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HYBRID DELAY EVOLUTION SYSTEMS WITH NONLINEAR CONSTRAINTS

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ABSTRACT: Motivated by the importance of reaction-diffusion systems in modeling real processes with memory, we are interested in the existence of mild solutions for systems of abstract delay evolution equations subjected to general nonlinear constraints. Wishing to allow the system nonlinearities to behave independently as much as possible, we use a vector approach based on matrices, vector-valued norms and a vector version of Krasnoselskii's fixed point theorem for a sum of two operators. The hybrid character of the systems comes from the different nature of the metrical and topological conditions imposed to the component equations. Also, the assumptions are put in connection with the support of the nonlinear constraints. Two examples are given to illustrate the theory.

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1. INTRODUCTION

In this paper, we are concerned with the existence of solutions to the semilinear system of abstract delay evolution equations with constraints (nonlocal initial conditions), of the type

$$\begin{cases} u_i'(t) + A_i u_i(t) = F_i(t, u_t), & t \in [0,T] \\ g_i(u)(t) = 0, & t \in [-\tau, 0] \\ & i = 1, ..., n. \end{cases} \tag{1.1}$$

Here $n \geq 1$, and for each $i \in J := \{1, ..., n\}$, the linear operator $-A_i : D(A_i) \subseteq X_i \to X_i$ generates a C_0-semigroup of contractions $\{S_i(t); t \geq 0\}$ on a Banach space $\left(X_i, |.|_{X_i}\right)$, $\tau \geq 0$, $u \in C([-\tau, T], X)$, where $X = X_1 \times ... \times X_n$, and for each t, u_t is the restriction of u to $[t - \tau, t]$ shifted to the interval $[-\tau, 0]$, i.e., $u_t \in C([-\tau, 0], X)$ and

$$u_t(s) = u(t + s), \quad s \in [-\tau, 0]. \tag{1.2}$$

The nonlinear perturbations in equations are given by the continuous mappings F_i from $[0, T] \times C([-\tau, 0], X)$ to X_i, and the constraints are expressed by the continuous mappings g_i from $C([-\tau, T], X)$ to $C([-\tau, 0], X)$.

Here are some particular cases of equations that could appear in system (1.1):

(E1) If $\tau = 0$, then u_t reduces to $u(t)$ and the equations of the system are *without delay,* of the form

$$u_i'(t) + A_i u_i(t) = F_i(t, u(t)).$$

(E2) The following delay equations

$$u_i'(t) + A_i u_i(t) = G_i(t, u(t), u_1(t - \gamma_1), ..., u_n(t - \gamma_n)), \tag{1.3}$$

where $\gamma_1, ..., \gamma_n \geq 0$ are given numbers, are particular cases of the equations from (1.1). In this case $\tau = \max\{\gamma_1, ..., \gamma_n\}$ and the delays are *discretely distributed* in the interval $[0, \tau]$.

(E3) It may happen that the delays $\gamma_1, ..., \gamma_n$ in (1.3) are continuous nonnegative functions of time, so that (1.3) takes the form

$$u_i'(t) + A_i u_i(t) = G_i(t, u(t), u_1(t - \gamma_1(t)), ..., u_n(t - \gamma_n(t))).$$

Then $\tau = \max\left\{|\gamma_1|_{C[0,T]}, ..., |\gamma_n|_{C[0,T]}\right\}$ and the delays are *time-depending discretely distributed* in $[0, \tau]$.

(E4) The general form of the system (1.1) also covers, as a particular case, the integro-differential systems whose equations are of the form

$$u_i'(t) + A_i u_i(t) = G_i\left(t, u(t), \int_0^\tau h_1(t, s, u_1(t - s))\, ds, ..., \int_0^\tau h_n(t, s, u_n(t - s))\, ds\right),$$

with the delays *continuously distributed* on $[0, \tau]$.

As concerns the constraints in system (1.1), let us mention some particular cases:

(C1) *Initial condition*:

$$u_0 = \varphi,$$

where $\varphi \in C([-\tau, 0], X)$ is given. Explicitly, this means $u(s) = \varphi(s)$ for every $s \in [-\tau, 0]$. In this case $g(u) = \varphi - u_0$.

(C2) *Linear multi-point conditions* (linear nonlocal initial conditions of discrete type):

$$u_i(s) = \varphi_i(s) + \sum_{j=1}^{m_i} a_{ij}(s) u_i(s + t_{ij}), \quad s \in [-\tau, 0], \ i = 1, ..., n, \tag{1.4}$$

where $0 < t_{ij} < t_{i,j+1} \leq T$ for $j = 1, ..., m_i$ and $i = 1, ..., n$. In this case $g_i(u)(s) = \varphi_i(s) - u_i(s) + \sum_{j=1}^{m_i} a_{ij}(s) u_i(s + t_{ij})$. These conditions include in particular the initial condition, and the *periodicity condition*

$$u_0 = u_T,$$

which explicitly means that $u(s) = u(T + s)$ for every $s \in [-\tau, 0]$.

Physically, if $\tau = 0$ and $u(t)$ stands for the state at time t of a process taking place in a spatial domain, being a function of space variable x, i.e., $u(t)(x) = u(x, t)$, the periodic condition requires that the final state of the process coincides with the initial state, i.e., $u(x, T) = u(x, 0)$. However, due to stimulatory or inhibitory exterior sources, one may expect an increased or a decreased final state, that is

$$u(x, T) = \omega(x) u(x, 0),$$

with $\omega(x) \geq 1$, or $\omega(x) \leq 1$, respectively. Such kind of conditions were considered in [24], and are also covered by (1.4).

