Dynamic Systems and Applications, **28**, No. 2 (2019), 491-506 ISSN: 1056-2176

LINEAR FRACTIONAL SYSTEM OF INCOMMENSURATE TYPE WITH DISTRIBUTED DELAY AND BOUNDED LEBESGUE MEASURABLE INITIAL CONDITIONS

ANDREY ZAHARIEV[1], HRISTO KISKINOV[2], AND EVGENIYA ANGELOVA[3]

[1,2,3]Faculty of Mathematics and Informatics
University of Plovdiv
4 Tzar Asen, 4000, Plovdiv, BULGARIA

ABSTRACT: In the present work is considered a fractional linear system with distributed delay and derivatives in Caputo sense of incommensurate type. For this system is studied the important problem for existence and uniqueness of the solutions of an initial problem in the case of bounded Lebesgue measurable initial conditions.

AMS Subject Classification: 34A08, 34A12

Key Words: distributed delay; fractional derivatives; linear fractional differential system

Received: December 1, 2018 ; **Revised:** March 27, 2019 ;
Published (online): April 4, 2019 **doi:** 10.12732/dsa.v28i2.14
Dynamic Publishers, Inc., Acad. Publishers, Ltd. https://acadsol.eu/dsa

1. INTRODUCTION

To obtain a deep understanding about the fractional calculus and respectively the fractional differential equations in details see the monographs of Kilbas et al. [12], Kiryakova [13], Podlubny [26], Feckan et al. [9] and Abbas et al. [1]. For distributed order fractional differential equations see [11], for an application-oriented exposition Diethelm [7] and for fractional evolution equations in Banach spaces Bajlecova [4]. The impulsive differential and functional differential equations with fractional derivatives and some applications are considered in the monograph of Stamova and Stamov [28]. Also it is worth noting some new interesting results for fractional differential equations and systems obtained in [2], [20],[29], [35], [38] and [39].

The first detailed study of linear differential equations and system with distributed delay (fundamental theory, variation of constants formula, stability, etc.) was done by A.D. Myshkis in his fundamental monograph [22]. It may be noted that fractional systems of retarded and neutral type with distributed delays are studied (basically existence and stability) in [14], [21], [31]–[34] for single order fractional derivatives and in [5] for Caputo-type distributed order fractional derivatives. Note that a lot of results are obtained from many authors, using the definition of Caputo type derivative applicable only in the particular case when the functions are absolutely continuous. In this work, we use the definition of Caputo-type derivative without the assumption that the functions are absolutely continuous.

It is well known that the problem of establishing a formula for the general solution for linear fractional differential equations and/or systems with delay, as well as its integral representation (variation of constants formula) need theorem for existing of fundamental matrix, i.e. theorem for existence and uniqueness of the solutions of initial problem (IP) in the case of discontinuous initial functions (see for example [6], [10], [37]). It must be noted that this problem is more complicated in compare with the integer order differential equations with delay. We point out that this is conditioned that a distinguishing feature of the fractional differential equations with delay is that the evolution of the processes described by such equations depends on the past history inspired from two sources, first of them is the impact conditioned of the delays and the other one the impact conditioned from the availability of Volterra type integral in the definitions of the fractional derivatives, i.e. the memory of the fractional derivative. It must be noted that the first of them (conditioned by the delays) is independent from the derivative type (integer or fractional).

In the present work, we consider a nonautonomous linear fractional system with distributed delay and derivatives in Caputo sense of incommensurate type. For this system, we study the important problem for existence and uniqueness of the solutions of initial problem (IP) in the case of bounded Lebesgue measurable initial conditions.

As far as we know, there are only a few results concerning IP for fractional differential equations with delay and discontinuous initial function. In [15] is studied an IP with bounded Lebesgue measurable initial conditions for autonomous linear system with distributed delay in the case when all differentiations orders are equal. In [6] are obtained results for existence and uniqueness for nonautonomous fractional system with distributed delay and piecewise continuous initial functions with finite many jumps. The results in [6] are generalized in [37] for neutral systems. It must be noted, that the technique of the proofs in the present paper (inspired from [15]) is different in compare with the technique used in [6]. Since in our obtained results the fractional differentiation orders are of incommensurate type then our result extends the corresponding one in [15] even in the autonomous case too. The proposed conditions coincide with the conditions which guaranty the same result in the case of integer order linear differential equations with distributed delay.

The results obtained in this article would be a good basis for building models of different processes from the real world. Good examples of new studies with application in modeling are [3], [16]–[19],[23]–[25],[27], [30].

The paper is organized as follows. In Section 2, we recall some needed definitions of Riemann-Liouville and Caputo fractional derivatives, as well as the needed part of their properties. In this section also we present a slightly modified version of the Weissinger generalization of the Banach's fixed point theorem ([36], Fixpunktsatz, p. 195) which will be used by the proof of the main results and is presented the problem statement too. In Section 3 as main result are obtained sufficient conditions for the existence and the uniqueness of the solutions of the Cauchy problem for linear incommensurate fractional differential system with distributed delays in the cases of Caputo derivatives and with Lebesgue measurable, bounded initial function.

2. PRELIMINARIES AND PROBLEM STATEMENT

For convenience and to avoid possible misunderstandings, below we recall only the definitions of Riemann-Liouville and Caputo fractional derivatives and some needed their properties. For details and other properties we refer to [12, 13, 26].

Let $\alpha \in (0,1)$ be an arbitrary number and denote by $L_1^{loc}(\mathbb{R},\mathbb{R})$ the linear space of all locally Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{R}$. Then for each $t > a, a \in \mathbb{R}$ and $f \in L_1^{loc}(\mathbb{R},\mathbb{R})$ the left-sided fractional integral operator and the corresponding left side Riemann-Liouville and Caputo fractional derivatives of order α are defined by

$$(D_{a+}^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \mathrm{d}s,$$

$${}_{RL}D_{a+}^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left(D_{a+}^{-(1-\alpha)} f(t) \right)$$

$${}_{C}D_{a+}^{\alpha} f(t) = {}_{RL}D_{a+}^{\alpha}[f(s) - f(a)](t) = {}_{RL}D_{a+}^{\alpha} f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}$$

respectively. We will use also the following relations (see [12]):

(i) $(D_{a+}^{0} f)(t) = f(t)$;

(ii) ${}_{C}D_{a+}^{\alpha} D_{a+}^{-\alpha} f(t) = f(t)$;

(iii) $D_{a+}^{-\alpha}\ {}_{C}D_{a+}^{\alpha} f(t) = f(t) - f(a)$.

