SOLVING TWO-DIMENSIONAL NONLINEAR VOLTERRA-FREDHOLM FUZZY INTEGRAL EQUATIONS BY USING ADOMIAN DECOMPOSITION METHOD

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ABSTRACT: In this paper, we propose Adomian decomposition method (ADM) to approximate the solution of two-dimensional nonlinear Volterra-Fredholm fuzzy integral equation (2D-NVFFIE). We convert this integral equation to a nonlinear system of Volterra-Fredholm integral equations in crisp case. The aim of this paper is to find an approximate solution of this system using ADM. Hence, we obtain an approximation for the fuzzy solution of the nonlinear Volterra-Fredholm fuzzy integral equation. A numerical example is given to demonstrate the validity and applicability of the proposed technique.

1. INTRODUCTION

In recent years, fuzzy differential and integral equations are attracting ever greater interest. They are one of the important part of the fuzzy analysis theory that play major role in numerical analysis. Babolian, Goghary and Abbasbandy [4] proposed another numerical procedure for solving fuzzy linear Fredholm integral of the second kind using Adomian method. Moreover, Friedman et. al [15] and Seikkala in [25]

defined the fuzzy derivative and then some generalizations of that have been investigated in [7, 9]. Consequently, the fuzzy integral which is the same as that of Dubois and Prade in [10]. Shao and Zhang [26] chose to define the integral of fuzzy function, using the Lebesgue-type concept for integration. One of the first applications of fuzzy integration was given by Wu and Ma who investigated the fuzzy Fredholm integral equation of the second kind. Recently, some mathematicians have studied solution of fuzzy integral equations by numerical methods [3, 5, 14, 17, 19, 20, 23].

The problems posed in the study of fuzzy integral equations are: existence and uniqueness, boundedness of the solutions [12, 13, 18] and the construction of numerical methods for the approximate solution.

Adomian decomposition method (ADM) has been recently intensively studied by scientists and engineers and used for solving nonlinear differential and integral problems [3, 4, 5, 11]. The ADM introduced by Adomian [1, 2] for solving different kind of functional equations and has been subject of extensive numerical and analytical studies.

In this paper we consider the following 2D-NVFFIE

$$u(s,t) = g(s,t) \oplus (FR) \int_c^t k_1(t,\tau) \odot G_1(u(s,\tau)) d\tau$$

$$\oplus (FR) \int_a^b k_2(s,\xi) \odot G_2(u(\xi,t)) d\xi, \quad (1.1)$$

where g, $u : A = [a,b] \times [c,d] \to E^1$ are continuous fuzzy-number valued functions, $k_1 : [c,d] \times [c,d] \to \mathbb{R}_+$, $k_2 : [a,b] \times [a,b] \to \mathbb{R}_+$ are continuous functions and G_1, $G_2 : E^1 \to E^1$ are continuous functions on E^1. The set E^1 is the set of all fuzzy numbers.

Equation (1.1) is called Fredholm integral equation with respect to the position and Volterra with respect to the time. This type of equation appears in many problems of mathematical physics, theory of elasticity, contact problems and mixed problems of mechanics of continuous media.

The paper is organized as follows: in Section 2, are given some basic notations of fuzzy numbers, fuzzy functions and fuzzy integrals. In Section 3, is introduced the parametric form of 2D-NVFFIE and then ADM is applied for solving this equation. In Section 4, is proved existence and uniqueness of the solution and convergence of the proposed method . In Section 5, is illustrated the accuracy of the method by solving numerical example.

2. BASIC CONCEPTS

In this section, we review some notions and results about fuzzy numbers, fuzzy-number-valued functions and fuzzy integrals.

Definition 2.1.   [21] A fuzzy number is a function $u : \mathbb{R} \rightarrow [0,1]$ satisfying the following properties:

   (i) u is upper semi-continuous on $\mathbb{R}$,

   (ii) $u(x) = 0$ outside of some interval $[c,d]$,

   (iii) there are the real numbers a and b with $c \leq a \leq b \leq d$, such that u is increasing on $[c,a]$, decreasing on $[b,d]$ and $u(x) = 1$ for each $x \in [a,b]$.

As it is shown in [10], the fuzzy numbers have the following convexity: u is fuzzy convex set i. e. that is $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$ and possess compact support $[u]^0 = x \in \mathbb{R} : u(x) > 0$, where $\overline{A}$ denotes the closure of A.

The set of all fuzzy numbers is denoted by E^1. Any real number $a \in \mathbb{R}$ can be interpreted as a fuzzy number $\tilde{a} = \chi_{[a]}$ and therefore $\mathbb{R} \subset E^1$. For any $0 < r \leq 1$ we denote the $r-$level set $[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}$ that is a closed interval and $[u]^r = [u_-^r, u_+^r]$ for all $r \in [0,1]$. These lead to the usual parametric representation of a fuzzy number: $[u]^r = [u_-^r, u_+^r]$ for all $r \in [0,1]$, where u_-, u_+ can be considered as functions $u_-, u_+ : [0,1] \rightarrow \mathbb{R}$, such that u_- is increasing and u_+ is decreasing.