(C3) *Linear nonlocal initial conditions of continuous type*, given by integrals:

$$u_0 = \varphi + \int_0^T k(\cdot, t) u_t dt,$$

or explicitly

$$\begin{aligned} u_i(s) &= \varphi_i(s) + \int_0^T k_i(s, t) u_i(t + s) dt \\ &= \varphi_i(s) + \int_s^{T+s} k_i(s, t - s) u_i(t) dt, \quad s \in [-\tau, 0]. \end{aligned}$$

For $\tau = 0$, this condition reduces to

$$u_i(0) = \varphi_i^0 + \int_0^T k_i(t) u_i(t) dt,$$

and we can understand it as in [33], assuming that the concentration of some diffusing substance whose initial level is unspecified, is balanced against some weighted average of all levels of concentration along the time interval $[0, T]$.

Since Volterra's pioneering works on integro-differential equations with delayed effects in population dynamics and materials with memory, the theory of delay differential equations progressed dramatically stimulated by the development of functional analysis and its numerous real world applications, wherever (in physics, chemistry, biology, medicine, economy etc, see e.g., [25]) the evolution of a process depends on its history in an essential way.

Differential equations with general boundary conditions of multi-point or integral type, have a long history (see, e.g. Cioranescu [16], Whyburn [46], Conti [17]), and have gained a special attention in the last decades motivated by concrete applications in different domains. See, for example, [2], [3], [5], [6], [8], [19], [23], [30], [31], [36], [39], [45], the recent survey paper [40] and the references therein.

For parabolic problems with nonlocal initial conditions we mention the papers of Kerefov [24], Vabishchevich [41], Chabrowski [14], Deng [18], Pao [37], Olmstead and Roberts [33], and Chapter 10 in [28], where nonlocal versions of some deterministic models from physics, mechanics, biology and medicine are stated. Abstract evolution equations with nonlocal initial conditions were considered by Byszewski [12], Jackson [22], Lin and Liu [26]. For more recent contributions, we refer the readers to the papers [4], [7], [9], [10], [13], [21], [26], [27], [29], [32], [35], [44] and the very recent monograph [11].

Throughout the paper, by $[-\tau, a]$ we shall denote the *support* of the constraints, that is the smallest subinterval $[-\tau, a]$ with $0 \leq a \leq T$ such that

$$g_i\left(u\right) = g_i\left(v\right), \qquad i = 1, ..., n,$$
$$\text{for every} \quad u, v \in C\left(\left[-\tau, T\right], X\right) \quad \text{with} \quad u|_{[-\tau,a]} = v|_{[-\tau,a]}\,.$$

Here by $u|_{[-\tau,a]}$ we mean the restriction of the function u to the interval $[-\tau, a]$.

The notion of support plays an essential role in the existence results for nonlocal problems, as first shown in the papers [3] and [4]. More exactly, in these papers it is shown that stronger conditions on nonlinearities have to be asked on the support subinterval, compared to those required on the rest of the interval on which the problem is considered. Mathematically, the integral equation equivalent to the nonlocal problem is of Fredholm type on the support interval, and of Volterra type on the rest of the interval. From a physical point of view, the evolution of a process is subjected to some constraints until a given moment of time, and becomes free of any constraints after that moment.

This double nature Fredholm–Volterra of the nonlocal problems, due to the support of the nonlocal conditions, makes useful to consider a split norm on the functional space where the problem is investigated. Thus, for the delay system (1.1), we shall consider here for the first time the split norm on $C\left(\left[-\tau, T\right], X_i\right)$, defined by

$$|u|_{\tau} = \max\left\{|u|_{C([-\tau,a],X_i)}\,,\quad |u|_{C_\theta([a-\tau,T],X_i)}\right\}, \tag{1.5}$$

where $|u|_{C([-\tau,a],X_i)}$ is the usual max norm

$$|u|_{C([-\tau,a],X_i)} = \max_{t\in[-\tau,a]} |u(t)|_{X_i}\,,$$

while for any $\theta > 0$, $|u|_{C_\theta([a-\tau,T],X_i)}$ is the Bielecki type norm on $C\left([a-\tau,T],X_i\right)$,

$$|u|_{C_\theta([a-\tau,T],X_i)} = \max_{t\in[a,T]} \left(|u_t|_{C([-\tau,0],X_i)}\, e^{-\theta(t-a)}\right).$$

In the particular case of equations without delay, when $\tau = 0$, the norm (1.5) reduces to the split norm that we already considered in our previous papers [3]-[5], [30] and [31].

As we shall see in the sequel, the hybrid character of the system comes from the different compactness properties associated to the equations of the system. The system will be split into to subsystems: the first m equations, and the last $n-m$ equations $(0 \leq m \leq n)$, and two existence results will be proved. The first one assumes on g_i, Lipschitz condition for $i = 1,...,m$, and completely continuity and some growth condition for $i = m+1,...,n$; the second result requires the Lipschitz condition for all the mappings g_i $(i = 1,...,n)$. Both results require for $i = 1,...,m$, that the operator $-A_i$ generate a C_0-semigroup of contractions, and the mapping F_i satisfies a Lipschitz condition, while for $i = m+1,...,n$, that $-A_i$ generates a compact C_0-semigroup of contractions, and F_i satisfies only a growth condition. The proof of these two results is based on a vector version of Krasnoselskii's fixed point theorem for a sum of a compact map and a generalized contraction in Perov's sense.

