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A NOTE ON THE THREE–STAGE GROWTH MODEL

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ABSTRACT: In this paper we study the one–sided Hausdorff approximation of the generalized cut function by sigmoidal modified three–stage growth model. The model has a certain right of existence insofar as the theory of sigmoidal functions is well developed. The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study. We examine the small data for modeling the growth of red abalone *Haliotis Rufescens* in Northern California. Numerical examples are presented using *CAS MATHEMATICA*.

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Key Words: modified three–stage growth model, generalized cut function associated to the model, Hausdorff distance, upper and lower bounds

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1. INTRODUCTION

Let us examine the following three–stage growth model

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

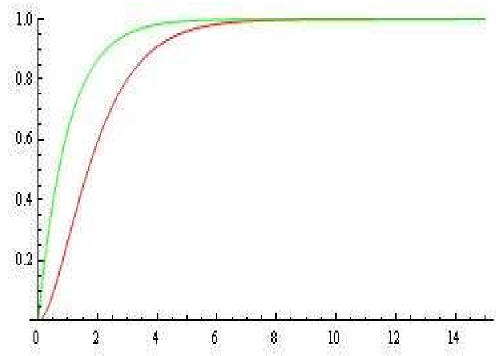

Figure 1: Three–stage growth model $\tilde{C}(t)$ (sigmoidal; red) for $n = k_2 - k_1$, $k_1 = 1$, $k_2 = 1.001$ and three–stage model $C(t)$ (first order; green) for $n = k_1 = k_2$, $k_1 = 1$, $k_2 = 2500$.

with two steps (k_1 and k_2) depending on the ratio of the growth parameters $\frac{k_1}{k_2}$.

For the mechanism the following system of ODEs is known [1]:

$$\begin{aligned} dA(t)/dt &= -k_1 A(t), \\ dB(t)/dt &= k_1 A(t) - k_2 B(t), \\ dC(t)/dt &= k_2 B(t), \\ A(0) &= A_0, \quad B(0) = 0, \quad C(0) = 0. \end{aligned}$$

Noticing that $dA/dt + dB/dt + dC/dt = 0$, hence $A + B + C = A_0$, and at any time, we find

$$C(t) = A_0 - B(t) - A(t)$$

or [4]:

$$C(t) = A_0 \left(1 - \frac{k_1}{k_1 - k_2} \left(e^{-k_1 t} - e^{-k_2 t} \right) - e^{-k_1 t} \right).$$

For some details, see [2], [3]. In [4], the authors debated to the following modified model for the individual growth of marine invertebrates:

$$\tilde{C}(t) = A_0 \left(1 - \frac{k_1}{n} \left(e^{-k_1 t} - e^{-k_2 t} \right) - e^{-k_2 t} \right)$$

where $n = k_2 - k_1$, and $\frac{k_1}{k_2}$ is close to 1.

The model $\tilde{C}$ predicts sigmoidal growth (see, Figure 1), i.e. in a three–stage growth model, the shape is controlled by the ratio $\frac{k_1}{k_2}$.

For $3D$–surface plot for the three–stage mechanism in the range $n = k_2 - k_1$, or $n = k_1 - k_2$, see, Figure 2.

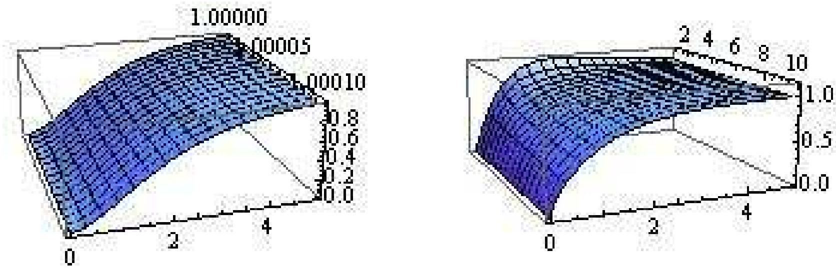

Figure 2: $3D$–surface plot for the three–stage mechanism in the range $n = k_2 - k_1$, or $n = k_1 - k_2$.