We will need a slightly modified version of the Weissinger generalization of the Banach's fixed point theorem (see [36], Fixpunktsatz, p. 195).

Theorem 1. *Let Ω be a complete metric space with metric d_Ω and let the following conditions hold:*

1. There exists a sequence $\gamma_q \geq 0, q \in \mathbb{N}$ with $\sum\limits_{q=1}^{\infty} \gamma_q < \infty$.

2. The operator $T : \Omega \to \Omega$ satisfies for each $q \in \mathbb{N}$ and for arbitrary $x, y \in \Omega$ the inequality

$$d_\Omega(T^q x, T^q y) \leq \gamma_q d_\Omega(x, y)$$

Then T has a unique fixed point $x^* \in \Omega$ and for every $x \in \Omega$ we have that

$$\lim_{q\to\infty} T^q x = x^*.$$

Remark 1. This modification of the Weissinger generalization of the Banach's fixed point theorem is not new, it is used in [8] in the case when Ω is a Banach space. But it is simply to be seen that the original Weissinger proof is correct in the presented in Theorem 1 case too, with some elementary modifications.

Consider the nonautonomous linear nonhomogeneous fractional system of incommensurate type with distributed delay in the following form

$$D^\alpha_{a+} X(t) = \sum_{i=0}^{m} \int_{-\sigma_i}^{0} [\mathrm{d}_\theta U^i(t,\theta)] X(t+\theta) + F(t), \tag{1}$$

and the corresponding homogeneous one

$$D^\alpha_{a+} X(t) = \sum_{i=0}^{m} \int_{-\sigma_i}^{0} [\mathrm{d}_\theta U^i(t,\theta)] X(t+\theta), \tag{2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n), \alpha_k \in (0,1), k \in \langle n \rangle = \{1, 2, \ldots, n\}, \sigma_i \in (0, \sigma], \sigma > 0$, $F : J_a \to \mathbb{R}^n, X : J_* \to \mathbb{R}^n, U^i : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}, J_a = [a, \infty), a \in \mathbb{R}, J_* = [a - \sigma, \infty)$, $X(t) = (x_1(t), \ldots, x_n(t))^T, F(t) = (f_1(t), \ldots, f_n(t))^T, U^i(t,\theta) = \{u^i_{kj}(t,\theta)\}^n_{k,j=1}$, $i \in \langle m \rangle_0 = \langle m \rangle \cup \{0\}, m \in \mathbb{N}, D^\alpha_{a+} X(t) = (D^{\alpha_1}_{a+} x_1(t), \ldots, D^{\alpha_n}_{a+} x_n(t))^T$ and $D^{\alpha_k}_{a+}$ denotes the left side Caputo fractional derivative ${}_C D^{\alpha_k}_{a+}$.

We will use also the following notations: $\mathbb{R}_+ = (0, \infty), \bar{\mathbb{R}}_+ = [0, \infty)$. Let $Y : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}, Y(t,\theta) = \{y_{ij}(t,\theta)\}^n_{i,j=1}$ and $|Y(t,\theta)| = \sum\limits_{k,j=1}^{n} |y_{kj}(t,\theta)|$. With $BV[-\sigma, 0]$ we will denote the linear space of matrix valued functions $Y(t,\theta)$ with bounded variation in θ on $[-\sigma, 0]$ for every $t \in J_a$ and $Var_{[-\sigma,0]} Y(t,\cdot) = \sum\limits_{k,j=1}^{n} Var_{[-\sigma,0]} y_{kj}(t,\cdot)$. For $W(t) = (w_1(t), \ldots, w_n(t))^T : J_a \to \mathbb{R}^n, \beta = (\beta_1, \ldots, \beta_n), \beta_k \in [-1, 1], k \in \langle n \rangle$ we will use the notation $I_\beta(W(t)) = diag((w_1(t))^{\beta_1}, \ldots, (w_n(t))^{\beta_n})$.

With $\mathfrak{C}^*_a$ we denote the Banach space of initial vector functions $\Phi = (\phi_1, \ldots, \phi_n)^T : [a - \sigma, a] \to \mathbb{R}^n, a \in \mathbb{R}$, which are bounded and Lebesgue measurable on the interval

$[a-\sigma, a]$ with norm

$$||\Phi|| = \sup_{t\in[a-\sigma,a]} |\Phi(t)| = \sum_{k=1}^{n} \sup_{t\in[a-\sigma,a]} |\phi_k(t)| < \infty.$$

Consider the following initial conditions for the system (1) ((2)) :

$$X(t) = \Phi(t) \quad (x_k(t) = \phi_k(t), k \in \langle n\rangle), t \in [a-\sigma, a], \Phi \in \mathfrak{C}_a^* \tag{3}$$

We say that for the kernels $U^i : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$ the conditions (S) are fulfilled if for each $i \in \langle m\rangle_0$ the following conditions hold:

(S1) The function $(t,\theta) \to U^i(t,\theta)$ is measurable in $(t,\theta) \in J_a \times \mathbb{R}$ and normalized so that $U^i(t,\theta) = 0$ for $\theta \geq 0$ and $U^i(t,\theta) = U^i(t,-\sigma_i)$ for $\theta \leq -\sigma_i, t \in J_a$.

(S2) For each $t \in J_a$ the kernel $U^i(t,\theta)$ is continuous from the left in θ on $(-\sigma_i, 0)$ and $U^i(t,\cdot) \in BV[-\sigma_i, 0]$.