2. PRELIMINARIES

For the treatment of systems we use the vector approach based on vector-valued metrics and norms, and matrices instead of constants.

Let us make the convention that the elements of $\mathbb{R}^n$ are seen as column vectors. By a *vector-valued metric* on a set Y we mean a mapping $d : Y \times Y \to \mathbb{R}^n_+$ such that $d(x,y) = 0$ if and only if $x = y$; $d(x,y) = d(y,x)$ for all $x,y \in Y$ and $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in Y$. Here by $\leq$ we mean the natural componentwise order relation of $\mathbb{R}^n$, more exactly, if $r,s \in \mathbb{R}^n$, $r = (r_1,...,r_n)$, $s = (s_1,...,s_n)$, then by $r \leq s$ one means that $r_i \leq s_i$ for $i = 1,...,n$. A set Y together with a vector-valued metric d is called a *generalized metric space.* For such a space, the notions of Cauchy sequence, convergence, completeness, open and closed set, are similar to those in usual metric spaces.

Similarly, a *vector-valued norm* on a linear space Y, is defined as being a mapping $\|\cdot\| : Y \to \mathbb{R}^n_+$ with $\|x\| = 0$ only for $x = 0$; $\|\lambda x\| = |\lambda| \|x\|$ for $x \in Y$, $\lambda \in \mathbb{R}$, and $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in Y$. To any vector-valued norm $\|.\|$ one can associate the vector-valued metric $d(x, y) := \|x - y\|$. A linear space Y endowed with a vector-valued norm $\|\cdot\|$ is called a *generalized Banach space* if Y is complete with respect to the associated vector-valued metric d.

If (Y, d) is a generalized metric space with d taking values in $\mathbb{R}^n$, we say that a mapping $N : Y \to Y$ is a *generalized contraction* (in Perov's sense) if there exists a square matrix M of size n with nonnegative entries such that its powers M^k tend to the zero matrix 0 as $k \to \infty$, and

$$d(N(x), N(y)) \leq M d(x, y) \text{ for all } x, y \in Y.$$

Such a matrix is said to be a *Lipschitz matrix*. For such type of mappings, the following generalization of Banach's contraction principle holds.

Theorem 2.1 (Perov). *If (Y, d) is a complete generalized metric space, then any generalized contraction $N : Y \to Y$ with the Lipschitz matrix M has a unique fixed point x^*, and*

$$d(N^k(x), x^*) \leq M^k(I - M)^{-1} d(x, N(x)),$$

for all $x \in Y$ and $k \in \mathbb{N}$ (where I stands for the identity matrix of the same size as M).

In this paper we use the following generalization of Theorem 2.1, a vector version of Krasnoselskii's fixed point theorem for a sum of two operators, due to Viorel [42], whose proof is sketched below for the reader's convenience.

Theorem 2.2. *Let $(Y, \|\cdot\|)$ be a generalized Banach space, $D \subset Y$ a nonempty bounded closed convex set and $N : D \to Y$ a mapping such that*

(i) *$N = A + B$ with $A : D \to Y$ a generalized contraction in Perov's sense, and $B : D \to Y$ a compact operator;*

(ii) *$A(u) + B(v) \in D$ for every $u, v \in D$.*

Then N has at least one fixed point in D.

Proof. For any fixed $v \in D$, the operator $u \in D \mapsto A(u) + B(v)$ is a generalized contraction from D to D, which by Perov's theorem has a unique fixed point u_v in D. Thus we have defined the operator $S : D \to D$, $S(v) = u_v$. For any two elements $v_1, v_2 \in D$, from

$$S(v_1) = A(S(v_1)) + B(v_1), \quad S(v_2) = A(S(v_2)) + B(v_2),$$

since A is a generalized contraction, we deduce that

$$\|S(v_1) - S(v_2)\| \leq (I - M)^{-1} \|B(v_1) - B(v_2)\|, \tag{2.1}$$

where M is the Lipschitz matrix of A. From this inequality, the continuity of S is obvious. To show that $S(D)$ is relatively compact, it is enough to prove that for any sequence (v_k) of elements from D, there exists a convergent subsequence of $(S(v_k))$. Since B is compact, there is a convergent subsequence of $(B(v_k))$, still denoted by $(B(v_k))$ for simplicity. Then from (2.1),

$$\|S(v_k) - S(v_{k+p})\| \leq (I - M)^{-1} \|B(v_k) - B(v_{k+p})\|,$$

which shows that $(S(v_k))$ is a Cauchy sequence. Therefore S is compact. Then Schauder's fixed point theorem implies that S has at least one fixed point in D, which is also a fixed point of N. $\square$

There are known several characterizations of matrices like that in Perov's theorem (see, e.g., [1] and [38]). More exactly, for a square matrix M of size n with nonnegative entries, i.e., $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$, the following statements are equivalent:

(a) $M^k \to 0$ as $k \to \infty$;

(b) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + ...$;

(c) the eigenvalues of M are located inside the unit disc of the complex plane, i.e., $\rho(M) < 1$, where $\rho(M)$ is the *spectral radius* of M;

(d) $I - M$ is nonsingular and inverse-positive, i.e., $(I - M)^{-1}$ has nonnegative entries.

