -\sum\limits_{i=1}^{3} G_i^T(x_3t)K_i^T \begin{pmatrix} 0 & 0 & B^T \end{pmatrix} (PD_2 + D_2^T P)x(t) \\ & +x^T(t)(PD_2 + D_2^T P)x(t) + \sum\limits_{i=1}^{3} [\frac{h_i^6}{3} x_3^T(t) K_i^T R_i K_i x_3(t) \\ & -2h_i^3 \int_0^{h_i} \int_{t-\theta}^{t} (s-t+\theta) x_3^T(s) K_i^T R_i K_i x_3(s) ds d\theta], \end{array}$$

then applying Lemma 2 and similarly to (47), we have

$$\begin{array}{l} -2h_i^3 \int_0^{h_i} \int_{t-\theta}^{t} (s-t+\theta) x_3^T(s) K_i^T R_i K_i x_3(s) ds d\theta \\ \leq -12 G_i^T(x_{3t}) \hat{R}_i G_i(x_{3t}). \end{array} \tag{54}$$

Denote

$$\zeta(t) = [x^T(t), -[K_1 G_1(t, x_{3t})]^T, -[K_2 G_2(t, x_{3t})]^T, -[K_3 G_3(t, x_{3t})]^T]^T.$$

Then we obtain $\dot{V}(x_t) \leq \zeta^T(t) \Psi_2 \zeta(t)$, where Ψ_2 is given by (51). The rest of the proof is similar to the proof of Theorem 2, which is omitted. □

5. NUMERICAL EXAMPLE

Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0.1 & 0.1 \end{bmatrix} x(t) + \sum_{i=1}^{3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \tag{55}$$

Clearly, (55) cannot be stabilized by a non-delayed feedback $u(t) = K x_1(t)$ for any K because the resulting matrix of the closed-loop system

$$G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2+K & 0.1 & 0.1 \end{bmatrix}$$

is not Hurwitz. In fact, we have

$$\varphi(\lambda) = det(\lambda I - G) = \lambda^3 - 0.1\lambda^2 - 0.1\lambda + (2 - K). \tag{56}$$

Notice $\Delta_1 = -0.1 < 0$. According to Routh-Hurwitz lemma, G is not Hurwitz for any gain K.

In this section, we will provide a numerical example to illustrate the effectiveness and the merits of the obtained results.

Example 1. Choose a delay feedback $u(t) = \sum\limits_{i=1}^{3} K_i x_1(t - h_i)$,then consider the following system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \sum_{i=1}^{3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} K_i x_1(t - h_i), \tag{57}$$

where $K_1 = -56.1718, K_2 = 108.8410, K_3 = -50.7693, h_1 = 0.01,$
$h_2 = 0.1105, h_3 = 0.2105.$

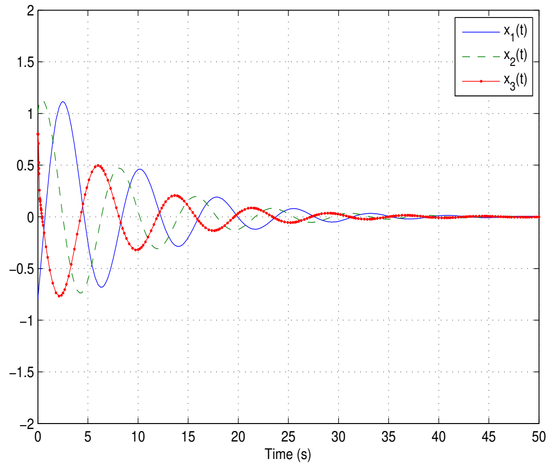

Figure 1: Evolution of system state in example.

Feasible solutions can be obtained by utilizing the MATLAB LMI Toolbox to solve the LMIs given in the Theorem 2, as follow:

$$P = \begin{bmatrix} 2.7135 & 2.1159 & 2.2227 \\ 2.1159 & 1.8869 & 1.8510 \\ 2.2227 & 1.8510 & 1.9306 \end{bmatrix},$$

$$R_1 = 1.0004, \quad R_2 = 0.2259, \quad R_3 = 2.8984, S_3 = 0.0523.$$

Furthermore, the system (57) is asymptotically stable. Simulation of solutions of the system are shown in Fig 1.

6. CONCLUSION

In this paper, we investigated the problems of stabilization for a class of the vector third-order systems by output feedback controller with artificial stabilizing delays. Using Writinger's inequality and model transformation methods, we have develop stability criteria to ensure that the closed-loop system is asymptotically stable. The example in last section has demonstrated the effectiveness of the proposed method. The controllers that we designed in this paper are useful to solve many control problems such as networked based control and delay-induced consensus in multi-agent systems.

ACKNOWLEDGMENTS

This work is supported by the Natural Science Foundation of China under Grants 61873186 and 61703307.

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