(S3) The Lebesgue decomposition of the kernel $U^i(t,\theta)$ for $t \in J_a$ and $\theta \in [-\sigma_i, 0]$ has the form:

$$U^i(t,\theta) = \aleph^i(t,\theta) + \int_{-\sigma_i}^{\theta} B(t,s)\mathrm{d}s + \Upsilon(t,\theta),$$

where $\aleph^i(t,\theta) = \{a^i_{kj}(t)H(\theta+\sigma_i(t))\}_{k,j=1}^n$, the functions
$A^i(t) = \{a^i_{kj}(t)\}_{k,j=1}^n \in L_1^{loc}(J_a, \mathbb{R}^n)$ are locally bounded on J_a,
$\Upsilon(t,\theta) = \{g_{kj}(t,\theta)\}_{k,j=1}^n \in C(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$, $\sigma_i(t) \in C(J_a, \bar{\mathbb{R}}_+)$ for $i \in \langle m\rangle$, $\sigma_0(t) \equiv 0$ for every $t \in J_a$, $H(t)$ is the Heaviside function and the function
$B(t,\theta) = \{b_{kj}(t,\theta)\}_{k,j=1}^n \in L_1^{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ is locally bounded on $J_a \times \mathbb{R}$.

(S4) There exists a locally bounded function $z_u \in L_1^{loc}(J_a, \mathbb{R}_+)$ such that $Var_{[-\sigma_i,0]}U^i(t,\cdot) \leq z_u(t)$ for each $t \in J_a$.

(S5) For each $t \in J_a$ the following relation holds:
$\int\limits_{-\sigma_i}^{0} |U^i(t,\theta) - U^i(t^*,\theta)|\mathrm{d}\theta \to 0$, when $t \to t^*$.

(S6) The sets $S^i_\Phi = \{t \in J_a | t - \sigma_i(t) \in S_\Phi\}$ for every $i \in \langle m\rangle$ do not have limit points, where with S_Φ is denoted the set of all jump points of the initial function Φ .

Definition 1. The vector function $X(t) = (x_1(t), \ldots, x_n(t))^T$ is a solution of the IP (1), (3) in $[a, a+b], b > 0(J_a)$ if $X \in C([a, a+b], \mathbb{R}^n)(X \in C(J_a, \mathbb{R}^n))$ satisfies the system (1) for all $t \in (a, a+b](t \in (a,\infty))$ and the initial condition (3) for $t \in [a-\sigma, a]$.

Consider the following auxiliary system

$$\begin{aligned} X(t) = \Phi(a) + I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-s)F(s)\mathrm{d}s \\ + I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-s) \sum_{i=0}^{m} \int_{-\sigma}^{0} [\mathrm{d}_\theta U(s,\theta)]X(s+\theta)\mathrm{d}s \end{aligned} \tag{4}$$

or in more detailed form for $k \in \langle n \rangle$

$$\begin{aligned} x_k(t) &= \phi_k(a) + \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} f_k(s) \mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} [\sum_{i=0}^{m} (\sum_{j=1}^{n} \int_{-\sigma}^{0} x_j(s+\theta) \mathrm{d}_\theta u_{kj}^i(s,\theta))] \mathrm{d}s \end{aligned} \tag{5}$$

with the initial condition (3), where $I_{-1}(\Gamma(\alpha)) = diag(\Gamma^{-1}(\alpha_1), \ldots, \Gamma^{-1}(\alpha_n))$.

Definition 2. The vector function $X(t) = (x_1(t), \ldots, x_n(t))^T$ is a solution of the IP (4), (3) in $[a, a+b], b > 0 (J_a)$ if $X \in C([a, a+b], \mathbb{R}^n) (X \in C(J_a, \mathbb{R}^n))$ satisfies the system (4) for all $t \in (a, a+b] (t \in (a, \infty))$ and the initial condition (3) for $t \in [a-\sigma, a]$.

Lemma 1. *Let the following conditions hold:*

1. *Conditions (S) hold.*
2. *The function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded.*

Then for each initial function $\Phi \in \mathfrak{C}_a^*$ every solution $X(t)$ of IP (1), (3) *is a solution of the IP* (4), (3) *and vice versa.*

The proof is analogical of the proof of Lemma 3.3 in [6] and will be omitted.

3. MAIN RESULTS

In this section we will obtain sufficient conditions for existence and uniqueness of the solutions of IP (1), (3). In virtue of Lemma 1 it is enough to study the IP (4), (3).

Let for every initial function $\Phi(t) = (\phi_1(t), \ldots, \phi_n(t)) \in \mathfrak{C}_a^*$ define the sets

$$\Omega_1^\Phi = \{G : [a-\sigma, a+1] \to \mathbb{R}^n \mid G|_{[a,a+1]} \in C([a, a+1], \mathbb{R}^n); G(t) = \Phi(t), t \in [a-\sigma, a]\}$$

and introduce in them a metric function $d_1^\Phi : \Omega_1^\Phi \times \Omega_1^\Phi \to \bar{R_+}$ for each $G, \bar{G} \in \Omega_1^\Phi$ as follows:

$$d_1^\Phi(G, \bar{G}) = \sum_{k=1}^{n} \sup_{t \in [a,a+1]} |g_k(t) - \bar{g}_k(t)|.$$

Obviously the set Ω_1^Φ equipped with d_1^Φ is a complete metric space in respect to the introduced metric function.

Introduce for each $G = (g_1, \ldots, g_n)^T \in \Omega_1^\Phi$ the operator

$(\mathfrak{R}G)(t) = (\mathfrak{R}_1 g_1(t), \ldots, \mathfrak{R}_n g_n(t))^T$ for $k \in \langle n \rangle$ with

$$\begin{aligned} \mathfrak{R}_k g_k(t) = {} & \phi_k(a) \\ & + \frac{1}{\Gamma(\alpha_k)}[\int_a^t (t-s)^{\alpha_k - 1}[\sum_{i=0}^m (\sum_{j=1}^n \int_{-\sigma}^0 g_j(s+\theta) \mathrm{d}_\theta u^i_{kj}(s,\theta))]\mathrm{d}s \\ & + \int_a^t (t-s)^{\alpha_k - 1} f_k(s)\mathrm{d}s] \end{aligned} \tag{6}$$

for $t \in (a, a+1]$ and with $\mathfrak{R}_k g_k(t) = \phi_k(t)$ for $t \in [a-\sigma, a]$.

Theorem 2. *Let the following conditions be fulfilled:*

1. *Conditions (S) hold.*

2. *The function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded.*

Then for every initial function $\Phi \in \mathfrak{C}^*_a$ the IP (4), (3) *has a unique solution in* $[a, a+1]$.

