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ULAM TYPE STABILITY RESULTS FOR NON-INSTANTANEOUS IMPULSIVE DIFFERENTIAL EQUATIONS WITH FINITE STATE DEPENDENT DELAY

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ABSTRACT: In this paper a system with state dependent bounded delay and non-instantaneous impulses is considered. An existence result based on the Banach contraction principle is given. Several sufficient conditions for Ulam-type stability are obtained. An example is given to illustrate our results.

Key Words: non-instantaneous impulses, differential equations, state dependent delay, existence, Ulam type stability

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1. INTRODUCTION

Impulsive differential equations are used to model physical phenomenon which experience instantaneous changes of the state at certain moments, such as in population dynamics, radio physics, pharmacokinetics, mathematical economy, ecology, industrial robotics, control theory and medicine. In [7] the authors studied Ulam-type stability of nonlinear impulsive ordinary differential equations and Hyers-Ulam stability for other classes of impulsive delay differential equations was studied in [8], [11]. Sometimes abrupt changes may stay for certain time intervals and such impulses are called non-instantaneous impulses. A well known application of non-instantaneous impulses is when one introduces insulin in the bloodstream which is an abrupt change and the consequent absorption is a gradual process as it remains active for a finite interval of time ([4]). Ulam type stability was studied for second order differential equations with non-instantaneous impulses in [5], and for first order differential equations with non-instantaneous impulses see [3]. For stability results for non-instantaneous fractional differential equation we refer the reader to [1], [2], [10] and the monograph [3].

In real world problems delay depends not only on the time but also on the unknown quantity (see, for example [1]). In this paper we study an initial value problem (IVP) for a nonlinear system of non-instantaneous impulsive differential equations with state dependent delay (NIDDE). We establish an existence result based on the Banach contraction principle. Also we obtain some sufficient conditions for Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. An example is given to illustrate our results.

2. STATEMENT OF THE PROBLEM AND DEFINITION OF THE SOLUTION

Let the positive constants r, T be given and the points $t_i, s_i \in [0, T]$ be such that $s_0 = 0$, $t_{k+1} = T$, $0 < t_i < s_i < t_{i+1}$, $i = 1, 2, \dots, k$.

Consider the space $PC_0 = C([-r, 0], E)$ endowed with the norm

$$||y||_{PC_0} = \sup_{t\in[-r,0]} \{||y(t)||_E : \ y \in PC_0\};$$

here E is a Banach space.

The intervals $(s_i, t_{i+1}), i = 0, 1, 2, \dots, k$ are the intervals on which the differential equation is given and the intervals $(t_i, s_i), i = 1, 2, \dots, k$ are called impulsive intervals and on these intervals the impulsive conditions are given.

Consider the initial value problem (IVP) for a nonlinear system of non-instantaneous impulsive differential equations with state dependent delay (NIDDE)

$$\begin{aligned} x'(t) =& f(t, x_{\rho(t,x_t)}), && \text{for } t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \dots, k, \\ x(t) =& g_i(t, x(t_i)), && \text{fot } t \in (t_i, s_i],\ i = 1, 2, \dots, k, \\ x(t) =& \phi(t), && \text{for } t \in [-r, 0], \end{aligned} \tag{1}$$

where the functions $f : \bigcup_{i=0}^{k}[s_i, t_{i+1}] \times PC_0 \to E$; $\rho : \bigcup_{i=0}^{k}[s_i, t_{i+1}] \times PC_0 \to [0, T]$, $\phi : [-r, 0] \to E$; $g_i : [t_i, s_i] \times E \to E$, $i = 1, 2, \dots, k$. Here for any $t \in [0, T]$ the notation $x_t(s) = x(t+s), s \in [-r, 0]$ is used, i.e. $x_t \in PC_0$ represents the history of the state $x(t)$ from time $t-r$ up to the present time t. Note that for any $t \in \bigcup_{i=0}^{k}(s_i, t_{i+1}]$ we let $x_{\rho(t,x_t)}(s) = x(\rho(t, x(t+s)) + s), s \in [-r, 0]$, i.e. the function ρ determines the state-dependent delay.

Remark 1. Note in the special case $\rho(t, x) \equiv t$ problem (1) reduces to an IVP for a delay non-instantaneous impulsive differential equation.

Let $\mathcal{PC}$ be the Banach space of all functions $y : [-r, T] \to E$ which are continuous on $[0, T]$ except for the points $t_i \in (0, T)$ at which $y(t_i+) = \lim_{t\downarrow t_i} y(t)$ and $y(t_i-) = y(t_i) = \lim_{t\uparrow t_i} y(t)$ exist and it is endowed with the norm $||y||_{\mathcal{PC}} = \sup_{t\in[-r,T]}\{||y(t)||_E : y \in \mathcal{PC}\}$.

We consider the assumptions:

A1. The function $f \in C(\bigcup\limits_{i=0}^{k}[s_i, t_{i+1}] \times PC_0, E)$.

A2. The function $\phi \in PC_0$.

