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THE DIRECT ELECTROMAGNETIC SCATTERING PROBLEM FOR A CRACK BURIED IN A PIECEWISE HOMOGENOUS MEDIUM

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ABSTRACT: In this paper, we consider the direct scattering problem of time-harmonic electromagnetic plane waves for a crack buried in a piecewise homogeneous medium. We solve a scattering problem for the Maxwell equations, where the scattering object is a crack Γ which is buried inside the bounded domain Ω in $\mathbb{R}^3$, and we have impedance type boundary condition on Γ and transmission boundary condition on the boundary of Ω. The problem can be reformulated as a boundary integral system. By employing the integral equation method and then used, in conjunction with the representation in a combination of layer potentials of the solution, we establish the well-posedness of the solution to the direct problem.

AMS Subject Classification: 35Q60, 35Q61

Key Words: electromagnetic scattering, crack, piecewise homogeneous medium, uniqueness, existence

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1. INTRODUCTION

In this paper, we consider the scattering problem of time-harmonic electromagnetic plane waves for a crack Γ buried in a piecewise homogeneous medium Ω in $\mathbb{R}^3$. This model corresponds to applications in biomedical imaging, non-destructive testing and geophysical explorations. In biomedical application, for example, a curved shrapnel in an arm can be modelled in terms of a crack, whereas the muscular structure over it can be considered as a homogeneous medium layer. The boundary conditions on Γ can be understood as that the boundary or more generally a portion of the boundary is coated with an unknown material with surface impedance λ in order to avoid detection.

Assume that Ω is a bounded and closed subset of $\mathbb{R}^3$ with C^2 boundary $\partial\Omega$ (we denote it by S_0, i.e., $\partial\Omega = S_0$). Let Ω_0 to denote the exterior region of Ω, that is $\Omega_0 = \mathbb{R}^3 \backslash \overline{\Omega}$. We suppose that the surface patch (or crack) Γ is buried inside Ω. In this paper, we also assume that the surface Γ can be extended to an arbitrary piecewise smooth, simply connected, closed surface $\partial\Omega_2$ (for simplicity, we denote it by S_1), enclosing a bounded domain Ω_2 such that Ω_2 is completely buried in the domain Ω. The normal vector ν on Γ coincides with the normal vector on S_1 that we again denote by ν, and ν is defined almost everywhere on Γ (except a finite number of points on Γ). We used Ω_1 to define the domain $\Omega \setminus \overline{\Omega}_2$. The electromagnetic properties of the homogeneous medium in Ω_0 are described by space-independent electric permittivity $\epsilon_0 > 0$, magnetic permeability $\mu_0 > 0$ and vanishing electric conductivity $\sigma_0 = 0$. The electromagnetic properties of the homogeneous medium in Ω are determined by space-independent electric permittivity $\epsilon_1 > 0$, magnetic permeability $\mu_1 > 0$ and electric conductivity $\sigma_1 \geq 0$.

We now give a brief description of the direct and inverse scattering problems.

The propagation of a time-harmonic electromagnetic wave (with the time variation of the form $e^{-\mathrm{i}\omega t}$, where $\omega > 0$ is the fixed frequency) in a piecewise homogeneous, is modelled by the time-harmonic Maxwell equations:

$$\mathrm{curl} E - \mathrm{i}k_0 H = 0, \qquad \mathrm{curl} H + \mathrm{i}k_0 E = 0 \quad \text{in} \quad \Omega_0, \tag{1}$$

$$\mathrm{curl} F - \mathrm{i}k_1 G = 0, \qquad \mathrm{curl} G + \mathrm{i}k_1 F = 0 \quad \text{in} \quad \Omega \setminus \overline{\Gamma}, \tag{2}$$

where k_j is the wave number in the corresponding region (Ω_0 or Ω) and satisfies the relation $k_j^2 = (\epsilon_j + \mathrm{i}\sigma_j/\omega)\mu_j w^2$ with $\mathrm{Re}k_j > 0$, $\mathrm{Im}k_j \geq 0$ $(j = 0, 1)$.