For $u, v \in E^1$, $k \in \mathbb{R}$, the addition and the scalar multiplication are defined by

$$[u \oplus v]^r = [u]^r + [v]^r = [u_-^r + v_-^r, u_+^r + v_+^r] \text{ and}$$

$$[k \odot u]^r = k.[u]^r = \begin{cases} [ku_-^r, ku_+^r], & \text{if } k \geq 0 \\ [ku_+^r, ku_-^r], & \text{if } k < 0 \end{cases} \text{ for all } r \in [0,1].$$

According to [27] we can summarize the following algebraic properties of addition and scalar multiplication of fuzzy numbers:

   (i) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ and $u \oplus v = v \oplus u$ for any $u,$ v $w \in E^1$,

   (ii) $u \oplus \tilde{0} = \tilde{0} \oplus u = u$ for any $u \in E^1$,

   (iii) with respect to $\tilde{0}$, none $u \in E^1 \setminus \mathbb{R}$, $u \neq \tilde{0}$ has opposite in $(E^1, \oplus)$,

   (iv) for any $a, b \in \mathbb{R}$ with $a, b \leq 0$ or $a, b \geq 0$, and any $u \in E^1$ we have $(a+b) \odot u = a \odot u \oplus b \odot u$,

   (v) for any $a \in \mathbb{R}$ and any $u, v \in E^1$ we have $a \odot (u \oplus v) = a \odot u \oplus a \odot v$,

   (vi) for any $a, b \in \mathbb{R}$ and any $u \in E^1$ we have $a \odot (b \odot u) = (ab) \odot u$ and $1 \odot u = u$.

As a distance between fuzzy numbers we use the Hausdorff metric [27] defined by
$D(u,v) = \sup_{r \in [0,1]} max(|u_-^r - v_-^r|, |u_+^r - v_+^r|)$ for any $u, v \in E^1$.

Lemma 2.1. *[6] The Hausdorff metric has the following properties:*

    (i) (E^1, D) *is a complete metric space,*

    (ii) $D(u \oplus w, v \oplus w) = D(u, v)$ *for all* $u, v, w \in E^1$,

    (iii) $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e)$ *for all* $u, v, w, e \in E^1$,

    (iv) $D(u \oplus v, \tilde{0}) \leq D(u, \tilde{0}) + D(v, \tilde{0})$ *for all* $u, v \in E^1$,

    (v) $D(k \odot u, k \odot v) = |k| D(u, v)$ *for all* $u, v \in E^1$, *for all* $k \in \mathbb{R}$.

Lemma 2.2. *[16] For any* k_1, $k_2 \in$, $k2\mathbb{R}$ *with* $k_1.k_2 \geq 0$ *and any* $u \in E^1$ *we have*

$$D(k_1 \odot u, k_2 \odot u) = |k_1 - k_2| D(u, \tilde{0}).$$

For any fuzzy-number-valued function $f : I \subset \mathbb{R} \to E^1$ we can define the functions $\underline{f}(., r)$, $\overline{f}(., r) : I \to \mathbb{R}$, by $\underline{f}(t, r) = (f(t))^r_-$, $\overline{f}(t, r) = (f(t))^r_+$ for each $t \in I$, for each $r \in [0, 1]$. These functions are called the left and right $r-$level functions of f.

Definition 2.2. *[28] A fuzzy-number-valued function* $f : [a, b] \to E^1$ *is said to be continuous at* $t_0 \in [a, b]$ *if for each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $D(f(t), f(t_0)) < \varepsilon$ *whenever* $|t - t_0| < \delta$. *If* f *is continuous for each* $t \in [a, b]$ *then we say that* f *is continuous on* $[a, b]$. *A fuzzy number* $u \in E^1$ *is upper bound for a fuzzy-number-valued function* $f : [a, b] \to E^1$ *if* $\underline{f}(., r) \leq u^r_-$ *and* $\overline{f}(., r) \leq u^r_+$ *for all* $t \in [a, b]$, $r \in [0, 1]$. *A fuzzy number* $u \in E^1$ *is lower bound for a fuzzy number-valued function* $f : [a, b] \to E^1$ *if* $u^r_+ \leq \overline{f}(., r)$ *and* $u^r_- \leq \underline{f}(., r)$ *for all* $t \in [a, b]$, $r \in [0, 1]$. *A fuzzy-number-valued function* $f : [a, b] \to E^1$ *is said to be bounded it has a lower bound and an upper bound.*

We can see that the above definition of the boundedness of a fuzzy-number-valued function can be expressed in the following equivalent form $f : [a, b] \to E^1$ is bounded iff there is $M \geq 0$ such that $D(f(t), \tilde{0}) \leq M$ for all $t \in [a, b]$. The constant M can be chosen as $M \geq max\{u^0_-, u^0_+\}$.

Lemma 2.3. *[28] If* $f : [a, b] \to E^1$ *is continuous then it is bounded and its supremum* $\sup_{t \in [a,b]} f(t)$ *must exist and is determined by* $u \in E^1$ *with* $u^r_- = \sup_{t \in [a,b]} f^r_-(t)$ *and* $u^r_+ = \sup_{t \in [a,b]} f^r_+(t)$. *A similar conclusion for the infinum is also true.*

On the set $C([a, b], E^1) = \{f : [a, b] \to E^1; \ f \text{ continuous}\}$ there is defined the metric $D^*(f, g) = \sup_{t \in [a,b]} D(f(t), g(t))$, for all $f, g \in C([a, b], E^1)$. We see that $(C([a, b], E^1), D^*)$ is a complete metric space.

In [27] Wu and Gong the notion of Henstock integral for fuzzy-number-valued functions is defined as follows:

Definition 2.3. Let $f : [a, b] \to E^1$. For $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ a partition of the interval $[a, b]$, we consider the points $\xi_i \in [x_{i-1}, x_i], \quad i = 1, ..., n$, and the function $\delta : [a, b] \to \mathbb{R}_+$. The partition $P = \{([x_{i-1}, x_i]; \xi_i); i = 1, ..., n\}$ denoted by $P = (\Delta_n, \xi)$ is called δ-fine iff $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$. For $I \in E^1$, the function f is fuzzy-Henstock integrable on $[a, b]$ if for any $\varepsilon > 0$ there is a function $\delta : [a, b] \to \mathbb{R}_+$ such that for any partition δ-fine P, $D(\sum\limits_{i=1}^{n} (x_i - x_{i-1}) \odot f(\xi_i), I) < \varepsilon$. The fuzzy number I is named the fuzzy-Henstock integral of f and will be denoted by $(FH) \int\limits_a^b f(t)dt$.