A3. The function $\rho \in C(\bigcup\limits_{i=0}^{k}[s_i, t_{i+1}] \times PC_0, [0, T])$ is such that for any $t \in \bigcup_{i=0}^{k}[s_i, t_{i+1}]$ and any function $u \in PC_0$ the inequality $\rho(t, u) \le t$ holds.

A4. The functions $g_i \in C([t_i, s_i] \times E, E)$, $i = 1, 2, \dots, k$.

Remark 2. Assumption A3 guarantees the delay of the argument in (1).

Definition 1. The function $x \in \mathcal{PC}$ is a solution of the IVP (1) iff it satisfies the following integral-algebraic equation

$$x(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0) + \displaystyle\int_0^t f(s, x_{\rho(s,x_s)})ds, & t \in (0, t_1], \\ g_i(t, x(t_i)), & t \in (t_i, s_i],\ i = 1, 2, \dots, k, \\ g_i(s_i, x(t_i)) + \displaystyle\int_{s_i}^t f(s, x_{\rho(s,x_s)})ds, & t \in (s_i, t_{i+1}],\ i = 1, 2, \dots, k. \end{cases} \tag{2}$$

Note in assumption (A1) the function f is defined for the time variable t only on the intervals without impulses $\bigcup_{i=0}^{k}[s_i, t_{i+1}]$. Without loss of generality let

$$f(t, x) = \begin{cases} f(t, x), & t \in \bigcup\limits_{i=0}^{k}[s_i, t_{i+1}],\ x \in E, \\ 0, & t \in \bigcup\limits_{i=1}^{k}(t_i, s_i),\ x \in E. \end{cases} \tag{3}$$

Define functions $h_i(t,x) = g_i(t,x) - x$ for $t \in [s_i, t_i]$ and $x \in E$. Then we obtain the following definition for the solution of (1):

Definition 2. Suppose Eq. (3) holds. The function $x \in \mathcal{PC}$ is a solution of the IVP (1) iff it satisfies the following integral-algebraic equation

$$x(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0) + \sum\limits_{j=1}^{i} h_j(s_j, x(t_j)) & \\ \qquad + \int_0^t f(s, x_{\rho(s,x_s)})ds, & t \in (s_i, t_{i+1}],\ i = 1, 2, \dots, k, \\ g_i(t, x(t_i)), & t \in (t_i, s_i],\ i = 1, 2, \dots, k. \end{cases} \tag{4}$$

Comments on Definition 1 and Definition 2. In the special case of instantaneous impulses, i.e. $s_i = t_i$, $i = 1, 2, \dots, k$, the function $h_i(t,x)$ gives the amount of the impulsive perturbation $\Delta x(t_i) = x(t_i + 0) - x(t_i - 0)$ at time t_i and the application of these functions in the definition of the solution of impulsive differential equations has meaning. In the case of non-instantaneous impulses the introduction of the function $h_i(t,x)$ is artificial and the application of these functions in Definition 2 is meaningless. In connection with this we will use Definition 1 in the study of properties of (1).

3. DEFINITIONS OF ULAM TYPES STABILITY

Let $\varepsilon > 0, \Psi \geq 0$ and $\Phi \in C(\bigcup_{i=1}^{k}[s_i, t_{i+1}], [0, \infty)$ be nondecreasing. We consider the following inequalities:

$$\begin{aligned} &||y'(t) - f(t, y_{\rho(t,y_t)})||_E \leq \varepsilon \quad \text{for } t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, k \\ &||y(t) - g_i(t, y(t_i))||_E \leq \varepsilon, \quad t \in (t_i, s_i],\ i = 1, 2, \dots, k, \end{aligned} \tag{5}$$

and

$$\begin{aligned} &||y'(t) - f(t, y_{\rho(t,y_t)})||_E \leq \Phi(t) \quad \text{for } t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, k \\ &||y(t) - g_i(t, y(t_i))||_E \leq \Psi, \quad t \in (t_i, s_i],\ i = 1, 2, \dots, k, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &||y'(t) - f(t, y_{\rho(t,y_t)})||_E \leq \varepsilon\Phi(t) \quad \text{for } t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, k \\ &||y(t) - g_i(t, y(t_i))||_E \leq \varepsilon\Psi, \quad t \in (t_i, s_i],\ i = 1, 2, \dots, k, \end{aligned} \tag{7}$$

Definition 3. ([6], [9]) The problem (1) is Ulam-Hyers stable if there exists a real number $c_{f,g_i} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in \mathcal{PC}$ of the inequality (5) there exists a solution $x \in \mathcal{PC}$ of the problem (1) with

$$||y(t) - x(t)||_E \leq \varepsilon\ c_{f,g_i}, \quad t \in [0, T]. \tag{8}$$

Definition 4. ([6], [9]) The problem (1) is generalized Ulam-Hyers stable if there exists function $K_{f,g_i} \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $K_{f,g_i}(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y \in \mathcal{PC}$ of the inequality (5) there exists a solution $x \in \mathcal{PC}$ of the problem (1) with

$$||y(t) - x(t)||_E \leq \; K_{f,g_i}(\varepsilon), \quad t \in [0, T]. \tag{9}$$