On the surface S_0, we have the transmission conditions

$$\nu \times E = \lambda_E \nu \times F, \qquad \nu \times H = \lambda_H \nu \times G \quad \text{on} \quad S_0, \tag{3}$$

where ν is the unit outward normal to the surface S_0. The constants λ_E and λ_H are given by $\lambda_E = \sqrt{\epsilon_0/(\epsilon_1 + \mathrm{i}\sigma_1/\omega)}$, $\lambda_H = \sqrt{\mu_0/\mu_1}$. On the surface S_0, the so-called transmission conditions (3) are imposed, which imply the continuity of the medium and equilibrium of the forces acting on it.

On the surface Γ, we have impedance type boundary conditions

$$\nu \times F_+ = 0, \qquad \nu \times G_- = \frac{\lambda}{k_1}(\nu \times F_-) \times \nu \quad \text{on} \quad \Gamma, \tag{4}$$

with a negative impedance constant λ. Similarly, ν is the unit outward normal to the surface Γ. F_+, F_- denote the limit of F on the surface Γ from the exterior (interior) of Γ, G_- denotes the limit of G on the surface Γ from the interior of Γ. That is

$$F_\pm(x) := \lim_{h \to 0^+} F(x \pm h\nu(x)),$$

$$G_-(x) := \lim_{h\to 0^+} G(x - h\nu(x))$$

for $x \in \Gamma$.

In the region Ω_0, the total field E, H must satisfy (1), (3) and satisfy the Silver-Müller radiation condition

$$\lim_{|x|\to\infty} (H \times x - |x|E) = 0 \tag{5}$$

uniformly with respect to all directions.

In the next section, an integral equation method is employed to establish the well-posedness of the direct problem and consider a general mixed boundary value problem. The radiation condition (5) ensures uniqueness of solutions to the exterior boundary value problem and leads to an asymptotic behaviour of the form

$$E(x) = \frac{e^{ik_0|x|}}{|x|}\left\{E^\infty(\widehat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad \text{as} \quad |x| \to \infty$$

uniformly in all directions $\widehat{x} = x/|x|$, where the vector field E^∞ is known as the electric far field pattern.

The inverse problem, we usually consider is, given the wave number $k_j (j = 0, 1)$, the constants λ_E, λ_H and the electric far field pattern $E^\infty(\widehat{x}, d, q)$ for all observation direction $\widehat{x}$, all incident directions d and all polarizations q, to determine the shape of the arc Γ, or the shape of Ω, or both Γ and Ω. As usual in most of the inverse problems, the first question to ask is the identifiability, i.e. whether an obstacle can be identified from knowledge of its far field pattern. Mathematically, the identifiability is the uniqueness issue which is of theoretical interest and is required in order to proceed to efficient numerical methods of solutions. However, studying an inverse problem always requires a solid knowledge of the corresponding direct problem. So in this paper we focus on direct scattering problem (1)-(5) in some Sobolev spaces, that is, to establish the existence and uniqueness of the solution by using the integral equation method.

In 1995, Kress (see [13]) considered the direct and inverse scattering problem for a perfectly conducting crack, and integral equation method was used to solve both the direct and the inverse problems for a sound-soft crack. The scattering problem in the unbounded domain is, thus, converted into a boundary integral equation. In 1997, Mönch (see [18]) extended this approach to a Neumann crack (sound-hard crack). In 2000, Kirsch and Ritter (see [12]) developed a linear sampling method for the inverse scattering of time-harmonic plane waves by open arcs. They derived a characterization of the scatterer in terms of the spectral data of the scattering matrix analogously to the case of the scattering by bounded open domains. In the same year, Ammari, Bao, and Wood considered the scattering problem with cracks for Maxwell equations (see [2]). In that paper, integral representations of the solutions to the model problem in both fundamental polarizations were derived and studied. Existence and uniqueness of the solutions for the integral equations were established. In 2003, Cakoni and Colton (see [3]) considered the inverse scattering problem of determining the shape of an infinite cylinder having an open arc as cross section. They assumed that the arc is a (possibly) partially coated perfect conductor and developed the linear sampling method. Extending to the impedance problem, Kuo-Ming Lee (see [16]) considered the direct and inverse scattering problem for an impedance crack in 2008. In 2009, Kress, Yaman, Yapar, and Akduman (see [14]) considered the scattering problem for an impedance cylinder buried in a dielectric cylinder, that is, an inpenetrable obstacle with impedance boundary condition buried in a homogeneous medium. They used a boundary integral equation approach for the corresponding direct and inverse problem. The boundary value problem for the Helmholtz equation is studied outside arcs (or slits) in a plane by Krutitskii (see [15]) in 2009, and the impedance boundary conditions are specified on the slits. In 2012, Yan and Ye (see [22]) considered the scattering of time-harmonic acoustic plane waves by a crack buried in a piecewise homogeneous medium. They solved a problem for the Helmholz equation and obtained the existence and uniqueness of the solution. For more related works, we refer to the monographs (see [4], [6], [7], [8], [11], [19], [20], [21]) and the reference therein.