The following two obvious propositions will be used in the proof of the main result:

Proposition 2.3. *If $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is a matrix with $\rho(M) < 1$, then $\rho\left(\widetilde{M}\right) < 1$ for every matrix $\widetilde{M} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ whose elements are close enough to the corresponding elements of M.*

Proposition 2.4. *If $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is a matrix with $\rho(M) < 1$, then $\rho\left(\widehat{M}\right) < 1$ for every matrix $\widehat{M} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that $\widehat{M} \leq M$ componentwise.*

The role of matrices with spectral radius less than one in the study of operator systems was pointed out in [38], in connection with several abstract principles from nonlinear functional analysis.

Basic notions and results from the semigroup theory that are frequently used in the present work can be found, for example, in [15], [20] and [43].

3. MAIN RESULTS

It is convenient that the constraints in problem (1.1) are written in the equivalent form

$$u_i(t) = \alpha_i(u)(t), \quad t \in [-\tau, 0],$$

where $\alpha_i(u) = g_i(u) + (u_i)_0$. The meaning of the notation $(u_i)_0$ is that given by (1.2), i.e., $(u_i)_0 \in C([-\tau,0], X_i)$, $(u_i)_0(s) = u_i(0+s) = u_i(s)$ for $s \in [-\tau, 0]$. In what follows the conditions on the constraints will be given in terms of the mappings $\alpha_i : C([-\tau,T], X_i) \to C([-\tau,0], X_i)$.

Looking for mild solutions to the problem (1.1), we are led in a standard way to the following integral system $C([-\tau,T], X)$,

$$\begin{cases} u_i(t) = \alpha_i(u)(t), & t \in [-\tau, 0], \\ u_i(t) = S_i(t)\alpha_i(u)(0) + \int_0^t S_i(t-s) F_i(s, u_s)\, ds, & t \in [0, T], \\ & i = 1, ..., n. \end{cases} \tag{3.1}$$

We first give a list of conditions in terms of the mappings A_i, F_i and α_i and of a number $p > 1$:

(H1) The linear operator $-A_i : D(A_i) \subseteq X_i \to X_i$ generates a C_0-semigroup of contractions on the Banach space X_i.

(H1*) The linear operator $-A_i : D(A_i) \subseteq X_i \to X_i$ generates a compact C_0-semigroup of contractions on the Banach space X_i.

(H2) The mapping $F_i : [0,T] \times C([-\tau,0], X) \to X_i$ is continuous and there exist $a_{ij} \in L^1(0,T;\mathbb{R}_+)$ with $a_{ij}|_{[0,a]} \in L^p(0,a)$, and $c_i \in C([0,T], \mathbb{R}_+)$ for $j \in J$, such that

$$|F_i(t,u)|_{X_i} \leq \sum_{j=1}^n a_{ij}(t)\, |u_j|_{C([-\tau,0],X_j)} + c_i(t) \tag{3.2}$$

for all $u \in C([-\tau,0], X)$ and $t \in [0,T]$.

(H2*) The mapping $F_i : [0,T] \times C([-\tau,0], X) \to X_i$ is continuous and there exists $a_{ij} \in L^1(0,T;\mathbb{R}_+)$ with $a_{ij}|_{[0,a]} \in L^p(0,a)$ for every $j \in J$, such that

$$|F_i(t,u) - F_i(t,v)|_{X_i} \leq \sum_{j=1}^n a_{ij}(t)\, |u_j - v_j|_{C([-\tau,0],X_j)} \tag{3.3}$$

for $u, v \in C([-\tau,0], X)$ and $t \in [0,T]$.

(H3) The mapping $\alpha_i : C([-\tau, T], X) \to C([-\tau, 0], X_i)$ is completely continuous and there exist $b_{ij} \in \mathbb{R}_+$ and $d_i \in \mathbb{R}_+$ for every $j \in J$, such that

$$|\alpha_i(u)|_{C([-\tau,0],X_i)} \leq \sum_{j=1}^{n} b_{ij} |u_j|_{C([-\tau,a],X_j)} + d_i \tag{3.4}$$

for all $u \in C([-\tau, T], X)$.

(H3*) There exist $b_{ij} \in \mathbb{R}_+$ for every $j \in J$, such that

$$|\alpha_i(u) - \alpha_i(v)|_{C([-\tau,0],X_i)} \leq \sum_{j=1}^{n} b_{ij} |u_j - v_j|_{C([-\tau,a],X_j)} \tag{3.5}$$

for all $u, v \in C([-\tau, T], X)$.

Our assumptions will be given differently on two sets of indices,

$$J_1 := \{1, ..., m\} \quad \text{and} \quad J_2 := \{m+1, ..., n\},$$

where $0 \leq m \leq n$, and it is understood that $J_1 = \emptyset$ if $m = 0$, and $J_2 = \emptyset$ if $m = n$.