Definition 5. ([6], [9]) The problem (1) is Ulam-Hyers-Rassias stable with respect to Φ, Ψ if there exists a real number $c_{f,g_i} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in \mathcal{PC}$ of the inequality (7) there exists a solution $x \in \mathcal{PC}$ of the problem (1) with

$$||y(t) - x(t)||_E \leq \varepsilon \; c_{f,g_i}(\Psi + \Phi(t)), \quad t \in [0, T]. \tag{10}$$

Definition 6. ([6], [9]) The problem (1) is generalized Ulam-Hyers-Rassias stable with respect to Φ, Ψ if there exists a real number $c_{f,g_i} > 0$ such that for each solution $y \in \mathcal{PC}$ of the inequality (6) there exists a solution $x \in \mathcal{PC}$ of the problem (1) with

$$||y(t) - x(t)||_E \leq \; c_{f,g_i}(\Psi + \Phi(t)), \quad t \in [0, T]. \tag{11}$$

Remark 3. If assumptions A1, A3, A4 are satisfied then the function $y \in \mathcal{PC}$ is a solution of the inequality (5) if and only if there exist a function $G \in C(\bigcup_{i=0}^{k}[s_i, t_{i+1}], E)$ and a sequence $G_i \in E, \; i = 1, 2, \dots, k$ which depend on y such that

(i) $||G(t)||_E \leq \varepsilon, \;$ for $t \in \bigcup_{i=0}^{k}[s_i, t_{i+1}]$, and $||G_i||_E \leq \varepsilon, \; i = 1, 2, \dots, k$;

(ii) $y'(t) = f(t, y_{\rho(t,y_t)}) + G(t), \;$ for $t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, k$;

(iii) $y(t) = g_i(t, y(t_i)) + G_i, \; t \in (t_i, s_i], \; i = 1, 2, \dots, k$.

Remark 4. If assumptions A1, A3, A4 are satisfied then the function $y \in \mathcal{PC}$ is a solution of the inequality (6) if and only if there exist a function $G \in C(\bigcup_{i=0}^{k}[s_i, t_{i+1}], E)$ and a sequence $G_i \in E, \; i = 1, 2, \dots, k$ which depend on y such that

(i) $||G(t)||_E \leq \Phi(t)$ for $t \in \bigcup_{i=0}^{k}[s_i, t_{i+1}]$, and $||G_i||_E \leq \Psi, \; i = 1, 2, \dots, k$;

(ii) $y'(t) = f(t, y_{\rho(t,y_t)}) + G(t) \;$ for $t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, k$;

(iii) $y(t) = g_i(t, y(t_i)) + G_i, \; t \in (t_i, s_i], \; i = 1, 2, \dots, k$.

Remark 5. If assumptions A1, A3, A4 are satisfied then the function $y \in \mathcal{PC}$ is a solution of the inequality (7) if and only if there exist a function $G \in C(\bigcup_{i=0}^{k}[s_i, t_{i+1}], E)$ and a sequence $G_i \in E, \; i = 1, 2, \dots, k$ which depend on y such that

(i) $||G(t)||_E \leq \varepsilon\Phi(t)$ for $t \in \bigcup_{i=0}^{k}[s_i, t_{i+1}]$, and $||G_i||_E \leq \varepsilon\Psi, \; i = 1, 2, \dots, k$;

(ii) $y'(t) = f(t, y_{\rho(t,y_t)}) + G(t) \;$ for $t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, k$;

(iii) $y(t) = g_i(t, y(t_i)) + G_i, \quad t \in (t_i, s_i],\ i = 1, 2, \dots, k.$

Note a similar remark for the inequality (7) applies.

4. ULAM TYPES STABILITY

Based on Remark 3 and Definition 1 we get the following result:

Lemma 1. *Suppose assumptions A1, A3, A4 are satisfied. If $y \in \mathcal{PC}$ is a solution of inequalities (5) then it satisfies the following integral-algebraic inequalities*

$$\begin{cases} ||y(t) - \phi(0) - \int_0^t f(s, y_{\rho(s,y_s)})ds||_E \le \varepsilon t, & t \in (0, t_1], \\ ||y(t) - g_i(t, y(t_i))||_E \le \varepsilon, & t \in (t_i, s_i],\ i = 1, 2, \dots, k, \\ ||y(t) - g_i(s_i, y(t_i)) - \int_{s_i}^t f(s, y_{\rho(s,y_s)})ds||_E & \\ \qquad \le \varepsilon + \varepsilon(t - s_i), & t \in (s_i, t_{i+1}],\ i = 1, 2, \dots, k. \end{cases} \tag{12}$$

Proof. From Remark 3 it follows that for any $i = 1, 2, \dots, k$ and $t \in (s_i, t_{i+1}]$ we have

$$y(t) - g_i(s_i, y(t_i)) - \int_{s_i}^t f(s, y_{\rho(s,y_s)})ds = G_i + \int_{s_i}^t G(s)ds. \tag{13}$$