As far as we all know, our result in this article is the first uniqueness result about the electromagnetic scattering problem by a crack buried in a piecewise homogeneous medium. We suppose that the scatterer is a surface patch Γ which is buried in a piecewise homogeneous medium. In other words, the interface S_0 is penetrable, and the surface Γ is buried in the domain Ω. For more knowledge of electromagnetic scattering problem, we can refer to the monographs (see [1], [5], [9], [10], [17]) and the reference therein. The aim of the present paper is to obtain the existence and uniqueness of the solution to problem (1)-(5) in some Sobolev spaces.

The outline of the paper is as follows. In Section 2, we will introduce the direct scattering problem, establish uniqueness to the problem. In Section 3, we will show the main results on the existence and uniqueness of the solution to the boundary integral system by using the integral equation method.

2. UNIQUENESS

In this section, we first show that the direct electromagnetic scattering problem has a unique solution.

2.1. PRELIMINARIES

In order to formulate our scattering problem more precisely, we need introduce some related spaces.

Let D be a bounded domain and ∂D be the boundary of D. We denote by $C_t(\partial D)$ and $C_t^{0,\alpha}(\partial D)$, $0 < \alpha < 1$, the spaces of all continuous and uniformly Hölder continuous tangential fields equipped with the supremum norm and the Hölder norm, respectively. Then we introduce the normed spaces of tangential fields possessing a surface divergence (see section 6.3 in [7]) by

$$C(\mathrm{Div}, \partial D) := \{a \in C_t(\partial D) : \mathrm{Div} a \in C(\partial D)\}$$

and

$$C^{0,\alpha}(\mathrm{Div}, \partial D) := \{a \in C_t^{0,\alpha}(\partial D) : \mathrm{Div} a \in C^{0,\alpha}(\partial D)\}$$

equipped with the norms

$$\|a\|_{C(\mathrm{Div},\partial D)} := \|a\|_{\infty,\partial D} + \|\mathrm{Div} a\|_{\infty,\partial D},$$

$$\|a\|_{C^{0,\alpha}(\mathrm{Div},\partial D)} := \|a\|_{\alpha,\partial D} + \|\mathrm{Div} a\|_{\alpha,\partial D}.$$

Now we consider the following mixed boundary value problem for the Maxwell equations: given four tangential fields $T_1, T_2 \in C^{0,\alpha}(\mathrm{Div}, S_0)$, $T_3 \in C^{0,\alpha}(\mathrm{Div}, \Gamma)$ and $T_4 \in C_t^{0,\alpha}(\Gamma)$, the direct problem consists in finding a solution $E, H \in C^1(\Omega_0) \cap C(\overline{\Omega}_0)$, $F, G \in C^1(\Omega \setminus \overline{\Gamma}) \cap C(\overline{\Omega \setminus \overline{\Gamma}})$ such that

$$\begin{cases} \mathrm{curl} E - \mathrm{i}k_0 H = 0, \qquad \mathrm{curl} H + \mathrm{i}k_0 E = 0 & \text{in} \quad \Omega_0, \\ \mathrm{curl} F - \mathrm{i}k_1 G = 0, \qquad \mathrm{curl} G + \mathrm{i}k_1 F = 0 & \text{in} \quad \Omega \setminus \overline{\Gamma}, \\ \nu \times E - \lambda_E \nu \times F = T_1 & \text{on} \quad S_0, \\ \nu \times H - \lambda_H \nu \times G = T_2 & \text{on} \quad S_0, \\ \nu \times F_+ = T_3 & \text{on} \quad \Gamma, \\ \nu \times G_- - \frac{\lambda}{k_1}(\nu \times F_-) \times \nu = T_4 & \text{on} \quad \Gamma, \\ \lim_{|x| \to \infty} (H \times x - |x| E) = 0. \end{cases} \tag{6}$$

Here, the definition of k_0, k_1, λ_E, λ_H, λ is mentioned above.