Theorem 3.1. *Assume that the conditions* (H1), (H2*) and (H3*) *hold for every* $i \in J_1$, *and that* (H1*), (H2) and (H3) *hold for every* $i \in J_2$. *In addition assume that the spectral radius of the* $n \times n$ *square matrix* $M = [m_{ij}]$, *where*

$$m_{ij} := |a_{ij}|_{L^1(0,a)} + b_{ij}, \tag{3.6}$$

is less than one. Then the problem (1.1) *has at least one mild solution on* $[-\tau, T]$. *In case that* $m = n$, *the solution is unique.*

Proof. The integral system (3.1) can be seen as a fixed point equation $u = N(u)$ in $C([-\tau, T], X)$ for the nonlinear operator N from the space $C([-\tau, T], X)$ to itself, $N = (N_1, ..., N_n)$, where $N_i : C([-\tau, T], X) \to C([-\tau, T], X_i)$ are defined by

$$N_i(u)(t) = \begin{cases} \alpha_i(u)(t), & t \in [-\tau, 0], \\ S_i(t)\alpha_i(u)(0) + \int_0^t S_i(t-s) F_i(s, u_s) ds, & t \in [0, T]. \end{cases}$$

Clearly, the operator N admits the representation $N = A + B$, where

$$A = (N_1, ..., N_m, 0, ..., 0), \quad B = (0, ..., 0, N_{m+1}, ..., N_n).$$

We shall apply the vector version of Krasnoselskii's fixed point theorem to the operator N on the space

$$C([-\tau, T], X) = C([-\tau, T], X_1) \times ... \times C([-\tau, T], X_n)$$

endowed with the vector-valued norm

$$\|u\| = \left(|u_1|_\tau, ..., |u_n|_\tau\right)^{tr},$$

where for each i, by $|u_i|_\tau$ we mean the norm in $C([-\tau, T], X_i)$ given by (1.5), with $\theta > 0$ chosen below. The result will follow from Theorem 2.2 once the following lemmas have been proved: □

Lemma 3.2. *There exist $R_1, ..., R_n \geq 0$ such that $A(u) + B(v) \in D$ for every $u, v \in D$, where*

$$D = \{u = (u_1, ..., u_n) \in C([-\tau, T], X) : \ |u_i|_\tau \leq R_i \ \ \textit{for } i \in J\}.$$

Lemma 3.3. *The operator A is a generalized contraction in Perov's sense on $C([-\tau, T], X)$.*

Lemma 3.4. *The operator B is completely continuous on $C([-\tau, T], X)$.*

Proof of Lemma 3.2. The fact that $A(u) + B(v) \in D$ for every $u, v \in D$ is equivalent to the inequalities

$$|N_i(u)|_\tau \leq R_i \quad \text{for} \quad u \in D, \ i \in J. \tag{3.7}$$

First note that from (3.3) for $v = 0$,

$$|F_i(t, u)|_{X_i} \leq \sum_{j=1}^{n} a_{ij}(t) |u_j|_{C([-\tau,0],X_j)} + |F_i(t, 0)|_{X_i}.$$

Hence the condition (3.2) also holds for $i \in J_1$, with $c_i(t) = |F_i(t, 0)|_{X_i}$. Similarly, from (3.5), (3.4) also holds for $i \in J_1$, with $d_i = |\alpha_i(0)|_{C([-\tau,0],X_i)}$.

In order to prove (3.7), take any $u \in D$ and $i \in J$. For $t \in [-\tau, 0]$,

$$|N_i(u)(t)|_{X_i} = |\alpha_i(u)(t)|_{X_i} \leq |\alpha_i(u)|_{C([-\tau,0],X_i)} \leq \sum_{j=1}^{n} b_{ij} |u_j|_{C([-\tau,a],X_j)} + d_i. \tag{3.8}$$

For $t \in [0, a]$, since the semigroups are of contractions,

$$|N_i(u)(t)|_{X_i} \leq |\alpha_i(u)(0)|_{X_i} + \int_0^t |F_i(s, u_s)|_{X_i} ds.$$

The term $|\alpha_i(u)(0)|_{X_i}$ is evaluated as in (3.8), while for the integral we have

$$\begin{aligned}\int_0^t |F_i(s, u_s)|_{X_i} ds &\leq \int_0^t \left(\sum_{j=1}^{n} a_{ij}(s) \left|(u_j)_s\right|_{C([-\tau,0],X_j)} + c_i(s) \right) ds \\ &= \int_0^t \left(\sum_{j=1}^{n} a_{ij}(s) |u_j|_{C([s-\tau,s],X_j)} + c_i(s) \right) ds.\end{aligned}$$

Since $0 \le s \le t \le a$, $|u_j|_{C([s-\tau,s],X_j)} \le |u_j|_{C([-\tau,a],X_j)}$. Then

$$\int_0^t |F_i(s,u_s)|_{X_i}\, ds \le \sum_{j=1}^n |a_{ij}|_{L^1(0,a)} |u_j|_{C([-\tau,a],X_j)} + |c_i|_{L^1(0,a)}.$$

Hence for $t \in [-\tau,a]$,

$$|N_i(u)(t)|_{X_i} \le \sum_{j=1}^n \left(|a_{ij}|_{L^1(0,a)} + b_{ij} \right) |u_j|_{C([-\tau,a],X_j)} + \gamma_i,$$

where $\gamma_i = |c_i|_{L^1(0,a)} + d_i$. As a result, for every $i \in J$, one has the following estimation of the first component from the definition of the norm $|N_i(u)|_\tau$,

$$|N_i(u)|_{C([-\tau,a],X_i)} \le \sum_{j=1}^n m_{ij} |u_j|_{C([-\tau,a],X_j)} + \gamma_i. \tag{3.9}$$