This proves the claim of Lemma 1. □

Lemma 2. *Suppose assumptions A1, A3, A4 are satisfied. If $y \in \mathcal{PC}$ is a solution of inequalities (6) then it satisfies the following integral-algebraic inequalities*

$$\begin{cases} ||y(t) - \phi(0) - \int_0^t f(s, y_{\rho(s,y_s)})ds||_E & \\ \qquad \le \int_0^t \Phi(s)ds, & t \in (0, t_1], \\ ||y(t) - g_i(t, y(t_i))||_E \le \Psi, & t \in (t_i, s_i],\ i = 1, 2, \dots, k, \\ ||y(t) - g_i(s_i, y(t_i)) - \int_{s_i}^t |f(s, y_{\rho(s,y_s)})ds||_E & \\ \qquad \le \Psi + \int_{s_i}^t \Phi(s)ds, & t \in (s_i, t_{i+1}],\ i = 1, 2, \dots, k. \end{cases} \tag{14}$$

The proof of Lemma 2 is based on Remark 4.

Lemma 3. *Suppose assumptions A1, A3, A4 are satisfied. If $y \in \mathcal{PC}$ is a solution* *of inequalities (7) then it satisfies the following integral-algebraic inequalities*

$$\begin{cases} ||y(t)-\phi(0)-\displaystyle\int_0^t f(s,y_{\rho(s,y_s)})ds||_E & \\ \qquad \le \varepsilon \displaystyle\int_0^t \Phi(s)ds, & t\in(0,t_1], \\ ||y(t)-g_i(t,y(t_i))||_E \le \varepsilon\Psi, & t\in(t_i,s_i],\ i=1,2,\dots,k, \\ ||y(t)-g_i(s_i,y(t_i))-\displaystyle\int_{s_i}^t |f(s,y_{\rho(s,y_s)})ds||_E & \\ \qquad \le \varepsilon\Psi+\varepsilon\displaystyle\int_{s_i}^t \Phi(s)ds, & t\in(s_i,t_{i+1}],\ i=1,2,\dots,k. \end{cases} \tag{15}$$

Now we will study the existence of a solution of (1), given by Definition 1, using the Banach contraction principle.

Theorem 1. *(*Existence result*). Suppose the following conditions are satisfied:*

1. *Assumption A1 is satisfied and there exists a constant* $L_f > 0$, *such that for any* $t \in \bigcup_{i=1}^k [s_i, t_{i+1}]$ *and any functions* $u, v \in \mathcal{PC}$ *the inequality*

$$||f(t,u_{\rho(t,u_t)})-f(t,v_{\rho(t,v_t)})||_E \le L_f||u_{\rho(t,u_t)}-v_{\rho(t,v_t)}||_{PC_0}$$

   holds.

2. *Assumption A4 is satisfied and there exist constants* $L_{g_i} > 0, i = 1, 2, \dots, k$, *such that*

$$||g_i(t,x)-g_i(t,y)||_E \le L_{g_i}||x-y||_E,\quad t\in[t_i,s_i],\ x,y\in E,\ i=1,2,\dots,k.$$

3. *Assumptions A2, A3 are satisfied.*

4. *The inequality*

$$\gamma = \max_{i=1,2,\dots,k} L_{g_i} + \eta L_f < 1 \tag{16}$$

   holds where $\eta = \max\{t_{i+1}-s_i,\ i=0,1,\dots,k\}$ *represents the maximal length of the intervals without impulses.*

Then the initial value problem (1) has a unique solution $x \in \mathcal{PC}$ *(as defined in Definition 1).*

Proof. Define the operator $\Omega : \mathcal{PC} \to \mathcal{PC}$ by

$$\Omega(y(t)) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0) + \int_0^t f(s, y_{\rho(s,y_s)})ds, & t \in (0, t_1], \\ g_i(s, y(t_i - 0)), & t \in (t_i, s_i],\ i = 1, 2, \dots, k, \\ g_i(s_i, y(t_i - 0)) & \\ \qquad + \int_{s_i}^t f(s, y_{\rho(s,y_s)})ds, & t \in (s_i, t_{i+1}],\ i = 1, 2, \dots, k. \end{cases} \tag{17}$$

We use induction w.r.t. the intervals to prove the claim.

Let $y, y^* \in \mathcal{PC}$ and $t \in [0, t_1]$. Then for any $t \in [0, t_1]$ it follows that $y_t, y_t^* \in PC_0$, $\rho(t, y_t), \rho(t, y_t^*) \in [0, T]$ and $y_{\rho(t,y_t)}, y^*_{\rho(t,y_t^*)} \in PC_0$. Then we obtain

$$||\Omega(y(t)) - \Omega(y^*(t))||_E \le \int_0^t L_f ||y_{\rho(s,y_s)} - y^*_{\rho(s,y_s^*)}||_{PC_0} ds \le L_f t_1 ||y - y^*||_{\mathcal{PC}}. \tag{18}$$

For any $y, y^* \in \mathcal{PC}$ and $t \in (t_i, s_i]$ we obtain

$$||\Omega(y(t)) - \Omega(y^*(t))||_E \le L_{g_i} ||y - y^*||_{\mathcal{PC}}. \tag{19}$$

