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DIFFERENTIAL EQUATIONS ASSOCIATED WITH DEGENERATE TANGENT POLYNOMIALS AND COMPUTATION OF THEIR ZEROS

C.S. RYOO

Department of Mathematics
Hannam University
Daejeon, 306-791, KOREA

ABSTRACT: In this paper, we study differential equations arising from the generating functions of degenerate tangent polynomials. We give explicit identities for the degenerate tangent polynomials. Finally, we observe an interesting phenomenon of "scattering" of the zeros of degenerate tangent polynomials of higher order.

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1. INTRODUCTION

Recently, many mathematicians have studied in the area of the degenerate Euler numbers and polynomials, degenerate Bernoulli numbers and polynomials, degenerate Genocchi numbers and polynomials, and degenerate tangent numbers and polynomials(see [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13]). In [1], L. Carlitz introduced the degenerate Bernoulli polynomials. Recently, the degenerate tangent numbers and polynomials were introduced by Ryoo(see [6]). The degenerate tangent polynomials $\mathcal{T}_n(x,\lambda)$ are defined by the generating function:

$$\frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}\mathcal{T}_n(x,\lambda)\frac{t^n}{n!}. \tag{1.1}$$

In [9], degenerate tangent numbers of higher order, $\mathcal{T}_n^{(k)}(x,\lambda)$ are defined by means of the following generating function

$$\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^k (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}. \tag{1.2}$$

Let $x = x_1 + x_2 + \cdots + x_N$. Then for $1 \le N \le n$, we have

$$\sum_{\substack{l_1+l_2+\cdots+l_N=n\\ l_1,l_2,\ldots,l_N\ge 0}} \binom{n}{l_1,l_2,\ldots,l_N} \mathcal{T}_{l_1}(x_1,\lambda)\mathcal{T}_{l_2}(x_2,\lambda)\cdots\mathcal{T}_{l_N}(x_N,\lambda) = \mathcal{T}_n^{(N)}(x,\lambda),$$

where $\binom{n}{l_1,l_2,\ldots,l_N}$ are the multinomial coefficients defined by

$$\binom{n}{l_1,l_2,\ldots,l_N} = \frac{n!}{l_1!l_2!\cdots l_N!}.$$

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations(see [13])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. We also have

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.3}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.4}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{1.5}$$

The generalized rising factorial $< x|\lambda >_n^{(N)}$ is defined by

$$< x|\lambda >_n^{(N)} = \prod_{k=0}^{n-1}(x-(N-k)\lambda) \tag{1.6}$$

for positive integer n, with the convention $< x|\lambda >_0^{(N)} = 1$.

Many mathematicians have studied in the area of the linear and nonlinear differential equations arising from the generating functions of special numbers and polynomials in order to give explicit identities for special polynomials(see [3, 7, 10]). In this paper, we study differential equations arising from the generating functions of the degenerate tangent polynomials. We give explicit identities for the degenerate tangent polynomials. Finally, we observe an interesting phenomenon of "scattering" of the zeros of degenerate tangent polynomials of higher order.

2. DIFFERENTIAL EQUATIONS ASSOCIATED WITH DEGENERATE TANGENT POLYNOMIALS

In this section, we study differential equations arising from the generating functions of degenerate tangent polynomials. Let

$$H = H(t,\lambda) = \frac{2}{(1+\lambda t)^{2/\lambda}+1} = \sum_{n=0}^{\infty} \mathcal{T}_n(\lambda)\frac{t^n}{n!}. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned} H^{(1)} &= \frac{d}{dt}H(t,\lambda) = \frac{d}{dt}\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right) \\ &= \frac{1}{1+\lambda t}\left(\frac{-4}{(1+\lambda t)^{2/\lambda}+1} + \left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^2\right) \\ &= \frac{H^2-2H}{1+\lambda t}. \end{aligned} \tag{2.2}$$

Let

$$F = F(t,x,\lambda) = \frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{T}_n(x,\lambda)\frac{t^n}{n!}. \tag{2.3}$$