Furthermore, we estimate $|N_i(u)|_{C_\theta([a-\tau,T],X_i)}$. To this aim, let $t \in [a,T]$. One has

$$|N_i(u)_t|_{C([-\tau,0],X_i)} = |N_i(u)|_{C([t-\tau,t],X_i)}.$$

Take any $s \in [t-\tau,t]$ and try to estimate $|N_i(u)(s)|_{X_i}$. For $s \le a$, we already have the estimation given by (3.9). Let $s \in [a,t]$. Then

$$\begin{aligned} |N_i(u)(s)|_{X_i} &\le |\alpha_i(u)(0)|_{X_i} + \int_0^a |F_i(\xi,u_\xi)|_{X_i}\, d\xi + \int_a^s |F_i(\xi,u_\xi)|_{X_i}\, d\xi \\ &\le \sum_{j=1}^n m_{ij} |u_j|_{C([-\tau,a],X_j)} + \gamma_i + \int_a^s |F_i(\xi,u_\xi)|_{X_i}\, d\xi. \end{aligned} \tag{3.10}$$

Also

$$\begin{aligned} \int_a^s |F_i(\xi,u_\xi)|_{X_i}\, d\xi &\le \int_a^s \left(\sum_{j=1}^n a_{ij}(\xi) \left|(u_j)_\xi\right|_{C([-\tau,0],X_j)} + c_i(\xi) \right) d\xi \\ &\le \sum_{j=1}^n \int_a^s a_{ij}(\xi) \left|(u_j)_\xi\right|_{C([-\tau,0],X_j)} d\xi + |c_i|_{L^1(a,T)}. \end{aligned}$$

Now we estimate the integral terms

$$\begin{aligned} \int_a^s a_{ij}(\xi) \left|(u_j)_\xi\right|_{C([-\tau,0],X_j)} d\xi &= \int_a^s a_{ij}(t) e^{\theta(t-a)} \left|(u_j)_t\right|_{C([-\tau,0],X_j)} e^{-\theta(t-a)} dt \\ &\le |u_j|_{C_\theta([a-\tau,T],X_j)} \int_a^s a_{ij}(t) e^{\theta(t-a)} dt. \end{aligned}$$

Let q by the conjugate exponent of p, i.e., $1/p + 1/q = 1$. Using Hölder's inequality,

$$\int_a^s a_{ij}(t) e^{\theta(t-a)} dt \le \frac{1}{(q\theta)^{1/q}} |a_{ij}|_{L^p(a,T)} e^{\theta(s-a)},$$

one can continue the estimation in (3.10) and obtain

$$|N_i(u)(s)|_{X_i} \leq e^{\theta(s-a)} \sum_{j=1}^{n} \left(m_{ij} + \frac{1}{(q\theta)^{1/q}} |a_{ij}|_{L^p(a,T)} \right) |u_j|_\tau + \widetilde{\gamma}_i,$$

where $\widetilde{\gamma}_i = \gamma_i + |c_i|_{L^1(a,T)} = |c_i|_{L^1(0,a)} + d_i$. Taking the maximum for $s \in [t-\tau, t]$ yields

$$|N_i(u)_t|_{C([-\tau,0],X_i)} \leq e^{\theta(t-a)} \sum_{j=1}^{n} \left(m_{ij} + \frac{1}{(q\theta)^{1/q}} |a_{ij}|_{L^p(a,T)} \right) |u_j|_\tau + \widetilde{\gamma}_i.$$

Dividing by $e^{-\theta(t-a)}$ and taking the maximum for $t \in [a,T]$ gives

$$|N_i(u)|_{C_\theta([a-\tau,T],X_i)} \leq \sum_{j=1}^{n} \widetilde{m}_{ij} |u_j|_\tau + \widetilde{\gamma}_i,$$

where

$$\widetilde{m}_{ij} = m_{ij} + \frac{1}{(q\theta)^{1/q}} |a_{ij}|_{L^p(a,T)}.$$

This together with (3.9) proves that

$$|N_i(u)|_\tau \leq \sum_{j=1}^{n} \widetilde{m}_{ij} |u_j|_\tau + \widetilde{\gamma}_i, \quad i \in J,$$

or under the matricial form

$$\|N(u)\| \leq \widetilde{M} \|u\| + \widetilde{\gamma},$$

where $\widetilde{M}$ is the $n \times n$ matrix $[\widetilde{m}_{ij}]$, and $\widetilde{\gamma}$ is the column vector $(\widetilde{\gamma}_1, ..., \widetilde{\gamma}_n)^{tr}$. Since the spectral radius of the matrix M is assumed less than one, and the elements of the matrix $\widetilde{M}$ can be assumed as close to the elements of M as we wish (by taking a sufficiently large $\theta > 0$), from Proposition 2.3 we may consider that the spectral radius of $\widetilde{M}$ is less than one. The proof is finished if there is a vector $R = (R_1, ..., R_n)^{tr}$ of nonnegative numbers such that

$$\widetilde{M}R + \widetilde{\gamma} \leq R,$$

that is $\left(I - \widetilde{M}\right) R \geq \widetilde{\gamma}$. Indeed we can take $R = \left(I - \widetilde{M}\right)^{-1} \widetilde{\gamma}$ whose components are nonnegative since the matrix $I - \widetilde{M}$ is inverse-positive. □