For any $y, y^* \in \mathcal{PC}$ and $t \in (s_i, t_{i+1}]$ we obtain

$$\begin{aligned} ||\Omega(y(t)) - \Omega(y^*(t))||_E \le & \Big(L_{g_i} + (t_{i+1} - s_i)L_f\Big)||y - y^*||_{\mathcal{PC}} \\ \le & \Big(L_{g_i} + \eta L_f\Big)||y - y^*||_{\mathcal{PC}}. \end{aligned} \tag{20}$$

From inequalities (18), (19) and (20) it follows that $||\Omega(y) - \Omega(y^*)||_{\mathcal{PC}} \le \gamma ||y - y^*||_{\mathcal{PC}}$, i.e. the operator Ω is a contraction. $\square$

We will use Definition 1 to study the Ulam types stability of problem (1).

Theorem 2. *(*Stability results*). Suppose the conditions of Theorem 1 are satisfied.*

(i) Assume for any $\varepsilon > 0$ inequality (5) has at least one solution. Then problem (1) is Ulam-Hyers stable.

(ii) Assume for any $\varepsilon > 0$ inequality (5) has at least one solution. Then problem (1) is generalized Ulam-Hyers stable.

(iii) Assume there exist constants $\Psi \ge 0$, $\Lambda_\Phi > 0$ and a function

$$\Phi \in C(\bigcup_{i=1}^k [s_i, t_{i+1}], [0, \infty))$$

such that for any $t \in [s_i, t_{i+1}],\ i = 0, 1, 2, \dots, k,$ inequality

$$\int_{s_i}^t \Phi(s)ds \le \Lambda_\Phi \Phi(t)$$

holds and for any $\varepsilon > 0$ inequality (7) has at least one solution. Then problem (1) is Ulam-Hyers-Rassias stable with respect to Φ, Ψ.

(iv) Assume there exist constants $\Psi \geq 0$, $\Lambda_\Phi > 0$ and a function

$$\Phi \in C(\bigcup_{i=1}^{k}[s_i, t_{i+1}], [0, \infty))$$

such that for any $t \in [s_i, t_{i+1}],\ i = 0, 1, 2, \ldots, k,$ inequality

$$\int_{s_i}^{t} \Phi(s)ds \leq \Lambda_\Phi \Phi(t)$$

holds and inequality (6) has at least one solution. Then problem (1) is generalized Ulam-Hyers-Rassias stable with respect to Φ, Ψ.

Proof. (i). Let $\varepsilon > 0$ be an arbitrary number and $y \in \mathcal{PC}$ be a solution of inequality (5) satisfying $y(t) = \phi(t),\ t \in [-r, 0]$. Therefore, the integral-algebraic inequalities (12) hold.

For any $t \in [0, T]$ we define the function $\gamma(t) = \sup_{s \in [-r,t]} ||x(s) - y(s)||_E$.

We use induction w.r.t. the intervals to prove that

$$\gamma(t) \leq c_{f,g_i}\varepsilon,\ t \in [0, T], \tag{21}$$

where

$$c_{f,g_i} = \begin{cases} 1 + (1+\eta)\Big(\sum_{j=1}^{p-1} e^{jL_f\eta} \prod_{m=0}^{j-1} L_{g_{p-m}}\Big) & \\ \quad +\eta e^{pL_f\eta} \prod_{j=1}^{p} L_{g_j}, & t \in (t_p, s_p], p = 1, 2, \ldots, k, \\ (1+\eta)\sum_{j=1}^{p}\Big(\prod_{m=0}^{j-2} L_{g_{p-m}}\Big)e^{jL_f\eta} & \\ \quad +\eta e^{(p+1)L_f\eta} \prod_{j=1}^{p} L_{g_j}, & t \in (s_p, t_{p+1}], p = 0, 1, 2, \ldots, k, \end{cases}$$

$\eta = \max\{t_{i+1} - s_i,\ i = 0, 1, \ldots, k\}$.

Let $t \in [0, t_1]$ be an arbitrary fixed point. Denote by $t^* \in [-r, t]$ the point such that $||x(t^*) - y(t^*)||_E \geq ||x(s) - y(s)||_E$ for all $s \in [-r, t]$. If $t^* \in [-r, 0]$ then $||x(t^*) - y(t^*)||_E = 0$ and $\gamma(t) = 0$. If $t^* \in (0, t]$ then according to Definition 1, Lemma 1, condition 1 of Theorem 1 and inequalities (12) we obtain

$$\begin{aligned}||x(t^*)-y(t^*)||_E \leq &|| \int_0^{t^*} \Big(f(s,x_{\rho(s,x_s)})-f(s,y_{\rho(s,y_s)})\Big)ds||_E \\ &+||y(t^*)-\phi(0)-\int_0^{t^*} f(s,y_{\rho(s,y_s)})ds||_E \\ \leq &L_f \int_0^{t^*} ||x_{\rho(s,x_s)})-y_{\rho(s,y_s)})||_{PC_0}ds+\varepsilon t^*.\end{aligned} \tag{22}$$