Then, by (2.3), we have

$$\begin{aligned} F^{(1)} &= \frac{d}{dt}F(t,x,\lambda) \\ &= \frac{d}{dt}\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\right) \\ &= \frac{1}{1+\lambda t}\left(H+(x-2)\right)F. \end{aligned} \tag{2.4}$$

By (2.4), we have

$$F^{(1)} = H(1+\lambda t)^{-1}F + (x-2)(1+\lambda t)^{-1}F. \tag{2.5}$$

Taking the derivative with respect to t in (2.5), we obtain

$$\begin{aligned} F^{(2)} = {} & 2H^2(1+\lambda t)^{-2}F + (2x-6-\lambda)(1+\lambda t)^{-2}F \\ & + \left((x-2)(x-2-\lambda)\right)(1+\lambda t)^{-2}F \end{aligned} \tag{2.6}$$

Continuing this process, we can guess that

$$F^{(N)} = \sum_{i=0}^{N} a_i(N,x,\lambda)H^i(1+\lambda t)^{-N}F, \quad (N=0,1,2,\ldots), \tag{2.7}$$

where $F^{(i)} = \left(\frac{d}{dt}\right)^i F(t,x,\lambda)$. Differentiating (2.7) with respect to t, we have

$$\begin{aligned}
F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\
&= \sum_{i=0}^{N} a_i(N,x,\lambda) i H^{i-1} H^{(1)} (1+\lambda t)^{-N} F \\
&\qquad + \sum_{i=0}^{N} a_i(N,x,\lambda) H^i(-N\lambda)(1+\lambda t)^{-N-1} F \\
&\qquad + \sum_{i=0}^{N} a_i(N,x,\lambda) H^i (1+\lambda t)^{-N} F^{(1)} \\
&= \sum_{i=0}^{N} a_i(N,x,\lambda) i \left(H^{i+1} - 2H^i\right)(1+\lambda t)^{-N-1} F \\
&\qquad + \sum_{i=0}^{N} a_i(N,x,\lambda) H^i(-N\lambda)(1+\lambda t)^{-N-1} F \\
&\qquad + \sum_{i=0}^{N} a_i(N,x,\lambda) \left(H^{i+1} + (x-2)H^i\right)(1+\lambda t)^{-N-1} F \\
&= \sum_{i=1}^{N+1} a_{i-1}(N,x,\lambda) i H^i (1+\lambda t)^{-N-1} F \\
&\qquad + \sum_{i=0}^{N} a_i(N,x,\lambda)(x-2-2i-N\lambda) H^i (1+\lambda t)^{-N-1} F
\end{aligned} \tag{2.8}$$

.

On the other hand, by replacing N by $N+1$ in (2.7), we get

$$F^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,x,\lambda)H^i(1+\lambda t)^{-N-1}F. \tag{2.9}$$

By (2.8) and (2.9), we have

$$\begin{aligned}&\sum_{i=0}^{N+1} a_i(N+1,x,\lambda)H^i(1+\lambda t)^{-N-1}F\\&=\sum_{i=1}^{N+1} a_{i-1}(N,x,\lambda)iH^i(1+\lambda t)^{-N-1}F\\&\qquad+\sum_{i=0}^{N} a_i(N,x,\lambda)(x-2-2i-N\lambda)H^i(1+\lambda t)^{-N-1}F.\end{aligned} \tag{2.10}$$

Comparing the coefficients on both sides of (2.10), we obtain

$$\begin{aligned}a_0(N+1,x,\lambda)&=(x-2-N\lambda)a_0(N,x,\lambda),\\a_{N+1}(N+1,x,\lambda)&=(N+1)a_N(N,x,\lambda),\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}a_i(N+1,x,\lambda)=(x-2-2i-N\lambda)a_i(N,x,\lambda)\\+ia_{i-1}(N,x,\lambda).\end{aligned} \tag{2.12}$$