Proof of Lemma 3.3. Similar estimations to those in the proof of Lemma 3.2 give for $i \in J_1$ and any $u, v \in C([-\tau, T], X)$,

$$|N_i(u) - N_i(v)|_{C([-\tau,a],X_i)} \leq \sum_{j=1}^{n} m_{ij} |u_j - v_j|_{C([-\tau,a],X_j)}$$

and

$$|N_i(u) - N_i(v)|_{C_\theta([a-\tau,T],X_i)} \le \sum_{j=1}^{n} \widetilde{m}_{ij} |u_j - v_j|_\tau .$$

Hence

$$|N_i(u) - N_i(v)|_\tau \le \sum_{j=1}^{n} \widetilde{m}_{ij} |u_j - v_j|_\tau .$$

Since for $i \in J_2$ the component A_i of the operator A are zero, Lipschitz inequalities as above also hold with all the Lipschitz constants equal to zero. Consequently,

$$\|A(u) - A(v)\| \le \widehat{M} \|u - v\| , \tag{3.11}$$

where $\widehat{M}$ is the $n \times n$ square matrix $[\widehat{m}_{ij}]$, with

$$\widehat{m}_{ij} = \begin{cases} \widetilde{m}_{ij} & \text{for } i \in J_1,\ j \in J \\ 0 & \text{for } i \in J_2,\ j \in J. \end{cases}$$

Clearly $\widehat{M} \le \widetilde{M}$, hence according to Proposition 2.4, the spectral radius of $\widehat{M}$ is less than one. Then (3.11) shows that A is a generalized contraction in Perov's sense. □

Proof of Lemma 3.4. The first components B_i for $i \in J_1$ are zero, so compact. The continuity and growth conditions for F_i $(i \in J_2)$ make that the mappings $u \mapsto F$ $_i(t, u_t)$ are continuous and bounded (send bounded sets into bounded sets). Also from (H4), the operators α_i and $S_i(t)$ are compact for $i \in J_2$ and $t > 0$. Then the generalized version of the Arzelà–Ascoli theorem for functions with values in a metric space (see, e.g., [34, pp 72-74] and [43, p 296]), guarantees that the operator B_i is completely continuous for every $i \in J_2$. □

For the next result, none of the mappings α_i is assumed to be completely continuous.

Theorem 3.5. *Assume that the conditions* (H1), (H2*) and (H3*) *hold for every* $i \in J_1$, *and that* (H1*), (H2) and (H3*) *hold for every* $i \in J_2$. *In addition assume that the spectral radius of the* $n \times n$ *square matrix* $M = [m_{ij}]$, *where*

$$m_{ij} := |a_{ij}|_{L^1(0,a)} + b_{ij}, \tag{3.12}$$

is less than one. Then the problem (1.1) *has at least one mild solution on* $[-\tau, T]$. *In case that* $m = n$, *the solution is unique.*

Proof. The only one difference is the way that the operator N is split as $N = A + B$. Now we take

$$A = (N_1, ..., N_m, A_{m+1}, ..., A_n), \quad B = (0, ..., 0, B_{m+1}, ..., B_n),$$

where for $i \in J_2$,

$$A_i(u)(t) = \begin{cases} \alpha_i(u)(t), & t \in [-\tau, 0], \\ S_i(t)\,\alpha_i(u)(0), & t \in [0, T], \end{cases}$$

and

$$B_i(u)(t) = \begin{cases} 0, & t \in [-\tau, 0], \\ \int_0^t S_i(t-s)\,F_i(s, u_s)\,ds, & t \in [0, T]. \end{cases}$$

The operator B is completely continuous, while A satisfies

$$\|A(u) - A(v)\| \le \widehat{M}^0 \|u - v\|,$$

where $\widehat{M}^0$ is the $n \times n$ square matrix $\left[\widehat{m}_{ij}^0\right]$, with

$$\widehat{m}_{ij}^0 = \begin{cases} \widetilde{m}_{ij} & \text{for } i \in J_1,\ j \in J \\ b_{ij} & \text{for } i \in J_2,\ j \in J, \end{cases}$$

which also satisfies $\widehat{M}^0 \le \widetilde{M}$. □

We conclude by two examples illustrating our main result.

Example 1. Consider the semilinear delay transport system

$$\begin{cases} u_1'(t) + d^1 \cdot \nabla u_1(t) = \alpha_{11}(t) u_1(t-\tau) + \alpha_{12}(t) \sin u_2(t-\tau), & t \in [0, T], \\ u_2'(t) + d^2 \cdot \nabla u_2(t) = \alpha_{21}(t) \cos u_1(t-\tau) + \alpha_{22}(t) u_2(t-\tau), & t \in [0, T], \\ u_1(t) = \int_0^a (\beta_{11}(t)\,u_1(t+s) + \beta_{12}(t)\,u_2(t+s))\,ds, & t \in [-\tau, 0], \\ u_2(t) = \int_0^a (\beta_{21}(t)\,u_1(t+s) + \beta_{22}(t)\,u_2(t+s))\,ds, & t \in [-\tau, 0]. \end{cases} \tag{3.13}$$

Here, $d^1, d^2 \in \mathbb{R}^n$; $\alpha_{ij} \in C[0, T]$ and $\beta_{ij} \in L^1(-\tau, 0)$, for $i, j = 1, 2$; $a \in [0, T]$; and $u_i(t)(x) = u_i(t, x)$ for $x \in \mathbb{R}^n$, $i = 1, 2$. Also, by a notation like $\sin u_2(t-\tau)$ we mean the function $x \mapsto \sin(u_2(t-\tau, x))$.