When the function $\delta : [a, b] \to \mathbb{R}_+$ is constant, then we obtain the Riemann integrability for fuzzy-number-valued functions [21]. In this case, $I \in E^1$ is called the fuzzy-Riemann integral of f on $[a, b]$, being denoted by $(FR) \int\limits_a^b f(t)dt$. Consequently, the fuzzy-Riemann integrability is a particular case of the fuzzy-Henstock integrability, and therefore the properties of the integral (FH) will be valid for the integral (FR), too.

Lemma 2.4.   [22] Let $f : [a, b] \to E^1$. Then f is (FH) integrable if and only if $\underline{f}(., r)$ and $\overline{f}(., r)$ are Henstock integrable for any $r \in [0, 1]$. Furthermore, for any $r \in [0, 1]$,

$$\left[(FH) \int\limits_a^b f(t)dt \right]^r = \left[(H) \int\limits_a^b \underline{f}(t, r)dt, \ (H) \int\limits_a^b \overline{f}(t, r)dt \right].$$

Remark 2.1. If $f : [a, b] \to E^1$ is continuous, then $\underline{f}(., r)$ and $\overline{f}(., r)$ are continuous for any $r \in [0, 1]$ and consequently, they are Henstock integrable. According to Lemma 2.2 we infer that f is (FH) integrable

Lemma 2.5.   [8] If f and g are fuzzy-Henstock integrable functions and if the function given by $D(f(t), g(t))$ is Lebesgue integrable, then

$$D \left((FH) \int\limits_a^b f(t)dt, \ (FH) \int\limits_a^b g(t)dt \right) \leq (L) \int\limits_a^b D(f(t), g(t))dt.$$

Definition 2.4.   [24] A function $f : A \to E^1$ is called:
(i) continuous in $(s_0, t_0) \in A$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $(s, t) \in A$ with $|s - s_0| < \delta$, $|t - t_0| < \delta$ it follows that $D(f(s, t), f(s_0, t_0)) < \varepsilon$. The function f is continuous on A if it is continuous in each $(s, t) \in A$.
(ii) bounded if there exists $M \geq 0$ such that $D(f(s, t), \tilde{0}) \leq M$ for all $(s, t) \in A$.

On the set $C(A, E^1) = \{f : A \to E^1; \ f \text{ continuous}\}$ there is defined the metric

$D^*(f,g) = \sup\limits_{(s,t)\in A} D(f(s,t),g(s,t))$, for all $f,g \in C(A,E^1)$.

We see that $(C(A,E^1),D^*)$ is a complete metric space.

3. ADM FOR SOLVING 2D-NVFFIE

In this section we introduce the parametric form of integral equation (1.1) and then apply ADM for solving this equation. For solving in parametric form of equation (1.1), consider $u(s,t,r) = (\underline{u}(s,t,r),\overline{u}(s,t,r))$ and
$g(s,t,r) = (\underline{g}(s,t,r),\overline{g}(s,t,r))$, $0 \le r \le 1$ and $(s,t) \in A$ are parametric form of $u(s,t)$ and $g(s,t)$, respectively. So the parametric form of equation (1.1) is as follows:

$$\underline{u}(s,t,r) = \underline{g}(s,t,r) + \int_c^t \underline{k_1(t,\tau)G_1(u(s,\tau,r))}d\tau + \int_a^b \underline{k_2(s,\xi)G_2(u(\xi,t,r))}d\xi.$$

$$\overline{u}(s,t,r) = \overline{g}(s,t,r) + \int_c^t \overline{k_1(t,\tau)G_1(u(s,\tau,r))}d\tau + \int_a^b \overline{k_2(s,\xi)G_2(u(\xi,t,r))}d\xi.$$

Let for $(s,t) \in A$, we have

$$H_1(\underline{u}(s,t,r),\overline{u}(s,t,r)) = \min\{G_1(\beta) : \underline{u}(s,t,r) \le \beta \le \overline{u}(s,t,r)\},$$

$$H_2(\underline{u}(s,t,r),\overline{u}(s,t,r)) = \min\{G_2(\beta) : \underline{u}(s,t,r) \le \beta \le \overline{u}(s,t,r)\},$$

$$F_1(\underline{u}(s,t,r),\overline{u}(s,t,r)) = \max\{G_1(\beta) : \underline{u}(s,t,r) \le \beta \le \overline{u}(s,t,r)\},$$

$$F_2(\underline{u}(s,t,r),\overline{u}(s,t,r)) = \max\{G_2(\beta) : \underline{u}(s,t,r) \le \beta \le \overline{u}(s,t,r)\}.$$