Proof. Let $\Phi \in \mathfrak{C}^*_a$ be an arbitrary initial function. Then since Φ is bounded and Lebesgue measurable, then from conditions (S) it follows that for every $t \in [a, a+1]$ the functions $t \to \int\limits_{-\sigma}^{0} g_j(t+\theta)\mathrm{d}_\theta u^i_{k,j}(t,\theta)$ are bounded and at least Lebesgue integrable for each $k, j \in \langle n \rangle, i \in \langle m \rangle_0$. Then (6) implies that the functions $\mathfrak{R}_k g_k(t)$ are continuous for each $t \in (a, a+1]$ and $\lim\limits_{t \to a+} \mathfrak{R}_k g_k(t) = \phi_k(a)$ for $k \in \langle n \rangle$. Thus $\mathfrak{R}_k g_k(t) \in C([a, a+1], \mathbb{R}^n)$ for $k \in \langle n \rangle$ and hence $\mathfrak{R}\Omega^\Phi_1 \subset \Omega^\Phi_1$, i.e. the operator $\mathfrak{R}$ maps Ω^Φ_1 into Ω^Φ_1.

We remind that $\Gamma(z), z \in \mathbb{R}_+$ has a local minimum at $z_{min} \approx 1.46163$, where it attains the value $\Gamma(z_{min}) \approx 0.885603$. Introduce the notations $\alpha_m = \min\limits_{k \in \langle n \rangle} \alpha_k$, $\alpha_M = \max\limits_{k \in \langle n \rangle} \alpha_k$, $q_0 = [2\alpha_m^{-1}] + 1$ and for every $q \in \langle q_0 \rangle$ denote with α_q that number among the numbers $\alpha_1, ..., \alpha_n$ for which $\Gamma(1 + q\alpha_q) = \min\limits_{k \in \langle n \rangle} \Gamma(1 + q\alpha_k)$.

Let $G, \bar{G} \in \Omega^\Phi_1$ be arbitrary. Then from (6) for every $t \in [a, a+1]$ and $k \in \langle n \rangle$ we have the estimation

$$
\begin{aligned}
&|\mathfrak{R}_k g_k(t) - \mathfrak{R}_k \bar{g}_k(t)| \\
&\leq \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} [\sum_{i=0}^{m} (\sum_{j=1}^{n} | \int_{-\sigma}^{0} (g_j(s+\theta) - \bar{g}_j(s+\theta)) \mathrm{d}_\theta u^i_{kj}(s,\theta)|)] \mathrm{d}s \\
&\leq \frac{(t-a)^{\alpha_k}}{\Gamma(1+\alpha_k)} \sum_{i=0}^{m} (\sum_{j=1}^{n} \sup_{s\in[a,a+1]} Var_{\theta\in[-\sigma,0]} u^i_{k,j}(s,\theta) \sup_{t\in[a,a+1]} |g_j(t) - \bar{g}_j(t)|) \qquad (7)\\
&\leq \frac{(t-a)^{\alpha_k}}{\Gamma(1+\alpha_k)} \sum_{j=1}^{n} (\sup_{t\in[a,a+1]} |g_j(t) - \bar{g}_j(t)| \sum_{i=0}^{m} \sum_{l,j=1}^{n} \sup_{s\in[a,a+1]} Var_{\theta\in[-\sigma,0]} u^i_{l,j}(s,\theta)) \\
&\leq \frac{(t-a)^{\alpha_k}}{\Gamma(1+\alpha_k)} U_1 d_1^{\Phi}(G,\bar{G})
\end{aligned}
$$

where $U_1 = \sum\limits_{i=0}^{m} \sum\limits_{l,j=1}^{n} \sup\limits_{s\in[a,a+1]} Var_{\theta\in[-\sigma,0]} u^i_{l,j}(s,\theta)$.

Let assume that for $k \in \langle n \rangle$ and every $t \in [a, a+1]$ the estimate

$$
|\mathfrak{R}^q_k g_k(t) - \mathfrak{R}^q_k \bar{g}_k(t)| \leq \frac{(t-a)^{q\alpha_k} U_1^q}{\Gamma(1+q\alpha_k)} d_1^{\Phi}(G,\bar{G}) \qquad (8)
$$

holds for some $q \in \mathbb{N}$. Note that inequality (7) implies that (8) holds for $q = 1$ and every $t \in [a, a+1], k \in \langle n \rangle$. Using the notations $\mathfrak{R}^q G(t) = Y(t) = (y_1(t), \ldots, y_n(t))^T$ and $\mathfrak{R}^q \bar{G}(t) = \bar{Y}(t) = (\bar{y}_1(t), \ldots, \bar{y}_n(t))^T$ we have that

$$
|\mathfrak{R}^{q+1}_k g_k(t) - \mathfrak{R}^{q+1}_k \bar{g}_k(t)| = |\mathfrak{R}\mathfrak{R}^q_k g_k(t) - \mathfrak{R}\mathfrak{R}^q_k \bar{g}_k(t)| = |\mathfrak{R} y_k(t) - \mathfrak{R} \bar{y}_k(t)|. \qquad (9)
$$

According the induction hypothesis (8) we have

$$
\begin{aligned}
&|\mathfrak{R}^{q+1}_k g_k(t) - \mathfrak{R}^{q+1}_k \bar{g}_k(t)| = |\mathfrak{R} y_k(t) - \mathfrak{R} \bar{y}_k(t)| \\
&\leq \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} [\sum_{i=0}^{m} (\sum_{j=1}^{n} | \int_{-\sigma}^{0} (y_j(s+\theta) - \bar{y}_j(s+\theta)) \mathrm{d}_\theta u^i_{kj}(s,\theta)|)] \mathrm{d}s \\
&\leq \frac{U_1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} \sum_{j=1}^{n} \sup_{\eta\in[a,a+s]} |(y_j(\eta) - \bar{y}_j(\eta))| \mathrm{d}s \qquad (10)\\
&\leq \frac{U_1 U_1^q}{\Gamma(\alpha_k)\Gamma(1+q\alpha_k)} d_1^{\Phi}(G,\bar{G}) \int_a^t (t-s)^{\alpha_k - 1} (s-a)^{q\alpha_k} \mathrm{d}s
\end{aligned}
$$