In addition, by (2.5), we have

$$F=a_0(0,x,\lambda)F, \tag{2.13}$$

which gives

$$a_0(0,x,\lambda)=1. \tag{2.14}$$

It is not difficult to show that

$$\begin{aligned}F^{(1)}&=a_0(1,x,\lambda)(1+\lambda t)^{-1}F+a_1(1,x,\lambda)H(1+\lambda t)^{-1}F\\&=H(1+\lambda t)^{-1}F+(x-2)(1+\lambda t)^{-1}F.\end{aligned} \tag{2.15}$$

Thus, by (2.15), we also find

$$a_0(1,x,\lambda)=x-2,\quad a_1(1,x,\lambda)=1. \tag{2.16}$$

From (2.11), we note that

$$\begin{aligned}a_0(N+1,x,\lambda)&=(x-2-N\lambda)a_0(N,x,\lambda)\\&=(x-2-N\lambda)(x-2-(N-1)\lambda)a_0(N-1,x,\lambda)\\&=\cdots=<x-2|\lambda>_{N+1}^{(N)}=(x-2|\lambda)_{N+1},\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}a_{N+1}(N+1,x,\lambda)&=(N+1)a_N(N,x,\lambda)\\&=(N+1)Na_{N-1}(N-1,x,\lambda)\\&=\cdots=(N+1)!.\end{aligned} \tag{2.18}$$

For $i = 1, 2, 3$ in (2.12), we find that

$$
\begin{aligned}
a_1(N+1, x, \lambda) &= \sum_{k=0}^{N} < x-4|\lambda >_k^{(N)} \ a_0(N-k, x, \lambda), \\
a_2(N+1, x, \lambda) &= \sum_{k=0}^{N-1} < x-6|\lambda >_k^{(N)} \ a_1(N-k, x, \lambda), \\
a_3(N+1, x, \lambda) &= \sum_{k=0}^{N-2} < x-8|\lambda >_k^{(N)} \ a_2(N-k, x, \lambda).
\end{aligned} \tag{2.19}
$$

Continuing this process, we can deduce that, for $1 \leq i \leq N$,

$$
a_i(N+1, x, \lambda) = \sum_{k=0}^{N-i+1} < x-2-2i|\lambda >_k^{(N)} \ a_{i-1}(N-k, x, \lambda).
$$

Note that, here the matrix $a_i(j, x, \lambda)_{0 \leq i, j \leq N+1}$ is given by

$$
\begin{pmatrix}
1 & (x-2|\lambda)_1 & (x-2|\lambda)_2 & (x-2|\lambda)_3 & \cdots & (x-2|\lambda)_{N+1} \\
0 & 1 & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 1 & \cdot & \cdots & \cdot \\
0 & 0 & 0 & 1 & \cdots & \cdot \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

Now, we give explicit expressions for $a_i(N+1, x, \lambda)$. By (2.17), (2.18), and (2.19), we have

$$
\begin{aligned}
a_1(N+1, x, \lambda) &= \sum_{k_1=0}^{N} < x-4|\lambda >_{k_1}^{(N)} \ a_0(N-k_1, x, \lambda) \\
&= \sum_{k_1=0}^{N} < x-4|\lambda >_{k_1}^{(N)} < x-2|\lambda >_{N-k_1}^{(N-k_1-1)},
\end{aligned}
$$

$$
\begin{aligned}
& a_2(N+1, x, \lambda) \\
&= 2 \sum_{k_2=0}^{N-1} < x-6|\lambda >_{k_2}^{(N)} \ a_1(N-k_2, x, \lambda) \\
&= \sum_{k_2=0}^{N-1} \sum_{k_1=0}^{N-k_2-1} < x-6|\lambda >_{k_2}^{(N)} < x-4|\lambda >_{k_1}^{(N-k_2-1)} < x-2|\lambda >_{N-k_2 k_1-1}^{(N-k_2-k_1-2)},
\end{aligned}
$$

and

$$
\begin{aligned}
&a_3(N+1,x,\lambda)\\
&= 3\sum_{k_3=0}^{N-2} < x-8|\lambda >_{k_3}^{(N)} a_2(N-k_3,x,\lambda)\\
&= 3\cdot 2\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-k_3-2}\sum_{k_1=0}^{N-k_3-k_2-2} < x-8|\lambda >_{k_3}^{(N)} < x-6|\lambda >_{k_2}^{(N-k_3-1)}\\
&\qquad \times < x-4|\lambda >_{k_1}^{(N-k_3-k_2-2)} < x-2|\lambda >_{N-k_3-k_2-k_1-2}^{(N-k_3-k_2-k_1-3)}
\end{aligned}
$$