From the definition of x_t and the function ρ we get

$$\begin{aligned}||x_{\rho(s,x_s)})-y_{\rho(s,y_s)})|| = &\sup_{\xi\in[-r,0]} ||x(\rho(s,x_s)+\xi)-y(\rho(s,y_s)+\xi)||_E \\ &\leq \sup_{\xi\in[-r,s]} ||x(\xi)-y(\xi)||_E=\gamma(s)\end{aligned}$$

and from (22) it follows that for any $t\in[0,t_1]$,

$$\gamma(t)\leq \varepsilon t_1+L_f\int_0^t \gamma(s)ds.$$

Therefore, according to Gronwall's inequality we have

$$\gamma(t)\leq \varepsilon t_1 e^{L_f t_1}, \quad t\in[0,t_1]. \tag{23}$$

Let $t\in(t_1,s_1]$ be an arbitrary fixed point. Denote by $t^*\in[-r,t]$ the point such that $||x(t^*)-y(t^*)||_E\geq ||x(s)-y(s)||_E$ for all $s\in[-r,t]$. If $t^*\in[-r,t_1]$ then from above $||x(t^*)-y(t^*)||_E\leq \varepsilon t_1 e^{L_f t_1}$ and $\gamma(t)\leq \varepsilon t_1 e^{L_f t_1}\leq \varepsilon(1+L_{g_1}t_1e^{L_f t_1})$. If $t^*\in(t_1,s_1]$ from Definition 1, Lemma 1, condition 2 of Theorem 1 and inequalities (12) and (23) for $t=t_1$ we get

$$\begin{aligned}||x(t^*)-y(t^*)||_E &\leq ||y(t^*)-g_1(t^*,y(t_1))||_E+||g_1(t^*,x(t_1))-g_1(t^*,y(t_1))||_E \\ &\leq \varepsilon+L_{g_1}||x(t_1)-y(t_i)||_E\leq(1+L_{g_1}t_1e^{L_f t_1})\varepsilon\end{aligned}$$

or

$$\gamma(t)\leq(1+L_{g_1}t_1e^{L_f t_1})\varepsilon, \qquad t\in(t_1,s_1]. \tag{24}$$

Let $t\in(s_1,t_2]$ be an arbitrary fixed point. Denote by $t^*\in[-r,t]$ the point such that $||x(t^*)-y(t^*)||_E\geq ||x(s)-y(s)||_E$ for all $s\in[-r,t]$. If $t^*\in[-r,s_1]$ then from above $||x(t^*)-y(t^*)||_E\leq \varepsilon(1+L_{g_1}t_1e^{L_f t_1})$ and $\gamma(t)\leq\varepsilon(1+L_{g_1}t_1e^{L_f t_1})<\varepsilon((1+\eta)e^{L_f\eta}+L_{g_1}t_1e^{2L_f\eta})$. If $t^*\in(s_1,t_2]$ then from Definition 1, Lemma 1, conditions

1,2 of Theorem 1 and inequalities (12), and (23) for $t = t_1$ we obtain

$$
\begin{aligned}
||x(t^*) - y(t^*)||_E \leq & ||y(t^*) - g_1(s_1, y(t_1)) - \int_{s_1}^{t^*} f(s, y_{\rho(s,y_s)})ds||_E \\
& + ||g_1(s_1, x(t_1)) - g_1(s_1, y(t_1))||_E \\
& + || \int_{s_1}^{t^*} \Big(f(s, x_{\rho(s,y_s)}) - f(s, y_{\rho(s,y_s)}) \Big) ds||_E \\
\leq & \varepsilon + \varepsilon(t^* - s_1) + L_{g_1} ||x(t_1) - y(t_1)||_E \\
& + L_f \int_{s_1}^{t} ||x_{\rho(s,y_s)}) - y_{\rho(s,y_s)})||_{PC_0} ds \\
\leq & \varepsilon + \varepsilon\eta + L_{g_1} t_1 e^{L_f t_1} \varepsilon + L_f \int_{s_1}^{t^*} ||x_{\rho(s,y_s)}) - y_{\rho(s,y_s)})||_{PC_0} ds.
\end{aligned}
\tag{25}
$$

From (25) it follows that for any $t \in (s_1, t_2]$,

$$
\gamma(t) \leq \varepsilon(1 + \eta + L_{g_1} t_1 e^{L_f t_1}) + L_f \int_{s_1}^{t} \gamma(s) ds.
$$

Therefore, according to Gronwall's inequality we have

$$
\gamma(t) \leq \varepsilon((1 + \eta) e^{L_f \eta} + L_{g_1} t_1 e^{2 L_f \eta}), \quad t \in [s_1, t_2]. \tag{26}
$$

Continue this process to prove inequality (21). Inequality (21) proves (i).

(ii) The proof is similar to the one in (i) where the function $K_{f,g_i}(x) = c_{f,g_i} x$ with the constant c_{f,g_i} defined in the proof of (i).