Substitute $s - a = z(t-a)$ in the right side of (10) and using the relation between the gamma and beta functions we obtain

$$
\begin{aligned}
|\mathfrak{R}_k^{q+1} g_k(t) - \mathfrak{R}_k^{q+1} \bar{g}_k(t)| &= |\mathfrak{R} y_k(t) - \mathfrak{R} \bar{y}_k(t)| \\
&\leq \frac{U_1^{q+1}(t-a)^{\alpha_k(q+1)}}{\Gamma(\alpha_k)\Gamma(1+q\alpha_k)} d_1^{\Phi}(G,\bar{G}) \int_0^1 (1-z)^{\alpha_k-1} z^{q\alpha_k} \mathrm{d}z \\
&\leq \frac{\Gamma(\alpha_k)\Gamma(1+q\alpha_k)(U_1(t-a)^{\alpha_k})^{q+1}}{\Gamma(\alpha_k)\Gamma(1+q\alpha_k)\Gamma(1+(q+1)\alpha_k)} d_1^{\Phi}(G,\bar{G}) \\
&\leq \frac{((t-a)^{\alpha_k} U_1)^{q+1}}{\Gamma(1+(q+1)\alpha_k)} d_1^{\Phi}(G,\bar{G})
\end{aligned} \tag{11}
$$

and hence by mathematical induction we have proved that (8) holds for each $q \in \mathbb{N}$, $k \in \langle n \rangle$ and every $t \in [a, a+1]$.

For $q \in \langle q_0 \rangle$ from (8) it follows that

$$
d_1^{\Phi}(\mathfrak{R}^q G, \mathfrak{R}^q \bar{G}) \leq \frac{nU_1^q}{\Gamma(1+q\alpha_q)} d_1^{\Phi}(G,\bar{G}).
$$

For all $q > q_0$ from (8) it follows that

$$
d_1^{\Phi}(\mathfrak{R}^q G, \mathfrak{R}^q \bar{G}) \leq \frac{nU_1^q}{\Gamma(1+q\alpha_m)} d_1^{\Phi}(G,\bar{G}).
$$

Let for $q \in \langle q_0 \rangle$ denote $\gamma_q = \frac{nU_1^q}{\Gamma(1+q\alpha_q)}$ and for every $q > q_0$ denote $\gamma_q = \frac{nU_1^q}{\Gamma(1+q\alpha_m)}$. Consider the one parameter Mittag-Leffler function

$$
E_{\alpha_m,1} = \sum_{q=1}^{\infty} \frac{z^q}{\Gamma(1+\alpha_m q)}, z \in \bar{\mathbb{R}}_+.
$$

It is simply to be seen that the series $\sum\limits_{q=1}^{\infty} \frac{U_1^q}{\Gamma(1+\alpha_m q)}$ is convergent because it is the considered Mittag-Leffler function evaluated at $z = U_1$. Then we have

$$
\sum_{q=1}^{\infty} \gamma_q = \sum_{q=1}^{q_0} \frac{nU_1^q}{\Gamma(1+\alpha_q q)} + n \sum_{q=q_0+1}^{\infty} \frac{U_1^q}{\Gamma(1+\alpha_m q)} < \infty
$$

and from Theorem 1 it follows that the IP (4), (3) has a unique solution in $[a, a+1]$. □

Theorem 3. *Let the following conditions be fulfilled:*

1. *Conditions (S) hold.*
2. *The function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded.*

Then for every initial function $\Phi \in \mathfrak{C}_a^*$ the IP (4), (3) has a unique solution in J_a.

Proof. Let $\Phi^0 \in \mathfrak{C}_a^*$ be an arbitrary initial function, $l = 1$ and denote by $X^1(t, \Phi^0)$ the unique solution of the IP (4), (3) in the interval $[a, a+1]$, existing according Theorem

2. Then we can define the function $\Phi^1(t) = (\phi_1^1(t), ..., \phi_n^1)^T : [a-\sigma, a+1] \to \mathbb{R}^n$ as follows: $\Phi^1|_{[a-\sigma,a]} = \Phi^0 \in \mathfrak{C}_\alpha^*, \Phi^1|_{[a,a+1]} = X^1(t,\Phi^0)$, where $X^1(t,\Phi^0)$ is the unique solution of the IP (4), (3) in the interval $[a, a+1]$ with the initial condition $X^1(t,\Phi^0) = \Phi^0(t)$ for $t \in [a-\sigma, a]$. By induction if the solution $X^l(t,\Phi^{l-1})$ exists, we can define for this $l \in \mathbb{N}$ the function $\Phi^l(t) = (\phi_1^l(t), \dots, \phi_n^l)^T : [a-\sigma, a+l] \to \mathbb{R}^n$ (which will be used as initial function in the next step) with $\Phi^l|_{[a-\sigma,a+(l-1)]} = \Phi^{l-1}, \Phi^l|_{[a+(l-1),a+l]} = X^l(t,\Phi^{l-1})$, where $X^l(t,\Phi^{l-1})$ is the unique solution of the IP (4), (3) in the interval $[a+(l-1), a+l]$ with the initial condition $X^l(t,\Phi^{l-1}) = \Phi^{l-1}(t)$ for $t \in [a-\sigma, a+(l-1)]$.

For the proof of the statement we will use the mathematical induction. Let $\Phi^0 \in \mathfrak{C}_a^*$ be an arbitrary initial function.

Assume that for some $l \in \mathbb{N}$ the statement holds, i.e. there exists $X^l(t,\Phi^{l-1})$ as the unique solution of the IP (4), (3) in the interval $[a+(l-1), a+l]$ under the initial condition $X^l(t,\Phi^{l-1}) = \Phi^{l-1}(t)$ for $t \in [a-\sigma, a+(l-1)]$. Note that according Theorem 2 our assumption is true for $l = 1$. Moreover our assumption allows us to define the next initial function $\Phi^l(t)$ with $\Phi^l|_{[a-\sigma,a+(l-1)]} = \Phi^{l-1}, \Phi^l|_{[a+(l-1),a+l]} = X^l(t,\Phi^{l-1})$.