Let us consider $X_i = L^{r_i}(\mathbb{R}^n)$, with $1 \le r_i < \infty$ for $i = 1, 2$. Then (see [43, p 88]), the operators A_i $(i = 1, 2)$ defined by

$$D(A_i) = \left\{v \in X_i :\ d^i \cdot \nabla v \in X_i\right\}$$
$$A_i v = -d^i \cdot \nabla v = -\sum_{j=1}^n d_j^i \frac{\partial v}{\partial x_j},$$

are generators of the C_0-groups of isometries $\{S_i(t);\ t \in \mathbb{R}\}$,

$$S_i(t)\,v(x) = v\left(x - td^i\right), \quad \text{for } v \in X_i,\ t \in \mathbb{R},\ x \in \mathbb{R}^n.$$

In this case all the assumptions of Theorem 3.1 are satisfied with $J = J_1 = \{1, 2\}$, $J_2 = \emptyset$, $a_{ij}(t) = |\alpha_{ij}(t)|$ (absolute value) and $b_{ij} = |\beta_{ij}|_{L^1(-\tau,0)}$. Therefore, if the spectral radius of the matrix

$$M = \begin{bmatrix} |\alpha_{11}|_{L^1(0,a)} + |\beta_{11}|_{L^1(-\tau,0)} & |\alpha_{12}|_{L^1(0,a)} + |\beta_{12}|_{L^1(-\tau,0)} \\ |\alpha_{21}|_{L^1(0,a)} + |\beta_{21}|_{L^1(-\tau,0)} & |\alpha_{22}|_{L^1(0,a)} + |\beta_{22}|_{L^1(-\tau,0)} \end{bmatrix}$$

is less than one, then the problem (3.13) has a unique mild solution in $C([-\tau,T], L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n))$.

Example 2. Let us consider a semilinear reaction-diffusion system of two Mackey-Glass type diffusive equations, with multi-point nonlocal initial conditions

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - d_1 \Delta u(t,x) = f(t,u_t,v_t)\, u(t-\tau,x) - \lambda_1 u(t,x) - a_{12}(t)\, v(t,x), & \text{in } Q, \\ \frac{\partial v}{\partial t}(t,x) - d_2 \Delta v(t,x) = g(t,u_t,v_t)\, v(t-\tau,x) - \lambda_2 v(t,x) - a_{21}(t)\, u(t,x), & \text{in } Q, \\ \frac{\partial u}{\partial \nu}(t,x) = \frac{\partial v}{\partial \nu}(t,x) = 0, & \text{on } \Sigma, \\ u(t,x) = \varphi(t)(x) + \sum_{k=1}^{p_1} \alpha_{1k} u(t_{1k}+t,x), & \text{in } Q_\tau, \\ v(t,x) = \psi(t)(x) + \sum_{j=k}^{p_2} \alpha_{2k} v(t_{2k}+t,x), & \text{in } Q_\tau, \end{cases} \tag{3.14}$$

where $Q = [0,T] \times \Omega$, $\Sigma = [0,T] \times \partial\Omega$, $Q_\tau = [-\tau,0] \times \Omega$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $d_1, d_2, \lambda_1, \lambda_2 > 0$, $\tau \geq 0$ and $0 < t_{i1} < ... < t_{ip_i} \leq T$ for $i = 1,2$.

We assume that $f, g : [0,T] \times C\left([-\tau,0], L^2(\Omega) \times L^2(\Omega)\right) \to \mathbb{R}$ are continuous; $\varphi, \psi \in C\left([-\tau,0], L^2(\Omega)\right)$; $|f(t,u,v)| \leq a_{11}(t)$, $|g(t,u,v)| \leq a_{22}(t)$ for all $t \in [0,T]$ and $u, v \in C[-\tau,0]$; and that $a_{ij} \in L^1(0,T;\mathbb{R}_+)$ with $a_{ij}|_{[0,a]} \in L^p(0,a)$ for some $p > 1$, where $a = \max\{t_{ij} : j = 1,...,p_i;\ i = 1,2\}$.

We apply Theorem 3.5 with $X_1 = X_2 = L^2(\Omega)$, and to the operators $A_i : D(A_i) \to L^2(\Omega)$ $(i = 1,2)$ given by

$$D(A_i) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\},$$
$$A_i u = d_i \Delta u - \lambda_i u,$$

which generate compact semigroups [11, Theorem 1.11.8]. Here $J_1 = \emptyset$ and $J_2 = J = \{1,2\}$, $b_{ii} = \sum_{k=1}^{p_i} |\alpha_{ik}|$ $(i = 1,2)$ and $b_{12} = b_{21} = 0$. Therefore, if the spectral radius of the matrix

$$M = \left[|a_{ij}|_{L^1(0,a)} + b_{ij} \right]_{i,j=1,2}$$

is less than one, then the problem (3.14) has at least one mild solution in $C([-\tau,T], L^2(\Omega) \times L^2(\Omega))$.

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