(iii) The proof is similar to the one in (i) where instead of Lemma 1 we apply Lemma 2 to prove

$$
\gamma(t) \leq \begin{cases}
e^{L_f \eta} \Big(1 + \sum_{i=1}^{p-1} \prod_{m=0}^{i-1} (L_{g_{p-m}} e^{L_f \eta}) \Big) \Psi & \\
+ \Lambda_\Phi \Big(\sum_{i=1}^{p} \prod_{m=0}^{i-1} (L_{g_{p-m}} e^{L_f \eta}) \Big) \Phi(t), & t \in (t_p, s_p],\ p = 1, 2, \dots, k, \\
e^{L_f \eta} \Big(1 + \sum_{i=1}^{p-1} \prod_{m=0}^{i-1} (L_{g_{p-m}} e^{L_f \eta}) \Big) \Psi & \\
+ \Lambda_\Phi e^{L_f \eta} \Big(1 + \sum_{i=1}^{p} \prod_{m=0}^{i-1} (L_{g_{p-m}} e^{L_f \eta}) \Big) \Phi(t), & t \in (s_p, t_{p+1}],\ p = 0, 1, 2, \dots, k.
\end{cases}
\tag{27}
$$

(iv) Let $\varepsilon > 0$ be an arbitrary number and $y \in \mathcal{PC}$ be a solution of inequality (7) satisfying $y(t) = \phi(t),\ t \in [-r, 0]$. Therefore, the integral-algebraic inequalities (15) hold. The rest of the proof is similar to the one in (i)where instead of Lemma 1 we apply Lemma 3 to prove that

$$\gamma(t) \le \begin{cases} \varepsilon e^{L_f \eta}\Big(1+\sum_{i=1}^{p-1}\prod_{m=0}^{i-1}(L_{g_{p-m}}e^{L_f\eta})\Big)\Psi \\ +\varepsilon\Lambda_\Phi\Big(\sum_{i=1}^{p}\prod_{m=0}^{i-1}(L_{g_{p-m}}e^{L_f\eta})\Big)\Phi(t), & t\in(t_p,s_p],\ p=1,2,\dots,k, \\ \varepsilon e^{L_f \eta}\Big(1+\sum_{i=1}^{p-1}\prod_{m=0}^{i-1}(L_{g_{p-m}}e^{L_f\eta})\Big)\Psi \\ +\varepsilon\Lambda_\Phi e^{L_f\eta}\Big(1+\sum_{i=1}^{p}\prod_{m=0}^{i-1}(L_{g_{p-m}}e^{L_f\eta})\Big)\Phi(t), & t\in(s_p,t_{p+1}], p=0,1,2,\dots,k. \end{cases}$$

□

Remark 6. The case of instantaneous impulses is a special case of non-instantaneous impulses with $t_i = s_i \in [0,T]: \ t_0 = 0, t_{k+1} = T, 0 < t_i = s_i < t_{i+1}, i = 1,2,\dots,k$ and from the above results we obtain new results for impulsive differential equation.

Remark 7. Note in the case of a system of impulsive differential equations without delays the generalized Ulam-Hyers-Rassias stability for the impulsive differential equations is studied in [7].

5. APPLICATION

Consider the special case $E=\mathbb{R}$, $r=1.5$, $q=0.5$, $s_0=0, t_1=1, s_1=2, t_2=T=3$, $\phi(t)=1, t\in[-1.5,0]$, $\rho(t,y)=t\sin^2(y)\le t$ for $(t,y)\in: [0,3]\times\mathbb{R}$, $g(t,y)=0.25ty, t\in[1,2]$, $f(t,y)=0.5y$. Let $x_t(s)=x(t-1.5)$ for all $s\in[-1.5,0]$. Then $y_{\rho(t,y_t)} = t\sin^2(y(t-1.5))-1.5<t$.

In this particular case the problem (1) could be written in the form

$$\begin{aligned} &x'(t) = 0.5x(t\sin^2(x(t-1.5))-1.5) \ \text{ for } t\in(0,1]\bigcup(2,3] \\ &x(t) = 0.25x(1), \quad t\in(1,2], \\ &x(t) = 1 \ \text{ for } t\in[-1.5,0], \end{aligned} \tag{28}$$

Then $L_g = 0.25$, $L_f = 0.5$, $\eta = 1$ and the inequality (16) holds. i.e. the IVP (28) has a unique solution. We will give the formula for the exact solution of (28).

Case 1. Let $t\in[0,1]$. Then $t\sin^2(t-1.5)-1.5\in[-1.5,0]$ and $x(t\sin^2(y(t-1.5))-1.5)=1$. Therefore, $x'(t)=0.5$ for $t\in[0,1]$ and we get $x(t)=1+0.5t$.

Case 2. Let $t\in(1,2]$. Then $x(t)=0.25x(1)=0.375$.

Case 3. Let $t\in(2,3]$. Because of the delay we consider two cases.