Define the sets

$$\Omega_{l+1}^{\Phi^l} = \{G : [a-\sigma, a+(l+1)] \to \mathbb{R}^n \mid G|_{[a+l,a+(l+1)]} \in C([a+l, a+(l+1)], \mathbb{R}^n); \\ G(t) = \Phi^l(t), t \in [a-\sigma, a+l]\}$$

with a metric function $d_{l+1}^{\Phi^l} : \Omega_{l+1}^{\Phi^l} \times \Omega_{l+1}^{\Phi^l} \to \bar{R_+}$ for each $G, \bar{G} \in \Omega_{l+1}^{\Phi^l}$ as follows:

$$d_{l+1}^{\Phi^l}(G,\bar{G}) = \sum_{k=1}^{n} \sup_{t\in[a+l,a+(l+1)]} |g_k(t) - \bar{g}_k(t)|.$$

Obviously the set $\Omega_{l+1}^{\Phi^l}$ equipped with $d_{l+1}^{\Phi^l}$ is a complete metric space in respect to the introduced metric function.

Define the operator $(\tilde{\mathfrak{R}}G)(t) = (\tilde{\mathfrak{R}}_1 g_1(t), ..., \tilde{\mathfrak{R}}_n g_n(t))^T$ for $G = (g_1, ..., g_n)^T \in \Omega_{l+1}^{\Phi^l}$ and $k \in \langle n \rangle$ with

$$\begin{aligned} \tilde{\mathfrak{R}}_k g_k(t) &= \phi_k^l(a) \\ &+ \frac{1}{\Gamma(\alpha_k)}[\int_a^t (t-s)^{\alpha_k-1}[\sum_{i=0}^{m}(\sum_{j=1}^{n}\int_{-\sigma}^{0} g_j(s+\theta)\mathrm{d}_\theta u_{kj}^i(s,\theta))]\mathrm{d}s \\ &+ \int_a^t (t-s)^{\alpha_k-1} f_k(s)\mathrm{d}s] \end{aligned} \tag{12}$$

for $t \in (a+l, a+(l+1)]$ and with $\tilde{\mathfrak{R}}_k g_k(t) = \phi_k^l(t)$ for $t \in [a-\sigma, a+l]$.

Taking into account that $\lim_{t\to(a+l)+} \tilde{\mathfrak{R}}_k g_k(t) = \phi_k^l(a+l)$ as in the proof of Theorem 2 we can conclude that $\tilde{\mathfrak{R}}_k g_k(t) \in C([a, a+(l+1)], \mathbb{R}^n)$ for $k \in \langle n \rangle$ and hence $\tilde{\mathfrak{R}}\Omega_{l+1}^{\Phi^l} \subset \Omega_{l+1}^{\Phi^l}$, i.e. the operator $\tilde{\mathfrak{R}}$ maps $\Omega_{l+1}^{\Phi^l}$ into $\Omega_{l+1}^{\Phi^l}$.

For arbitrary $G, \bar{G} \in \Omega_{l+1}^{\Phi^l}$ from (12) for every $t \in [a+l, a+(l+1)]$ and $k \in \langle n \rangle$ the same way as in Theorem 2 we have the estimation

$$\begin{aligned}
&|\tilde{\mathfrak{R}}_k g_k(t) - \tilde{\mathfrak{R}}_k \bar{g}_k(t)| \\
&\le \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} [\sum_{i=0}^{m} (\sum_{j=1}^{n} | \int_{-\sigma}^{0} (g_j(s+\theta) - \bar{g}_j(s+\theta)) \mathrm{d}_\theta u_{kj}^i(s,\theta)|)] \mathrm{d}s \\
&\le \frac{(t-(a+l))^{\alpha_k} U_{l+1}}{\Gamma(1+\alpha_k)} d_{l+1}^{\Phi^l}(G, \bar{G})
\end{aligned} \tag{13}$$

where $U_{l+1} = \sum\limits_{i=0}^{m} \sum\limits_{r,j=1}^{n} \sup\limits_{s \in [a, a+(l+1)]} Var_{\theta \in [-\sigma, 0]} u_{r,j}^i(s,\theta)$.

As in Theorem 2 we assume that for $k \in \langle n \rangle$ and every $t \in [a+l, a+(l+1)]$ the inequality

$$|\tilde{\mathfrak{R}}_k g_k(t) - \tilde{\mathfrak{R}}_k \bar{g}_k(t)| \le \frac{((t-(a+l))^{\alpha_k} U_{l+1})^q}{\Gamma(1+q\alpha_k)} d_{l+1}^{\Phi^l}(G, \bar{G}) \tag{14}$$

holds for some $q \in \mathbb{N}$. Note that the inequality (13) implies that (14) holds for every $t \in [a+l, a+(l+1)]$, $k \in \langle n \rangle$ and $q = 1$. Then using the same notations as in Theorem 2 we have that

$$|\tilde{\mathfrak{R}}_k^{q+1} g_k(t) - \tilde{\mathfrak{R}}_k^{q+1} \bar{g}_k(t)| = |\tilde{\mathfrak{R}} \tilde{\mathfrak{R}}_k^q g_k(t) - \tilde{\mathfrak{R}} \tilde{\mathfrak{R}}_k^q \bar{g}_k(t)| = |\tilde{\mathfrak{R}} y_k(t) - \tilde{\mathfrak{R}} \bar{y}_k(t)|. \tag{15}$$

In the next calculations, using the induction hypothesis (14) we obtain

$$\begin{aligned}
&|\tilde{\mathfrak{R}}_k^{q+1} g_k(t) - \tilde{\mathfrak{R}}_k^{q+1} \bar{g}_k(t)| = |\tilde{\mathfrak{R}} y_k(t) - \tilde{\mathfrak{R}} \bar{y}_k(t)| \\
&\le \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} [\sum_{i=0}^{m} (\sum_{j=1}^{n} | \int_{-\sigma}^{0} (y_j(s+\theta) - \bar{y}_j(s+\theta)) \mathrm{d}_\theta u_{kj}^i(s,\theta)|)] \mathrm{d}s \\
&\le \frac{U_{l+1}}{\Gamma(\alpha_k)} \int_{a+l}^t (t-s)^{\alpha_k - 1} \sum_{j=1}^{n} \sup_{\eta \in [a+l, a+s]} |(y_j(\eta) - \bar{y}_j(\eta))| \mathrm{d}s \\
&\le \frac{U_{l+1} U_{l+1}^q}{\Gamma(\alpha_k)\Gamma(1+q\alpha_k)} d_{l+1}^{\Phi^l}(G, \bar{G}) \int_{a+l}^t (t-s)^{\alpha_k - 1} (s-(a+l))^{q\alpha_k} \mathrm{d}s
\end{aligned} \tag{16}$$