Next, we will show the main theorem in this section.

Theorem 1. *The boundary value problem* (6) *admits at most one solution.*

2.2. THE PROOF OF THEOREM ??

Clearly, it is enough to show that E, H, F, G vanishes identically corresponding homogeneous boundary value problem (6), that is, $E = H = 0$ in Ω_0, $F = G = 0$ in $\Omega \setminus \overline{\Gamma}$ if $T_1 = T_2 = 0$ on S_0, $T_3 = T_4 = 0$ on Γ.

Proof. From Gauss' divergence theorem and the Maxwell equations (2), we have that

$$\begin{aligned}&\int_{S_0} \nu \times F \cdot \overline{G}\mathrm{d}s - \int_{S_1} \nu \times F_+ \cdot \overline{G}_+\mathrm{d}s \\ =&\int_{\Omega_1} (\mathrm{curl}F \cdot \overline{G} - F \cdot \mathrm{curl}\overline{G})\mathrm{d}x = \mathrm{i}\int_{\Omega_1} (k_1|G|^2 - \overline{k_1}|F|^2)\mathrm{d}x\end{aligned} \tag{7}$$

and

$$\begin{aligned}&\int_{S_1} \nu \times F_- \cdot \overline{G}_-\mathrm{d}s \\ =&\int_{\Omega_2} (\mathrm{curl}F \cdot \overline{G} - F \cdot \mathrm{curl}\overline{G})\mathrm{d}x = \mathrm{i}\int_{\Omega_2} (k_1|G|^2 - \overline{k_1}|F|^2)\mathrm{d}x.\end{aligned} \tag{8}$$

By using the boundary conditions on Γ and the sum of (7), (8), we have

$$\begin{aligned}&\int_{S_0} \nu \times F \cdot \overline{G}\mathrm{d}s \\ =&\mathrm{i}\int_{\Omega_1} (k_1|G|^2 - \overline{k_1}|F|^2)\mathrm{d}x + \mathrm{i}\int_{\Omega_2} (k_1|G|^2 - \overline{k_1}|F|^2)\mathrm{d}x \\ &+ \int_\Gamma \frac{\lambda|\nu \times F_-|^2}{\overline{k_1}}\mathrm{d}s.\end{aligned} \tag{9}$$

By the transmission boundary conditions on S_0 and (9), we also have

$$\begin{aligned}&\int_{S_0} \nu \times E \cdot \overline{H}\mathrm{d}s = \int_{S_0} \nu \times E \cdot [(\nu \times \overline{H}) \times \nu]\mathrm{d}s \\ =&\lambda_E\lambda_H \int_{S_0} \nu \times F \cdot [(\nu \times \overline{G}) \times \nu]\mathrm{d}s = \lambda_E\lambda_H \int_{S_0} \nu \times F \cdot \overline{G}\mathrm{d}s \\ =&\mathrm{i}\lambda_E\lambda_H \int_{\Omega_1} (k_1|G|^2 - \overline{k_1}|F|^2)\mathrm{d}x + \mathrm{i}\lambda_E\lambda_H \int_{\Omega_2} (k_1|G|^2 - \overline{k_1}|F|^2)\mathrm{d}x \\ &+ \frac{\lambda_E\lambda_H}{\overline{k_1}} \int_\Gamma \lambda|\nu \times F_-|^2\mathrm{d}s.\end{aligned} \tag{10}$$

Taking the real part of (10), we have, on noting that $\lambda_E\lambda_H = k_0/k_1$, $Rek_1 > 0$, $Imk_1 \geq 0$ and $\lambda < 0$ that

$$\begin{aligned}&\mathrm{Re}\int_{S_0} \nu \times E \cdot \overline{H}\mathrm{d}s \\ =&-\frac{2k_0\mathrm{Re}k_1\mathrm{Im}k_1}{|k_1|^2}\int_{\Omega_1} |F|^2\mathrm{d}x - \frac{2k_0\mathrm{Re}k_1\mathrm{Im}k_1}{|k_1|^2}\int_{\Omega_2} |F|^2\mathrm{d}x \\ &+ \frac{k_0}{|k_1|^2}\int_\Gamma \lambda|\nu \times F_-|^2\mathrm{d}s \leq 0.\end{aligned}$$