2. MAIN RESULTS

Without loosing of generality, for $A_0 = 1$ and $n = k_2 - k_1 > 0$, $\frac{k_1}{k_2} \to 1$ we consider the following family:

$$\tilde{C}(t) = 1 - \frac{k_1}{n}\left(e^{-k_1 t} - e^{-k_2 t}\right) - e^{-k_2 t}. \tag{2.1}$$

We find that the sigmoid (1) has an inflection at point:

$$t^* = \frac{1}{n}\ln\left(\frac{\left(-k_2^2 + \frac{k_1 k_2^2}{n}\right)n}{k_1^3}\right).$$

Definition 1. The associate to the (1) cut function $\tilde{C}^*$ is defined by

$$\tilde{C}^*(t) = \begin{cases} 0, & if \quad t < t_1, \\ \tilde{C}'(t^*)(t - t^*) + \tilde{C}(t^*), & if \quad t_1 \le t < t_2, \\ 1, & if \quad t \ge t_2. \end{cases} \tag{2.2}$$

The straight line $y = \tilde{C}'(t^*)(t - t^*) + \tilde{C}(t^*)$ cross the lines $y = 0$ and $y = 1$ at the points t_1 and t_2.

Definition 2. [6] The one–sided Hausdorff distance $\overrightarrow{\rho}(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the one–sided Hausdorff distance between their completed graphs $\mathcal{F}(f)$ and $\mathcal{F}(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\overrightarrow{\rho}(f, g) = \sup_{B \in \mathcal{F}(g)} \inf_{A \in \mathcal{F}(f)} ||A - B||,$$

where $|| \cdot ||$ is a norm in $\mathbb{R}^2$.

We recall that completed graph of f is the closure of the graph of f as a subset of $\Omega \times \mathbb{R}$. If the graph of an interval function f equals $\mathcal{F}(f)$, then the f is called S-continuous.

The Hausdorff distance $\rho(f,g) = \max\{\overrightarrow{\rho}(f,g), \overrightarrow{\rho}(g,f)\}$ defines a metric in the set of the S-continuous interval functions [7]–[10].

2.1. APPROXIMATION OF THE CUT FUNCTION (2) BY SIGMOID FUNCTION (1)

The one–sided Hausdorff distance d between the functions (1) and (2) satisfies the relation

$$\tilde{C}(t_2 + d) = 1 - d. \tag{2.3}$$

The following theorem gives upper and lower bounds for d

Theorem 1. *Let*

$$p = -e^{-k_2 t_2} - \frac{k_1}{n} e^{-k_1 t_2} + \frac{k_1}{n} e^{-k_2 t_2},$$

$$q = 1 + k_2 e^{-k_2 t_2} + \frac{k_1^2}{n} e^{-k_1 t_2} - \frac{k_1 k_2}{n} e^{-k_2 t_2},$$

$$r = -2\frac{q}{p};\ \ n = k_2 - k_1 > 0;\ \ \frac{k_1}{k_2} \to 1;\ \ \frac{2k_1 - k_2}{k_1} < e^{t_2(k_2 - k_1)}.$$

For the one–sided Hausdorff distance d between $\tilde{C}^*(t)$ and the sigmoidal function (1) the following inequalities hold for: $r > e^2$

$$d_l = \frac{1}{r} < d < \frac{\ln r}{r} = d_r. \tag{2.4}$$

Proof. Let us examine the function:

$$F(d) = \tilde{C}(t_2 + d) - 1 + d. \tag{2.5}$$

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = p + qd. \tag{2.6}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Figure 3).

In addition $G'(d) > 0$.

From the conditions of the theorem, we see that $p < 0$ and $q > 0$ and $G(d_l) = \frac{1}{2}p < 0$.

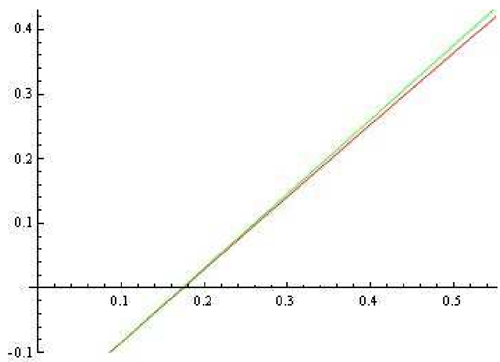

Figure 3: The functions $F(d)$ and $G(d)$ for $k_1 = 1; k_2 = 1.01$.