Then:

$$\underline{k_1(t,\tau)G_1(u(s,\tau,r))} = \begin{cases} k_1(t,\tau)H_1(\underline{u}(s,\tau,r),\overline{u}(s,\tau,r)), & \text{if } k_1(t,\tau) \ge 0 \\ k_1(t,\tau)F_1(\underline{u}(s,\tau,r),\overline{u}(s,\tau,r)), & \text{if } k_1(t,\tau) < 0 \end{cases}$$

$$\underline{k_2(s,\xi)G_2(u(\xi,t,r))} = \begin{cases} k_2(s,\xi)H_2(\underline{u}(\xi,t,r),\overline{u}(\xi,t,r)), & \text{if } k_2(s,\xi) \ge 0 \\ k_2(s,\xi)F_2(\underline{u}(\xi,t,r),\overline{u}(\xi,t,r)), & \text{if } k_2(s,\xi) < 0 \end{cases}$$

$$\overline{k_1(t,\tau)G_1(u(s,\tau,r))} = \begin{cases} k_1(t,\tau)F_1(\underline{u}(s,\tau,r),\overline{u}(s,\tau,r)), & \text{if } k_1(t,\tau) \ge 0 \\ k_1(t,\tau)H_1(\underline{u}(s,\tau,r),\overline{u}(s,\tau,r)), & \text{if } k(t,\tau) < 0 \end{cases}$$

$$\overline{k_2(s,\xi)G_2(u(\xi,t,r))} = \begin{cases} k_2(s,\xi)F_2(\underline{u}(\xi,t,r),\overline{u}(\xi,t,r)), & \text{if } k_2(s,\xi) \ge 0 \\ k_2(s,\xi)H_2(\underline{u}(\xi,t,r),\overline{u}(\xi,t,r)), & \text{if } k_2(s,\xi) < 0 \end{cases}$$

for $a \leq s, \xi \leq b$, $c \leq \tau \leq t \leq d$ and $0 \leq r \leq 1$. The ADM method is applied to solve 2D-NVFFIE. Let for all $a \leq s, \xi \leq b$, $c \leq \tau \leq t \leq d$ and $0 \leq r \leq 1$ the functions $G_1(\beta)$, $G_2(\beta)$ are increasing for $\beta \in [\underline{u}(s,t,r), \overline{u}(s,t,r)]$ and $k_1(t,\tau) \geq 0$, $k_2(s,\xi) \geq 0$. Then the parametric form of (1.1) is

$$
\begin{aligned}
\underline{u}(s,t,r) =& \underline{g}(s,t,r) + \int_c^t k_1(t,\tau)G_1(\underline{u}(s,\tau,r))d\tau \\
& + \int_a^b k_2(s,\tau)G_2(\underline{u}(\xi,t,r))d\xi \\
\overline{u}(s,t,r) =& \overline{g}(s,t,r) + \int_c^t k_1(t,\tau)G_1(\overline{u}(s,\tau,r))d\tau \\
& + \int_a^b k_2(s,\tau)G_2(\overline{u}(\xi,t,r))d\xi.
\end{aligned}
\tag{3.1}
$$

Now, we explain ADM as a numerical algorithm for approximating solution of this system of nonlinear integral equations in crisp case. Then, we find approximate solution for $u(s,t,r)$. The ADM assume an infinite series solution for the unknowns functions $(\underline{u}(s,t,r), \overline{u}(s,t,r))$, given by

$$
\underline{u}(s,t,r) = \sum_{i=0}^{\infty} \underline{u}_i(s,t,r), \quad \overline{u}(s,t,r) = \sum_{i=0}^{\infty} \overline{u}_i(s,t,r)
\tag{3.2}
$$

The nonlinear operator $G_1(\underline{u})$, $G_1(\overline{u})$, $G_2(\underline{u})$ and $G_2(\overline{u})$ into an infinite series of polynomials given by

$$
\begin{aligned}
G_1(\underline{u}) = \sum_{n=0}^{\infty} \underline{A}_n(\underline{u}_0, \underline{u}_1, ..., \underline{u}_n), \quad G_1(\overline{u}) = \sum_{n=0}^{\infty} \overline{A}_n(\overline{u}_0, \overline{u}_1, ..., \overline{u}_n), \\
G_2(\underline{u}) = \sum_{n=0}^{\infty} \underline{B}_n(\underline{u}_0, \underline{u}_1, ..., \underline{u}_n), \quad G_2(\overline{u}) = \sum_{n=0}^{\infty} \overline{B}_n(\overline{u}_0, \overline{u}_1, ..., \overline{u}_n),
\end{aligned}
\tag{3.3}
$$

where the $A_n = (\underline{A}_n, \overline{A}_n)$, $B_n = (\underline{B}_n, \overline{B}_n)$, $n \geq 0$ are the so-called Adomian polynomial defined by

$$
\begin{aligned}
\underline{A}_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[G_1 \left(\sum_{i=0}^{\infty} \lambda^i \underline{u}_i \right) \right]_{\lambda=0}, \quad \overline{A}_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[G_1 \left(\sum_{i=0}^{\infty} \lambda^i \overline{u}_i \right) \right]_{\lambda=0}, \\
\underline{B}_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[G_2 \left(\sum_{i=0}^{\infty} \lambda^i \underline{u}_i \right) \right]_{\lambda=0}, \quad \overline{B}_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[G_2 \left(\sum_{i=0}^{\infty} \lambda^i \overline{u}_i \right) \right]_{\lambda=0},
\end{aligned}
\tag{3.4}
$$

where λ is formal parameter.

Substituting equations (3.2) and (3.3) into equation (3.1) to get

$$\underline{u}_0(s,t,r) = \underline{g}(s,t,r)$$

$$\underline{u}_1(s,t,r) = \int_c^t k_1(t,\tau)\underline{A}_0 d\tau + \int_a^b k_2(s,\xi)\underline{B}_0 d\xi$$

$$\vdots$$

$$\underline{u}_{n+1}(s,t,r) = \int_c^t k_1(t,\tau)\underline{A}_n d\tau + \int_a^b k_2(s,\xi)\underline{B}_n d\xi$$

$$(3.5)$$

and

$$\overline{u}_0(s,t,r) = \overline{g}(s,t,r)$$

$$\overline{u}_1(s,t,r) = \int_c^t k_1(t,\tau)\overline{A}_0 d\tau + \int_a^b k_2(s,\xi)\overline{B}_0 d\xi$$