Substitute $s - (a+l) = z(t-(a+l))$ in the right side of (16) and using the relation between the gamma and beta functions we obtain

$$
\begin{aligned}
&|\tilde{\mathfrak{R}}_k^{q+1} g_k(t) - \tilde{\mathfrak{R}}_k^{q+1} \bar{g}_k(t)| = |\tilde{\mathfrak{R}} y_k(t) - \tilde{\mathfrak{R}} \bar{y}_k(t)| \\
&\le \frac{U_{l+1}^{q+1} (t-(a+l))^{\alpha_k(q+1)}}{\Gamma(\alpha_k)\Gamma(1+q\alpha_k)} d_{l+1}^{\Phi^l}(G,\bar{G}) \int\limits_0^1 (1-z)^{\alpha_k-1} z^{q\alpha_k} \mathrm{d}z \\
&\le \frac{\Gamma(\alpha_k)\Gamma(1+q\alpha_k) U_{l+1}^{q+1} (t-(a+l))^{\alpha_k(q+1)}}{\Gamma(\alpha_k)\Gamma(1+q\alpha_k)\Gamma(1+(q+1)\alpha_k)} d_{l+1}^{\Phi^l}(G,\bar{G}) \\
&\le \frac{U_{l+1}^{q+1} (t-(a+l))^{\alpha_k(q+1)}}{\Gamma(1+(q+1)\alpha_k)} d_{l+1}^{\Phi^l}(G,\bar{G})
\end{aligned} \tag{17}
$$

and hence by mathematical induction we have proved that (14) holds for each $q \in \mathbb{N}$, $k \in \langle n \rangle$ and every $t \in [a+l, a+(l+1)]$.

For $q \in \langle q_0 \rangle$ from (14) it follows that

$$d_{l+1}^{\Phi^l}(\mathfrak{R}^q G, \mathfrak{R}^q \bar{G}) \le \frac{nU_{l+1}^q}{\Gamma(1+q\alpha_q)} d_{l+1}^{\Phi^l}(G,\bar{G}).$$

For all $q > q_0$ from (14) it follows that

$$d_{l+1}^{\Phi^l}(\mathfrak{R}^q G, \mathfrak{R}^q \bar{G}) \le \frac{nU_{l+1}^q}{\Gamma(1+q\alpha_m)} d_{l+1}^{\Phi^l}(G,\bar{G}).$$

Let for $q \in \langle q_0 \rangle$ denote $\gamma_q = \frac{nU_{l+1}^q}{\Gamma(1+q\alpha_q)}$ and for every $q > q_0$ denote $\gamma_q = \frac{nU_{l+1}^q}{\Gamma(1+q\alpha_m)}$. Consider the one parameter Mittag-Leffler function

$$E_{\alpha_m,1} = \sum_{q=1}^{\infty} \frac{z^q}{\Gamma(1+\alpha_m q)}, z \in \bar{\mathbb{R}}_+.$$

It is simply to be seen that the series $\sum\limits_{q=1}^{\infty} \frac{U_{l+1}^q}{\Gamma(1+\alpha_m q)}$ is convergent because it is the considered Mittag-Leffler function evaluated at $z = U_{l+1}$. Then we have

$$\sum_{q=1}^{\infty} \gamma_q = \sum_{q=1}^{q_0} \frac{nU_{l+1}^q}{\Gamma(1+\alpha_q q)} + n \sum_{q=q_0+1}^{\infty} \frac{U_{l+1}^q}{\Gamma(1+\alpha_m q)} < \infty$$

and from Theorem 1 it follows that the IP (4), (3) has a unique solution in $[a+l, a+l+1]$.

Thus by mathematical induction (in respect to l) we have proved that the IP (4), (3) has a unique solution in J_a. □

Corollary 1. *Let the following conditions be fulfilled:*

1. Conditions (S) hold.

2. The function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded.

Then for every initial function $\Phi \in \mathfrak{C}_a^*$ the IP (1), (3) has a unique solution in J_a.

Proof. The statement follows immediately from Theorem 3 and Lemma 1. □

4. ACKNOWLEDGMENT

This paper has been supported by Grant FP19-FMI-002 and by the National Scientific Program "Information and Communication Technologies for a Single Digital Market in Science, Education and Security (ICTinSES)", financed by the Ministry of Education and Science.

REFERENCES

[1] S. Abbas, M. Benchohra, J. R. Graef, and J. Henderson, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018.

[2] T. Abdeljawad, J. Alzabut, The q-fractional analogue for Gronwall-type inequality, *Journal of Function Spaces and Applications*, (2013), Art. ID: 543839.

[3] A. Antonov, S. Nenov, T. Tsvetkov, Impulsive controlability of tumor growth, *Dynamic Systems and Applications*, **28**, No. 1 (2019), 93–109

[4] E. Bajlecova, *Fractional Evolution Equations in Banach Spaces*, Ph. D. Thesis, Eindhoven University of Technology, 2001.

[5] D. Boyadzhiev, H. Kiskinov, M. Veselinova, A. Zahariev, Stability analysis of linear distributed order fractional systems with distributed delays, *Fract Calc Appl Anal.* **20** 4, (2017) 914–935.

[6] D. Boyadzhiev, H. Kiskinov, A. Zahariev, Integral representation of solutions of fractional system with distributed delays, *Integral Transforms and Special Functions*, **29**, No. 8, (2018), doi:10.1080/10652469.2018.1497025.

[7] K. Diethelm, *The Analysis of Fractional Differential Equations, an Application-oriented Exposition Using Differential Operators of Caputo Type*, Springer-Verlag, (Lecture Notes in Mathematics; vol. 2004) Berlin, 2010.