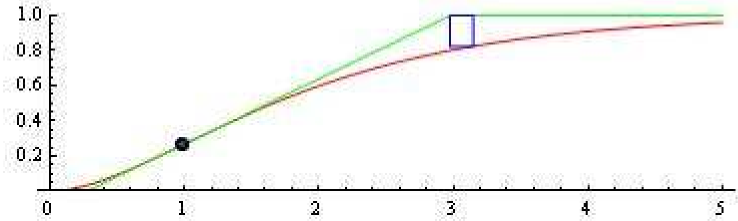

Figure 4: The cut function $\tilde{C}^*(t)$ and the sigmoidal function $\tilde{C}(t)$ with $k_1 = 1$, $k_2 = 1.01$, $t^* = 0.985033$, $t_1 = 0.27045$, $t_2 = 2.97525$; H– distance $d = 0.174444$, $d_l = 0.0865764$, $d_r = 0.211829$.

Further, for $r > e^2$ we have $G(d_r) > 0$.

This completes the proof of the theorem.

The model (1) for $k_1 = 1$, $k_2 = 1.01$, $t^* = 0.985033$, $t_1 = 0.27045$, $t_2 = 2.97525$ is visualized on Figure 4.

From the nonlinear equation (3) and inequalities (4) we have: $d = 0.174444$, $d_l = 0.0865764$, $d_r = 0.211829$.

2.2. NUMERICAL EXAMPLE

We examine the following data. (The small data for modeling the growth of red abalone is shown in Table 1. For more details, see [5]).

Age	*Length(mm)*
1	16.1
2	33.9
3	54.3
4	76.2
5	97.8
6	117.1
7	133.3
8	146.5
9	157.2
10	166
11	173.3
12	179.6

Table 1: The small data for modeling the growth of red abalone *Haliotis Rufescens* in Northern California [5]

The model (2) based on the data of Table 1 for the estimated parameters:

$$A_0 = 179.6;\ k_2 = 0.4384;\ k_1 = 0.434133;\ t^* = 2.26955;\ t_1 = 0.62341;\ t_2 = 6.854$$

is plotted on Figure 5.

Specifically, we will note that the growth model functions is checked by an additional six criteria, the consideration of which go beyond this article.

For example, for the predictive power (PP) criterion

$$PP = \sum_{i=1}^{n} \left(\frac{\tilde{C}(t_i) - y_i}{y_i} \right)^2$$

measures the distance of model actual data from the estimates against the actual data, we find $PP = 0.243413$.

Remarks. The model $\tilde{C}(t)$ has a certain right of existence insofar as the theory of sigmoidal functions is well developed.

Of course, its use in approximating such data is associated with a loss of accuracy when using the operator of the programming environment (for example, *CAS Mathematica*) to find a local extreme.

The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study.

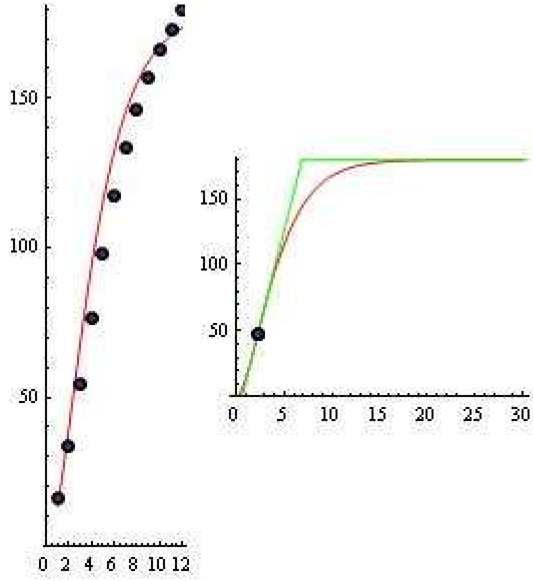

Figure 5: The model $\tilde{C}(t)$

For some approximation, computational and modelling aspects, see [11]–[32].

The results obtained in this paper can be used when controlling growth in Software Reliability Models, see [33], [34].

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