$$\vdots$$

$$\overline{u}_{n+1}(s,t,r) = \int_c^t k_1(t,\tau)\overline{A}_n d\tau + \int_a^b k_2(s,\xi)\overline{B}_n d\xi$$

$$(3.6)$$

We approximate $u(s,t,r) = (\underline{u}(s,t,r), \overline{u}(s,t,r))$ by

$$\underline{\phi}_n(s,t,r) = \sum_{i=0}^{n-1} \underline{u}_i(s,t,r), \overline{\phi}_n(s,t,r) = \sum_{i=0}^{n-1} \overline{u}_i(s,t,r),$$

where $\lim_{n\to\infty} \underline{\phi}_n(s,t,r) = \underline{u}(s,t,r)$ and $\lim_{n\to\infty} \overline{\phi}_n(s,t,r) = \overline{u}(s,t,r)$

4. EXISTENCE AND CONVERGENCE ANALYSIS

In this section we prove the existence, uniqueness of the solution of equation (1.1) and convergence of ADM.

We introduce the following conditions:

(i) $g \in C(A, E^1)$, $k_1 \in C([c,d] \times [c,d], \mathbb{R}_+)$, $k_2 \in C([a,b] \times [a,b], \mathbb{R}_+)$;

(ii) there exist $L_1 \geq 0$ and $L_2 \geq 0$ such that $D(G_1(u), G_1(v)) \leq L_1 D(u,v)$, $D(G_2(u), G_2(v)) \leq L_2 D(u,v)$ for all $u, v \in E^1$;

(iii) $\alpha = M_1 L_1(d-c) + M_2 L_2(b-a) < 1$, where $|k_1(t,\tau)| \leq M_1$ and $|k_2(s,\xi)| \leq M_2$ for all $a \leq s, \xi \leq b$, $c \leq t, \tau \leq d$ according to the continuity of k_1 and k_2.

Lemma 4.1. *Let the functions g_1, $g_2 \in C(A, E^1)$, $k_1 \in C([c, d] \times [c, d], \mathbb{R}_+)$ and $k_2 \in C([a, b] \times [a, b], \mathbb{R}_+)$. Then the functions $F_1, F_2 : A \to E^1$ defined by $F_1(s, t) = (FR) \int_c^t k_1(t, \tau) \odot g_1(s, \tau)d\tau$, $F_2(s, t) = (FR) \int_a^b k_2(s, \xi) \odot g_2(\xi, t)d\xi$ are continuous on A.*

Proof. Let arbitrary $(s_0, t_0) \in A$ and $\varepsilon > 0$. Since $g_1 \in C(A, E^1)$ and $k_1 \in C([c, d] \times [c, d], \mathbb{R}_+)$ we infer that there exist $\delta(\varepsilon) = \frac{\varepsilon}{3M} > 0$, where $D(k_1(t, \tau) \odot g_1(s, \tau), \tilde{0}) \leq M$, according to the continuity of k_1 and g_1.

Then for any $s \in [a, b]$, $t \in [c, d]$ with $|s - s_0| < \delta$ and $|t - t_0| < \delta$, it follows that $|k_1(t, \tau) - k_1(t_0, \tau)| < \frac{\varepsilon}{3M_{g_1}}$ and $D(g_1(s, \tau), g_1(s_0, \tau)) < \frac{\varepsilon}{3M_1}$, where M_1, $M_{g_1} \geq 0$ are such that $|k_1(t, \tau)| \leq M_1$ and $D(g_1(s, \tau), \tilde{0}) \leq M_{g_1}$ for all $a \leq s \leq b$, $c \leq t, \tau \leq d$. Then for $t > t_0$ we have

$$D(F_1(s, t), F_1(s_0, t_0)) \leq \leq \int_c^{t_0} D(k_1(t, \tau) \odot g_1(s, \tau), k_1(t_0, \tau) \odot g_1(s_0, \tau))d\tau$$

$$+ \int_{t_0}^t D(k_1(t, \tau) \odot g_1(s, \tau), \tilde{0})d\tau \leq$$

$$\leq \int_c^{t_0} |k_1(t, \tau)| D(g_1(s, \tau), g_1(s_0, \tau))d\tau$$

$$+ \int_c^{t_0} |k_1(t, \tau) - k_1(t_0, \tau)| D(g_1(s_0, \tau), \tilde{0})d\tau$$

$$+ \int_{t_0}^t D(k_1(t, \tau) \odot g_1(s, \tau), \tilde{0})d\tau$$

$$\leq M_1 \frac{\varepsilon}{3M_1} + \frac{\varepsilon}{3M_{g_1}} M_{g_1} + \frac{\varepsilon}{3M} M = \varepsilon.$$

Analogously is proved that the function F_2 is continuous on A. □

Theorem 4.1. *Let the conditions $(i) - (iii)$ are fulfilled. Then the integral equation (1.1) has a unique solution.*

Proof. Let $\mathfrak{F}(A, E^1) = \{f : A \to E^1\}$ and $X = C(A, E^1)$. We define the operator $A : X \to \mathfrak{F}(A, E^1)$ by

$$A(u)(s, t) = g(s, t) \oplus (FR) \int_c^t k_1(t, \tau) \odot G_1(u(s, \tau))d\tau$$

$$\oplus (FR) \int_a^b k_2(s,\xi) \odot G_2(u(\xi,t))d\xi, \quad (s,t) \in A, \ u \in X.$$

Firstly, we prove that $A(X) \subset X$. To this purpose, we show that the operator A is uniformly continuous. Let arbitrary $u \in X$, $(s_0, \tau_0) \in A$ and $\varepsilon > 0$. Since u is continuous it follows that for $\frac{\varepsilon}{L_1} > 0$ there exists $\delta(\varepsilon) > 0$ such that $D(u(s,\tau), u(s_0, \tau_0)) \leq \frac{\varepsilon}{L_1}$ for any $(s,t) \in A$ with $|s - s_0| + |\tau - \tau_0| < \delta$.