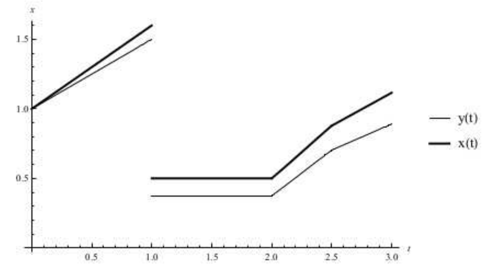

Figure 1: Graph of the solution $x(t)$ of (28) and the solution $y(t)$ of (31) for $\varepsilon = 0.1$

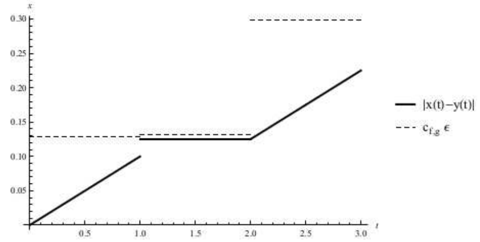

Figure 2: Graph of the difference $|x(t) - y(t)|$ for $\varepsilon = 0.1$ and the upper bound $c_{f,g}\varepsilon$

Case 3.1. If $t \in (2, 2.5]$ then $t - 1.5 \in (0.5, 1]$, $x(t-1.5) = 1 + 0.5(t-1.5) = 0.5t + 0.25$, $t\sin^2(x(t-1.5)) - 1.5 \in (0.5, 1]$, $x(t\sin^2(x(t-1.5)) - 1.5) = 1 + 0.5(t\sin^2(t-0.5) - 1.5) = 0.5t\sin^2(t-0.5) + 0.25$ and $x'(t) = 0.5\Big(0.5t\sin^2(t-0.5) + 0.25\Big) = 0.25t\sin^2(t-0.5) + 0.125$ for $t \in (2, 2.5]$ with $x(2) = 0.375$.

Case 3.2. If $t \in (2.5, 3]$ then $t - 1.5 \in (1, 2]$, $x(t-1.5) = 0.5$, $t\sin^2(x(t-1.5)) - 1.5 \in (1, 2]$ and $x(t\sin^2(x(t-1.5)) - 1.5) = 0.25x(1) = 0.25(1+0.5) = 0.375$. Therefore, $x'(t) = 0.375$ for $t \in (2.5, 3]$. The initial condition is chosen such that the solution is continuous at 2.5.

Then the exact solution of the IVP (28) is (see Figure 1)

$$x(t) = \begin{cases} 1, & t \in [-1.5, 0], \\ 1 + 0.5t, & t \in (0, 1], \\ 0.375, & t \in (1, 2], \\ 0.375 + \displaystyle\int_2^t \Big(0.25s\sin^2(s-0.5) + 0.125\Big)ds & \\ \quad = 0.125 - 0.125t + 0.25\Big(- 1.05319 + 0.25t^2 & \\ \quad +(-0.0675378 + 0.210368t)\cos(2t) & \\ \quad +(-0.105184 - 0.135076t)\sin(2t), & t \in (2, 2.5], \\ 0.375 + \displaystyle\int_2^{2.5} \Big(0.25s\sin^2(s-0.5) + 0.125\Big)ds & \\ \quad +0.375(t-2.5) = -0.233996 + 0.375t, & t \in (2.5, 3]. \end{cases} \tag{29}$$

Also,

$$c_{f,g_i} = \begin{cases} e^{0.25}, & t \in (0, t_1], \\ 1 + e^{0.25}0.25, & t \in (t_1, s_1], \\ (1+1)e^{0.25} + 0.25e^{0.5}, & t \in (s_1, t_2]. \end{cases} \tag{30}$$

According to Theorem 2 (i) problem (28) is Ulam-Hyers stable, i.e. the inequality $||y(t) - x(t)||_E \leq \varepsilon\, c_{f,g_i}, t \in [0, 3]$ holds where $y(t)$ is a solution of the inequalities (5), i.e. it is a solution of

$$\begin{aligned} &y'(t) = 0.5x(t\sin^2(y(t-1.5)) - 1.5) + \varepsilon \quad \text{for } t \in (0,1] \bigcup (2,3], \\ &y(t) = 0.25y(1) + \varepsilon, \quad t \in (1,2], \\ &y(t) = 1 \quad \text{for } t \in [-1.5, 0], \end{aligned} \tag{31}$$

with a solution (see Figure 1 for $\varepsilon = 0.1$)

$$y(t) = \begin{cases} 1, & t \in [-1.5, 0], \\ 1 + (0.5+\varepsilon)t, & t \in (0,1], \\ 0.375 + 1.25\varepsilon, & t \in (1,2], \\ 0.375 + 1.25\varepsilon & \\ \quad + \displaystyle\int_2^t \Big(0.25s\sin^2(s-0.5) + 0.125 + \varepsilon\Big)ds, & t \in (2, 2.5], \\ 0.375 + 1.25\varepsilon & \\ \quad + \displaystyle\int_2^{2.5} \Big(0.25s\sin^2(s-0.5) + 0.125 + \varepsilon\Big)ds & \\ \quad +(0.375+\varepsilon)(t-2.5), & t \in (2.5, 3]. \end{cases} \tag{32}$$

Conclusion: Problem (28) is Ulam-Hyers stable, i.e. the inequality $|x(t) - y(t) \leq c_{f,g_i}\varepsilon$ holds (see Figure 2 for $\varepsilon = 0.1$).

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