Therefore, by Rellich's lemma or Theorem 6.11 (see [7]), it follows that $E = H = 0$ in Ω_0. So we have $\nu \times E = \nu \times H = 0$, the transmission boundary conditions in (6) imply that $\nu \times F = \nu \times G = 0$. Then the Holmgren's uniqueness theorem or Theorem 6.5 (see [7]) implies that $F = G = 0$ in $\Omega \setminus \overline{\Gamma}$. So we complete the proof of the Theorem 1. $\square$

3. EXISTENCE

We are now ready to prove the existence of a solution to problem (6).

3.1. PRELIMINARIES

We first denote the fundamental solution of the Helmholtz equation with wave number $k_j (j = 0, 1)$ by Φ_j, that is,

$$\Phi_j(x, y) = \frac{e^{\mathrm{i}k_j|x-y|}}{4\pi|x-y|}, \quad x, y \in \mathbb{R}^3, \quad x \neq y.$$

Given two integrable vector fields a on S_0, b on Γ and an integral function ψ on Γ. Besides, we extend b and ψ to the whole boundary S_1 in the following:

$$\widetilde{b} = \begin{cases} 0 & \text{on} \quad S_1 \setminus \Gamma, \\ b & \text{on} \quad \Gamma, \end{cases} \qquad \widetilde{\psi} = \begin{cases} 0 & \text{on} \quad S_1 \setminus \Gamma, \\ \psi & \text{on} \quad \Gamma. \end{cases}$$

We introduce the magnetic dipole operator $M_{i,j}$, $\widetilde{M}_i$, respectively, by

$$(M_{i,j}a)(x) = 2\int_{S_0} \nu(x) \times \mathrm{curl}_x\{a(y)\Phi_j(x, y)\}\mathrm{d}s(y), \quad x \in S_i,$$

$$(\widetilde{M}_i\widetilde{b})(x) = 2\int_{S_1} \nu(x) \times \mathrm{curl}_x\{\widetilde{b}(y)\Phi_1(x, y)\}\mathrm{d}s(y), \quad x \in S_i,$$

i.e.

$$(\widetilde{M}_i b)(x) = 2\int_{\Gamma} \nu(x) \times \mathrm{curl}_x\{b(y)\Phi_1(x, y)\}\mathrm{d}s(y), \quad x \in S_i$$

for $i = 0, 1$; $j = 0, 1$. We also introduce the electric dipole operator $N_{i,j}$, $\widetilde{N}_i$, respectively, by

$$(N_{i,j}a)(x) = 2\nu(x) \times \mathrm{curlcurl}\int_{S_0} \nu(y) \times a(y)\Phi_j(x, y)\mathrm{d}s(y), \quad x \in S_i,$$

$$(\widetilde{N}_i\widetilde{b})(x) = 2\nu(x) \times \mathrm{curlcurl}\int_{S_1} \nu(y) \times \widetilde{b}(y)\Phi_1(x, y)\mathrm{d}s(y), \quad x \in S_i,$$

i.e.

$$(\widetilde{N}_i b)(x) = 2\nu(x) \times \mathrm{curlcurl}\int_{\Gamma} \nu(y) \times b(y)\Phi_1(x, y)\mathrm{d}s(y), \quad x \in S_i$$

for $i = 0, 1$; $j = 0, 1$. We also give the single-layer operator $\widetilde{S}$ and the double-layer operator $\widetilde{K}$, respectively, by

$$(\widetilde{S}\widetilde{\psi})(x) = 2\int_{S_1} \Phi_1(x, y)\widetilde{\psi}(y)\mathrm{d}s(y), \quad x \in S_1,$$

i.e.

$$(\widetilde{S}\psi)(x) = 2\int_{\Gamma} \Phi_1(x,y)\psi(y)\mathrm{d}s(y), \quad x \in S_1$$

and

$$(\widetilde{K}\widetilde{\psi})(x) = 2\int_{S_1} \frac{\partial \Phi_1(x,y)}{\partial \nu(y)}\widetilde{\psi}(y)\mathrm{d}s(y), \quad x \in S_1,$$

i.e.

$$(\widetilde{K}\psi)(x) = 2\int_{\Gamma} \frac{\partial \Phi_1(x,y)}{\partial \nu(y)}\psi(y)\mathrm{d}s(y), \quad x \in S_1.$$

Next, we will show the main theorem in this section.