From condition (ii) we have

$$D(G_1(u(s,\tau)), G_1(u(s_0, \tau_0))) \leq L_1 D(u(s,\tau), u(s_0, \tau_0)) \leq \varepsilon$$

and the function $G_1^u : A \to E^1$, defined by $G_1^u(s,\tau) = G_1(u(s,\tau))$ is continuous in (s_0, τ_0) for any $u \in X$. We conclude that G_1^u is continuous on A for any $u \in X$. Applying Lemma 4.1 it follows that the function $F_1^u : A \to E^1$, defined by $F_1^u(s,t) = (FR) \int_c^t k_1(t,\tau) \odot G_1^u(s,\tau)d\tau$ is continuous on A for any $u \in X$.

Let $G_2^u : A \to E^1$ be function defined by $G_2^u(s,\tau) = G_2(u(s,\tau))$ for any $u \in X$ and $(s,t) \in A$. Analogously, we conclude that the function $F_2^u(s,t) = (FR) \int_a^b k_2(s,\xi) \odot G_2^u(\xi,t)d\xi$ is continuous on A for any $u \in X$. Since $g \in X$, we conclude that the operator $A(F)$ is continuous on A for any $u \in X$.

Now, we prove that $A : X \to X$ is a contraction. Let arbitrary $u, v \in X$. From conditions (ii), (iii) and Lemma 2.5 we have

$$D(A(u)(s,t), A(v)(s,t)) \leq D((FR) \int_c^t k_1(t,\tau)$$

$$\odot G_1(u(s,\tau))d\tau \oplus (FR) \int_a^b k_2(s,\xi) \odot G_2(u(\xi,t))d\xi,$$

$$(FR) \int_c^t k_1(t,\tau) \odot G_1(v(s,\tau))d\tau$$

$$\oplus (FR) \int_a^b k_2(s,\xi) \odot G_2(v(\xi,t))d\xi)$$

$$\leq \int_c^d |k_1(t,\tau)| D(G_1(u(s,\tau)), G_1(v(s,\tau)))d\tau$$

$$+ \int_a^b |k_2(s,\xi)| D(G_2(v(\xi,t)), G_2(v(\xi,t)))d\xi$$

$$\leq M_1 L_1 \int_c^d D(u(s,\tau), v(s,\tau))d\tau$$

$$+ M_2 L_2 \int_a^b D(u(\xi,t), v(\xi,t)) d\xi \leq \alpha D^*(u,v),$$

for all $(s,t) \in A$.

Under the condition $\alpha < 1$ the operator A is contraction therefore, by the Banach fixed-point theorem for contraction, there exist a unique solution to problem (1.1) and this completes the proof. $\qquad\square$

Theorem 4.2. *The series solution* $\overline{u}(s,t,r) = \sum\limits_{i=0}^{\infty} \overline{u}_i(s,t,r)$ *of the integral equation (1.1) using ADM converges if* $0 < \alpha < 1$ *and* $|\overline{g}(s,t,r)| < \infty$.

Proof. Denoting $E = (C(A,\mathbb{R}), \|.\|)$ the Banach space of all continuous functions on A. Define the sequence of partial sums $\overline{s}_n(s,t,r) = \sum\limits_{i=0}^{n} \overline{u}_i(s,t,r)$ for all $(s,t) \in A$, $0 \leq r \leq 1$. Let $\overline{s}_n$ and $\overline{s}_m$ be arbitrary partial sums with $n \geq m$. We are going to prove that $\{\overline{s}_n\}$ is a Cauchy sequence in E.

$$\|\overline{s}_n - \overline{s}_m\| = \max_{(s,t)\in A} |\overline{s}_n(s,t,r) - \overline{s}_m(s,t,r)|$$

$$= \max_{(s,t)\in A} |\sum_{i=0}^{n} \overline{u}_i(s,t,r) - \sum_{i=0}^{m} \overline{u}_i(s,t,r)| = \max_{(s,t)\in A} |\sum_{i=m+1}^{n} \overline{u}_i(s,t,r)|$$

$$= \max_{(s,t)\in A} |\sum_{i=m+1}^{n} \int_c^t k_1(t,\tau)\overline{A}_{i-1}d\tau + \int_a^b k_2(s,\xi)\overline{B}_{i-1}d\xi|$$

$$= \max_{(s,t)\in A} |\int_c^t k_1(t,\tau) \sum_{i=m}^{n-1} \overline{A}_i d\tau + \int_a^b k_2(s,\xi) \sum_{i=m}^{n-1} \overline{B}_i d\xi|.$$

From [11], we have

$$\sum_{i=m}^{n-1} \overline{A}_i = G_1(\overline{s}_{n-1}) - G_1(\overline{s}_{m-1}), \quad \sum_{i=m}^{n-1} \overline{B}_i = G_2(\overline{s}_{n-1}) - G_2(\overline{s}_{m-1}).$$

Consequently, from conditions (ii) and (iii) we obtain

$$\|\overline{s}_n - \overline{s}_m\| = \max_{(s,t)\in A} |\int_c^t k_1(t,\tau)[G_1(\overline{s}_{n-1}) - G_1(\overline{s}_{m-1})]d\tau$$

$$+ \int_a^b k_2(s,\xi)[G_2(\overline{s}_{n-1}) - G_2(\overline{s}_{m-1})]d\xi|$$

$$\leq (M_1 L_1(d-c) + M_2 L_2(b-a))\|\overline{s}_{n-1} - \overline{s}_{m-1}\| = \alpha\|\overline{s}_{n-1} - \overline{s}_{m-1}\|.$$