Continuing this process, we have

$$
\begin{aligned}
a_i(N+1,\lambda) = i!&\sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-k_i-i+1}\cdots\sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i+1} < x-2-2i|\lambda >_{k_i}^{(N)}\\
&\times < x-2-2(i-1)|\lambda >_{k_{i-1}}^{(N-k_i-1)}\cdots\\
&\times < x-4|\lambda >_{k_1}^{(N-k_i-k_{i-1}-\cdots k_2-i+1)}\\
&\times < x-2|\lambda >_{N-k_i-k_{i-1}-\cdots-k_2-k_1-i+1}^{(N-k_i-k_{i-1}-\cdots k_2-k_1-i)}.
\end{aligned}
\tag{2.20}
$$

Therefore, by (2.20), we obtain the following theorem.

Theorem 1. For $N = 0, 1, 2, \ldots$, the functional equation

$$F^{(N)} = \sum_{i=0}^{N} a_i(N,x,\lambda)H^i(1+\lambda t)^{-N}F$$

has a solution

$$F = F(t,x,\lambda) = \frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda},$$

where

$$
\begin{aligned}
&a_0(N,x,\lambda) = (x-2|\lambda)_N,\\
&a_N(N,x,\lambda) = N!,\\
&a_i(N,\lambda) = i!\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i} < x-2-2i|\lambda >_{k_i}^{(N-1)}\\
&\qquad \times < x-2-2(i-1)|\lambda >_{k_{i-1}}^{(N-k_i-2)}\cdots\\
&\qquad \times < x-4|\lambda >_{k_1}^{(N-k_i-k_{i-1}-\cdots k_2-i)}\\
&\qquad \times < x-2|\lambda >_{N-k_i-k_{i-1}-\cdots-k_2-k_1-i}^{(N-k_i-k_{i-1}-\cdots k_2-k_1-i-1)}.
\end{aligned}
$$

Here is a plot of the surface for this solution. In Figure 1, we choose $\lambda = 1/10$, $-3 \le x \le 3$, and $0.1 \le t \le 5$.

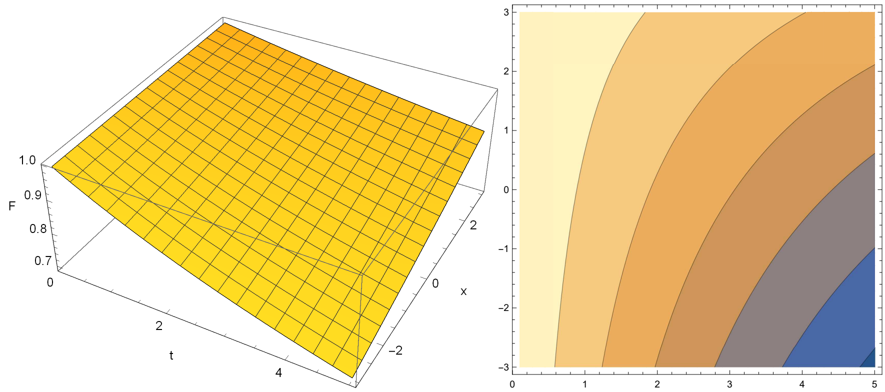

Figure 1: The surface for the solution $F(t, x, \lambda)$

In Figure 1(left), we plot of the surface for this solution. In Figure 1(right), we shows a higher-resolution density plot of the solution.