[8] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, *Journal of Mathematical Analysis and Applications*, **265** (2002), 229–248.

[9] M. Feckan, J. Wang, M. Pospisil, *Fractional-order Equations and Inclusions*, Walter De Gruyter, Berlin, 2017.

[10] A. Golev, M. Milev, Integral representation of the solution of the Cauchy problem for autonomous linear neutral fractional system, *International Journal of Pure and Applied Mathematics*, **119**, No. 1 (2018), 235–247.

[11] Z. Jiao, Y.Q. Chen, Y. Podlubny, *Distributed-order Dynamic Systems: Stability, Simulation, Applications and Perspectives*, Springer-Verlag (Springer Brief) Berlin, 2012.

[12] Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam, 2006.

[13] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman Sci. Techn., Harlow, & John Wiley and Sons, New York, 1994.

[14] H. Kiskinov, A. Zahariev, On fractional systems with Riemann-Liouville derivatives and distributed delays - choice of initial conditions, existence and uniqueness of the solutions, *Eur. Phys. J. Special Topics*, **226** (2017), 3473–3487.

[15] K. Krol, Asymptotic properties of fractional delay differential equations, *Applied Mathematics and Computation*, **218** (2011), 1515–1532.

[16] N. Kyurkchiev, A. Iliev, A. Rahnev, A new class of activation functions based on the correcting amendments of Gompertz-Makeham type, *Dynamic Systems and Applications*, **28**, No. 2 (2019), 243–257.

[17] S. Markov, A. Iliev, A. Rahnev, N. Kyurkchiev, A note on the Log-logistic and transmuted Log-logistic models. Some applications, *Dynamic Systems and Applications*, **27**, No. 3 (2018), 593–607.

[18] S. Markov, A. Iliev, A. Rahnev, N. Kyurkchiev, A Note On the Three-stage Growth Model, *Dynamic Systems and Applications*, **28**, No. 1 (2019), 63–72.

[19] S. Markov, N. Kyurkchiev, A. Iliev, A. Rahnev, On the approximation of the generalized cut functions of degree p+1 by smooth hyper-log-logistic function, *Dynamic Systems and Applications*, **27**, No. 4 (2018), 715–728.

[20] Mohammed M. Matar, Approximate controllability of fractional nonlinear hybrid differential systems via resolvent operators, *Journal of Mathematics*, **2019** (2019), Article ID 8603878, 7 pages.

[21] M. Milev, S. Zlatev, A note about stability of fractional retarded linear systems with distributed delays, *International Journal of Pure and Applied Mathematics*, **115**, 4 (2017), 873–881.

[22] A. Myshkis, *Linear Differential Equations with Retarded Argument*, Nauka, Moscow, 1972. (in Russian).

[23] S. Nenov, A. Antonov, T. Tsvetkov, Impulsive models: maximum yield of some biological systems, *Dynamic Systems and Applications*, **28**, No. 2 (2019), 317–328.

[24] J. Nieto, G. Stamov, I. Stamova, A fractional-order impulsive delay model of price fluctuations in commodity markets: almost periodic solutions, *Eur. Phys. J. Spec. Top.* **226**, Issue 16-18, (2017), 3811-3825.

[25] N. Pavlov, A. Iliev, A. Rahnev, N. Kyurkchiev, On Some Nonstandard Software Reliability Models, *Dynamic Systems and Applications*, **27**, No. 4 (2018), 757-771.

[26] Y. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[27] S. Saker, J. Alzabut, Periodic solutions, global attractivity and oscillation of an impulsive delay host-macroparasite model, *Mathematical and Computer Modelling*, **45**, (2007) no. 5-6, 531-543.

[28] I. Stamova, G. Stamov, *Functional and Impulsive Differential Equations of Fractional Order, Qualitative Analysis and Applications*, CRC Press, Boca Raton FL, 2017.

[29] I. Stamova, J. Alzabut, G. Stamov, Fractional dynamical systems: recent trends in theory and applications, *Eur. Phys. J. Special Topics*, **226**, (2017), 3327–3331.

[30] I. Stamova, G. Stamov, J. Alzabut, Global exponential stability for a class of impulsive BAM neural networks with distributed delays *Applied Mathematics and Information Sciences*, **7** (2013), no. 4, 1539-1546.

[31] M. Veselinova, H. Kiskinov, A. Zahariev, Stability analysis of linear fractional differential system with distributed delays, *AIP Conference Proceedings*, **1690** (2015), Article ID 040013.

[32] M. Veselinova, H. Kiskinov, A. Zahariev, Stability analysis of neutral linear fractional system with distributed delays, *Filomat*, **30** No. 3 (2016)m 841–851.

[33] M. Veselinova, H. Kiskinov, A. Zahariev, Explicit conditions for stability of neutral linear fractional system with distributed delays, *AIP Conference Proceedings*, **1789** (2016), Article ID 040005.

[34] M. Veselinova, H. Kiskinov, A. Zahariev, About stability conditions for retarded fractional differential systems with distributed delays, *Communications in Applied Analysis*, **20** (2016), 325–334.

[35] Yiheng Wei, Da-Yan Liu, Peter W. Tse, Yong Wang, On the Leibniz rule and Laplace transform for fractional derivatives, *arXiv:1901.11138v1 [math.GM]*, (2019).

[36] J. Weissinger, Zur Theorie und Anwendung des Iterationsverfahrens, *Math. Nachr.*, **8** (1952), 193–212.

[37] A. Zahariev, H. Kiskinov, Existence of fundamental matrix for neutral linear fractional system with distributed delays, *Int. J. Pure Appl. Math.*, **119**, No. 1 (2018), 31–51.

[38] Fengrong Zhang, Deliang Qian, Changpin Li, Finite-time stability analysis of fractional differential systems with variable coefficients. *Chaos*, **29** (2019), no. 1, 013110, 6 pp.

[39] Zhixin Zhang, Yufeng Zhang, Jia-Bao Liu, Jiang Wei, Global asymptotical stability analysis for fractional neural networks with time-varying delays, *Mathematics*, **7** (2019), No. 2, Article ID 138.