Let, $n = m + 1$ then

$$\|\overline{s}_{m+1} - \overline{s}_m\| \leq \alpha \|\overline{s}_m - \overline{s}_{m-1}\| \leq \alpha^2 \|\overline{s}_{m-1} - \overline{s}_{m-2}\| \leq ... \leq \alpha^m \|\overline{s}_1 - \overline{s}_0\|.$$

Using the triangle inequality we have

$$\begin{aligned}\|\overline{s}_n - \overline{s}_m\| &\leq \|\overline{s}_{m+1} - \overline{s}_m\| + \|\overline{s}_{m+2} - \overline{s}_{m+1}\| + ... + \|\overline{s}_n - \overline{s}_{n-1}\| \\ &\leq \left[\alpha^m + \alpha^{m+1} + ... + \alpha^{n-1}\right]\|\overline{s}_1 - \overline{s}_0\| \\ &\leq \alpha^m \left[1 + \alpha + \alpha^2 + ... + \alpha^{n-m-1}\right]\|\overline{s}_1 - \overline{s}_0\| \leq \alpha^m \left(\frac{1 - \alpha^{n-m}}{1 - \alpha}\right)\|\overline{u}_1\|.\end{aligned}$$

Since $0 < \alpha < 1$ so, $1 - \alpha^{n-m} \leq 1$ then

$$\|\overline{s}_n - \overline{s}_m\| \leq \frac{\alpha^m}{1 - \alpha} \max_{(s,t) \in A} |\overline{u}_1(s, t, r)| \tag{4.1}$$

But $|\overline{u}_1(s, t, r)| < \infty$ (since $\overline{g}(s, t, r)$ is bounded) then $\|\overline{s}_n - \overline{s}_m\| \to 0$ as $m \to \infty$, from which we conclude that $\{\overline{s}_n\}$ is a Cauchy sequence in E, therefore the series $\sum_{i=1}^{\infty} \overline{u}_i(s, t, r)$ converges. Similarly, we have $\{\underline{s}_n\}$ is a Cauchy sequence. □

Theorem 4.3. *The maximum absolute error of the series solution (3.2) to problem (1.1) is estimated to be*

$$\max_{(s,t) \in A} \left|\overline{u}(s, t, r) - \sum_{i=0}^{m} \overline{u}_i(s, t, r)\right| \leq \frac{\alpha^m}{1 - \alpha}(M_1(d - c)\overline{\phi_1} + M_2(b - a)\overline{\phi_2}), \tag{4.2}$$

$$\max_{(s,t) \in A} \left|\underline{u}(s, t, r) - \sum_{i=0}^{m} \underline{u}_i(s, t, r)\right| \leq \frac{\alpha^m}{1 - \alpha}(M_1(d - c)\underline{\phi_1} + M_2(b - a)\underline{\phi_2}), \tag{4.3}$$

where

$$\overline{\phi_1} = \max_{(s,t) \in A} |G_1(\overline{g}(s, t, r))|, \quad \overline{\phi_2} = \max_{(s,t) \in A} |G_2(\overline{g}(s, t, r))|,$$

$$\underline{\phi_1} = \max_{(s,t) \in A} |G_1(\underline{g}(s, t, r))|, \quad \underline{\phi_2} = \max_{(s,t) \in A} |G_2(\underline{g}(s, t, r))|.$$

Proof. From inequality (4.1) we have

$$\|\overline{s}_n - \overline{s}_m\| \leq \frac{\alpha^m}{1 - \alpha} \max_{(s,t) \in A} |\overline{u}_1(s, t, r)|.$$

As $n \to \infty$ then $\overline{s}_n(s, t, r) \to \overline{u}(s, t, r)$. Then

$$\max_{(s,t) \in A} |\overline{u}(s, t, r) - \overline{s}_m(s, t, r)| \leq \frac{\alpha^m}{1 - \alpha} \max_{(s,t) \in A} |\overline{u}_1(s, t, r)|.$$

From (3.4) and (3.6) for $\overline{u}_1(s, t, r)$ we obtain

$$\max_{(s,t) \in A} |\overline{u}_1(s, t, r)| \leq M_1(d - c)\overline{\phi_1} + M_2(b - a)\overline{\phi_2}.$$

Finally the maximum absolute error is

$$\max_{(s,t)\in A} |\overline{u}(s,t,r) - \sum_{i=0}^{m} \overline{u}_i(s,t,r)| \leq \frac{\alpha^m}{1-\alpha}(M_1(d-c)\overline{\phi}_1 + M_2(b-a)\overline{\phi}_2).$$

Analogously, we obtain the inequality (4.3).   □

5. NUMERICAL EXAMPLE

The Adomian's polynomials can be generated using the traditional formula (3.4). For example, if $G(u) = u^2$ the polynomials using the formula (3.4) are

$$\begin{aligned}
A_0 &= u_0^2, \\
A_1 &= 2u_0 u_1, \\
A_2 &= u_1^2 + 2u_0 u_2, \\
A_3 &= 2u_1 u_2 + 2u_0 u_3, \\
A_4 &= u_2^2 + 2u_1 u_3 + 2u_0 u_4, \\
A_5 &= 2u_2 u_3 + 2u_1 u_4 + 2u_0 u_5.
\end{aligned}$$

By induction we find for $m = 0$ $A_0 = u_0^2$ for $m = 1, 2, 3, ..., 2\nu+1$, where $\nu \geq 0$ $A_{2\nu+1} = 2\sum_{n=0}^{\nu} u_n u_{2\nu+1-n}$, and for $m = 2, 4, 6, ..., 2\nu$, where $\nu \geq 1$ $A_{2\nu} = u_\nu^2 + 2\sum_{n=0}^{\nu-1} u_n u_{2\nu-n}$.