From (1.1), we note that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t, x, \lambda) = \sum_{k=0}^{\infty} \mathcal{T}_{k+N}(x, \lambda)\frac{t^k}{k!}. \tag{2.21}$$

From Theorem 1, (1.3), and (2.21), we can derive the following equation:

$$\begin{aligned}\sum_{k=0}^{\infty} \mathcal{T}_{k+N}(x, \lambda)\frac{t^k}{k!} &= F^{(N)} = \sum_{i=0}^{N} a_i(N, x, \lambda)(1+\lambda t)^{-N} F \\ &= \sum_{i=0}^{N} a_i(N, x, \lambda)(1+\lambda t)^{-N}\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^{i+1}(1+\lambda t)^{x/\lambda} \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{N} a_i(N, x, \lambda)\sum_{l=0}^{k}\binom{k}{l}(-\lambda)^l\binom{N+l-1}{N-1} l!\mathcal{T}_{k-l}^{(i+1)}(x, \lambda)\right)\frac{t^k}{k!}.\end{aligned} \tag{2.22}$$

By comparing the coefficients on both sides of (2.22), we obtain the following theorem.

Theorem 2. For $k = 0, 1, \ldots,$ and $N = 0, 1, 2, \ldots,$ we have

$$\mathcal{T}_{k+N}(x, \lambda) = \sum_{i=0}^{N}\sum_{l=0}^{k} a_i(N, x, \lambda)\binom{k}{l}(-\lambda)^l\binom{N+l-1}{N-1} l!\mathcal{T}_{k-l}^{(i+1)}(x, \lambda), \tag{2.23}$$

where

$$
\begin{aligned}
&a_0(N,x,\lambda) = (x-2|\lambda)_N,\\
&a_N(N,x,\lambda) = N!,\\
&a_i(N,\lambda) = i!\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i} < x-2-2i|\lambda >_{k_i}^{(N-1)}\\
&\qquad\times < x-2-2(i-1)|\lambda >_{k_{i-1}}^{(N-k_i-2)}\cdots\\
&\qquad\times < x-4|\lambda >_{k_1}^{(N-k_i-k_{i-1}-\cdots k_2-i)}\\
&\qquad\times < x-2|\lambda >_{N-k_i-k_{i-1}-\cdots-k_2-k_1-i}^{(N-k_i-k_{i-1}-\cdots k_2-k_1-i-1)}.
\end{aligned}
$$

Let us take $k=0$ in (2.23). Then, we have the following corollary.

Corollary 3. For $N = 0,1,2,\ldots$, we have

$$\mathcal{T}_N(x,\lambda) = \sum_{i=0}^{N} a_i(N,x,\lambda)\mathcal{T}_0^{(i+1)}(x,\lambda).$$

3. ZEROS OF THE DEGENERATE TANGENT POLYNOMIALS OF HIGHER ORDER

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the higher order degenerate tangent polynomials of the higher order $\mathcal{T}_n^{(k)}(x,\lambda)$. By using computer, the higher order degenerate tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ can be determined explicitly. A few of them are

$$
\begin{aligned}
&\mathcal{T}_0^{(k)}(x,\lambda) = 1, \quad \mathcal{T}_1^{(k)}(x,\lambda) = -k+x,\\
&\mathcal{T}_2^{(k)}(x,\lambda) = -k+\lambda k+k^2-\lambda x-2kx+x^2,\\
&T_3^{(k)}(x,\lambda) = 3\lambda k-2\lambda^2 k+3k^2-3\lambda k^2-k^3+2\lambda^2 x-3kx+6\lambda kx+3k^2x\\
&\qquad -3\lambda x^2-3kx^2+x^3,\\
&\mathcal{T}_4^{(k)}(x,\lambda) = 2k-11\lambda^2 k+6\lambda^3 k+3k^2-18\lambda k^2+11\lambda^2k^2-6k^3+6\lambda k^3+k^4\\
&\qquad -6\lambda^3 x+18\lambda kx-22\lambda^2 kx+12k^2x-18\lambda k^2x-4k^3x+11\lambda^2x^2\\
&\qquad -6kx^2+18\lambda kx^2+6k^2x^2-6\lambda x^3-4kx^3+x^4.
\end{aligned}
$$