Example 5.1. Let $A = [0,1] \times [0,1]$. Consider the following 2D-NVFFIE

$$u(s,t) = g(s,t) \oplus (FR) \int_0^t \frac{t\tau}{3} \odot u(s,\tau)d\tau \oplus (FR) \int_0^1 \frac{s\xi}{2} \odot u^2(\xi,t)d\xi,$$

where

$$g(s,t,r) = (st(1-\frac{t^3}{9})(1+r) - \frac{st^2}{8}(1+r)^2, \quad st(1-\frac{t^3}{9})(3-r) - \frac{st^2}{8}(3-r)^2).$$

The exact solution is $u_{exact}(s,t,r) = (st(1+r), st(3-r))$.

The general form of the equation is

$$\underline{u}(s,t,r) = st(1-\frac{t^3}{9})(1+r) - \frac{st^2}{8}(1+r)^2 + \int_0^t \frac{t\tau}{3}\underline{u}(s,\tau,r)d\tau + \int_0^1 \frac{s\xi}{2}\underline{u}^2(\xi,t,r)d\xi,$$

$$\overline{u}(s,t,r) = st(1-\frac{t^3}{9})(3-r) - \frac{st^2}{8}(3-r)^2 + \int_0^t \frac{t\tau}{3}\overline{u}(s,\tau,r)d\tau + \int_0^1 \frac{s\xi}{2}\overline{u}^2(\xi,t,r)d\xi.$$

Using ADM, we have

$$\underline{u}_0(s,t,r) = st(1 - \frac{t^3}{9})(1+r) - \frac{st^2}{8}(1+r)^2,$$

$$\underline{u}_1(s,t,r) = \int_0^t \frac{t\tau}{3}\underline{u}_0(s,\tau,r)d\tau + \int_0^1 \frac{s\xi}{2}\underline{u}_0^2(\xi,t,r)d\xi$$

$$= \frac{st^4}{9}(1 - \frac{t^3}{18})(1+r) + \frac{st^2}{8}(1 - \frac{11t^3}{36} + \frac{t^6}{81})(1+r)^2$$

$$- \frac{st^3}{32}(1 - \frac{t^3}{9})(1+r)^3 + \frac{st^4}{512}(1+r)^4,$$

$$\underline{u}_2(s,t,r) = \int_0^t \frac{t\tau}{3}\underline{u}_1(s,\tau,r)d\tau + \int_0^1 \frac{s\xi}{2}2\underline{u}_0(\xi,t,r)\underline{u}_1(\xi,t,r)d\xi$$

$$= \frac{st^7}{162}(1 - \frac{t^3}{27})(1+r) + \frac{st^5}{72}(\frac{11}{4} - \frac{95t^3}{84} + \frac{16t^6}{405})(1+r)^2$$

$$+ \frac{st^3}{32}(1 - \frac{87t^3}{180} + \frac{11t^6}{216} - \frac{t^9}{729})(1+r)^3 + \frac{st^4}{256}(-\frac{109}{36} + \frac{3t^3}{4} - \frac{t^6}{27})(1+r)^4$$

$$+ \frac{3st^5}{2048}(1 - \frac{t^3}{9})(1+r)^5 - \frac{st^6}{16384}(1+r)^6.$$

Then we have

$$\underline{u}(s,t,r) = (st(1 - \frac{t^9}{4374}) + \frac{st^8}{72}(-\frac{257}{252} + \frac{16t^3}{405})(1+r)$$

$$+ \frac{st^6}{96}(-\frac{77}{60} + \frac{11t^3}{72} - \frac{t^6}{243})(1+r)^2 + \frac{st^4}{256}(-\frac{91}{36} + \frac{3t^3}{4} - \frac{t^6}{27})(1+r)^3$$

$$+ \frac{3st^5}{2048}(1 - \frac{t^3}{9})(1+r)^4 - \frac{st^6}{16384}(1+r)^5 + ...)(1+r).$$

With the same procedure we can obtain

$$\overline{u}(s,t,r) = (st(1 - \frac{t^9}{4374}) + \frac{st^8}{72}(-\frac{257}{252} + \frac{16t^3}{405})(3-r)$$

$$+ \frac{st^6}{96}(-\frac{77}{60} + \frac{11t^3}{72} - \frac{t^6}{243})(3-r)^2 + \frac{st^4}{256}(-\frac{91}{36} + \frac{3t^3}{4} - \frac{t^6}{27})(3-r)^3$$

$$+ \frac{3st^5}{2048}(1 - \frac{t^3}{9})(3-r)^4 - \frac{st^6}{16384}(3-r)^5 + ...)(3-r).$$

So the approximate solution $u(s,t,r) = (\underline{u}(s,t,r), \overline{u}(s,t,r))$. The numerical results obtained with the ADM with three iterations are presented in Table 1.

ACKNOWLEDGEMENTS

m	$\underline{u}_m$	$\overline{u}_m$	$\underline{u}$	$\overline{u}$	$\underline{u}_{exact}$	$\overline{u}_{exact}$
0	0,110147	0,173578	0,110147	0,173578	0,12	0,2
1	0,008385	0,020115	0,118532	0,193692	0,12	0,2
2	0,001203	0,004489	0,119735	0,198182	0,12	0,2

Table 1: Comparison of approximation solutions with exact solution for $(s_0, t_0) = (0.2, 0.4)$ and $r = 0.5$

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