We investigate the beautiful zeros of the $\mathcal{T}_n^{(k)}(x,\lambda)$ by using a computer. We plot the zeros of the degenerate tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ for $n=10, \lambda=1/10, k=2,4,6,8$, and $x\in\mathbb{C}$(Figure 1). In Figure 2(top-left), we choose $n=10, k=2$ and $\lambda=1/10$. In Figure 2(top-right), we choose $n=10, k=4$ and $\lambda=1/10$. In Figure

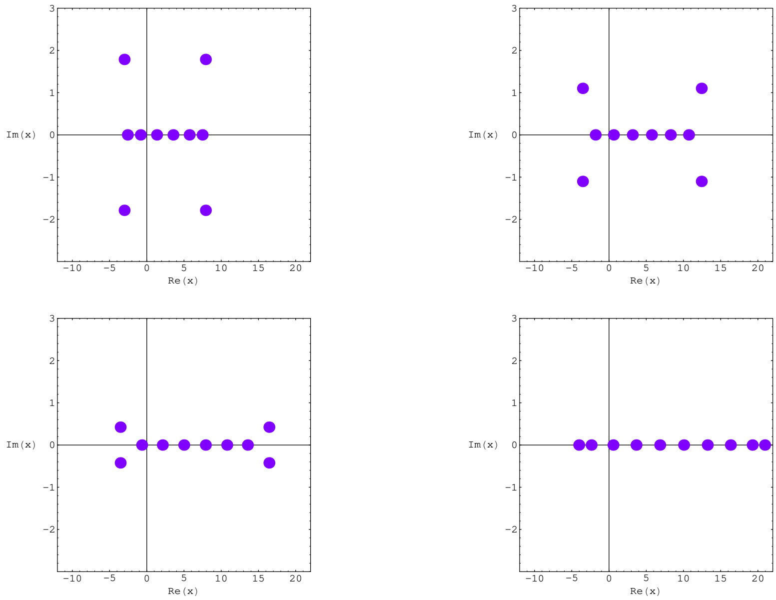

Figure 2: Zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$

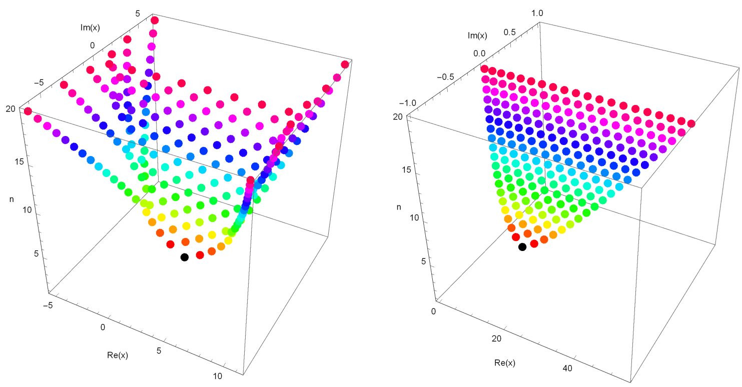

Figure 3: Stacks of zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \leq n \leq 20$

2(bottom-left), we choose $n = 10, k = 6$ and $\lambda = 1/10$. In Figure 2(bottom-right), we choose $n = 10, k = 8$ and $\lambda = 1/10$. Stacks of zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \leq n \leq 20$ from a 3-D structure are presented(Figure 3). In Figure 3(left), we choose $1 \leq n \leq 20, k = 2$, and $\lambda = 1/10$. In Figure 3(right), we choose $1 \leq n \leq 20, k = 8$, and $\lambda = 2$. Our numerical results for approximate solutions of real zeros of $\mathcal{T}_n^{(k)}(x,\lambda) = 0$ are displayed(Tables 1, 2).

degree n	$k=2, \lambda=1/10$ real zeros	$k=2, \lambda=1/10$ complex zeros	$k=8, \lambda=1/10$ real zeros	$k=8, \lambda=1/10$ complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	4	0	4	0
5	5	0	5	0
6	6	0	6	0
7	3	4	7	0
8	4	4	8	0
9	5	4	9	0
10	6	4	10	0

Table 1: Numbers of real and complex zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$

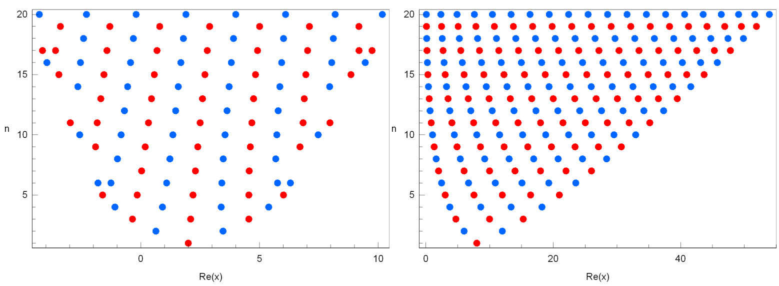

Figure 4: Real zeros of $\mathcal{T}_{n,\lambda}^{(k)}(x)$ for $k=2$ and $1 \leq n \leq 20$

Plot of real zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \leq n \leq 20$ structure are presented(Figure 4). In Figure 3(left), we choose $1 \leq n \leq 20, k=2$, and $\lambda=1/10$. In Figure 3(right), we choose $1 \leq n \leq 20, k=8$, and $\lambda=2$. We observe a remarkably regular structure of the complex roots of the partially degenerate tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$. We hope to verify a remarkably regular structure of the complex roots of the degenerate tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$(Table 1). Next, we calculated an approximate solution satisfying $\mathcal{T}_n^{(k)}(x,\lambda)=0, x \in \mathbb{C}$. The results are given in Table 2 and Table 3.

Finally, we shall consider the more general problems. How many zeros does $\mathcal{T}_n^{(k)}(x,\lambda)$ have? We are not able to decide if $\mathcal{T}_n^{(k)}(x,\lambda)=0$ has n distinct solutions. We would like to know the number of complex zeros $C_{\mathcal{T}_n^{(k)}(x,\lambda)}$ of $\mathcal{T}_n^{(k)}(x,\lambda), Im(x) \neq 0$. Since n is the degree of the polynomial $\mathcal{T}_n^{(k)}(x,\lambda)$, the number of real zeros $R_{\mathcal{T}_n^{(k)}(x,\lambda)}$ lying on the real plane $Im(x)=0$ is then $R_{\mathcal{T}_n^{(k)}(x,\lambda)} = n - C_{\mathcal{T}_n(x,\lambda)}$, where $C_{\mathcal{T}_n^{(k)}(x,\lambda)}$

degree n	x
1	2.0000
2	0.6349, 3.465
3	−0.3515, 2.100, 4.552
4	−1.0897, 0.913, 3.39, 5.39
5	−1.610, −0.1517, 2.200, 4.55, 6.01
6	−1.803, −1.259, 1.087, 3.41, 5.8, 6.3
7	0.03291, 2.30, 4.6

Table 2: Approximate solutions of $\mathcal{T}_n^{(k)}(x,\lambda)=0, k=2, \lambda=1/10, x\in\mathbb{R}$

degree n	x
1	8.0000
2	5.2211, 10.879
3	3.2000, 8.1000, 13.000
4	1.5769, 5.9548, 10.345, 14.723
5	0.21838, 4.1573, 8.2000, 12.243, 16.182
6	−0.94053, 2.5916, 6.3516, 10.148, 13.908, 17.441
7	−1.9351, 1.1953, 4.7076, 8.3000, 11.892, 15.405, 18.535

Table 3: Approximate solutions of $\mathcal{T}_n^{(k)}(x,\lambda)=0, k=8, \lambda=1/10, x\in\mathbb{C}$

denotes complex zeros. See Table 1 for tabulated values of $R_{\mathcal{T}_n^{(k)}(x,\lambda)}$ and $C_{\mathcal{T}_n^{(k)}(x,\lambda)}$.

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