Theorem 2. *The boundary value problem* (6) *has a unique solution. The solution depends continuously on the boundary data in the sense that the operator mapping the given boundary data onto the solution is continuous from* $C^{0,\alpha}(\mathrm{Div}, S_0) \times C^{0,\alpha}(\mathrm{Div}, S_0) \times C^{0,\alpha}(\mathrm{Div}, \Gamma) \times C_t^{0,\alpha}(\Gamma)$ *into* $C^{0,\alpha}(\overline{\Omega}_0) \times C^{0,\alpha}(\overline{\Omega}_0) \times C^{0,\alpha}(\overline{\Omega \setminus \overline{\Gamma}}) \times C^{0,\alpha}(\overline{\Omega \setminus \overline{\Gamma}})$.

3.2. THE PROOF OF THEOREM ??

In this section, we will prove the existence of solutions by using the integral equation method.

Proof. The uniqueness of solutions follows from Theorem 1. Now, we prove the existence of solutions. Let $a, b \in C^{0,\alpha}(\mathrm{Div}, S_0)$, $c \in C^{0,\alpha}(\mathrm{Div}, \Gamma)$, $d \in C_t^{0,\alpha}(\Gamma)$, $\psi \in C^{0,\alpha}(\Gamma)$ are five densities to be determined. Besides, we extend c, d and ψ to the whole boundary S_1 in the following:

$$\widetilde{c} = \begin{cases} 0 & \text{on} \quad S_1 \setminus \Gamma, \\ c & \text{on} \quad \Gamma, \end{cases} \qquad \widetilde{d} = \begin{cases} 0 & \text{on} \quad S_1 \setminus \Gamma, \\ d & \text{on} \quad \Gamma, \end{cases}$$

and

$$\widetilde{\psi} = \begin{cases} 0 & \text{on} \quad S_1 \setminus \Gamma, \\ \psi & \text{on} \quad \Gamma. \end{cases}$$

We will seek a solution in the form

$$\begin{aligned} E(x) =& \frac{\lambda_H k_0}{k_1}\mathrm{curl}\int_{S_0} a(y)\Phi_0(x,y)\mathrm{d}s(y) \\ &+ \lambda_E \mathrm{curlcurl}\int_{S_0} b(y)\Phi_0(x,y)\mathrm{d}s(y), \end{aligned} \tag{11}$$

$$\begin{aligned} H(x) = \frac{1}{\mathrm{i}k_0}\mathrm{curl}E(x) =& \frac{\lambda_H}{\mathrm{i}k_1}\mathrm{curlcurl}\int_{S_0} a(y)\Phi_0(x,y)\mathrm{d}s(y) \\ &+ \frac{\lambda_E k_0}{\mathrm{i}}\mathrm{curl}\int_{S_0} b(y)\Phi_0(x,y)\mathrm{d}s(y) \end{aligned} \tag{12}$$

for $x \in \Omega_0$ and

$$\begin{aligned} F(x) =& \operatorname{curl} \int_{S_0} a(y)\Phi_1(x,y)\mathrm{d}s(y) + \operatorname{curl}\operatorname{curl} \int_{S_0} b(y)\Phi_1(x,y)\mathrm{d}s(y) \\ & + \operatorname{curl} \int_{S_1} \widetilde{c}(y)\Phi_1(x,y)\mathrm{d}s(y) + \frac{\mathrm{i}}{k_1^2}\operatorname{curl}\operatorname{curl} \int_{S_1} \nu(y) \\ & \times (\widehat{S}^2\widetilde{c})(y)\Phi_1(x,y)\mathrm{d}s(y) \\ =& \operatorname{curl} \int_{S_0} a(y)\Phi_1(x,y)\mathrm{d}s(y) + \operatorname{curl}\operatorname{curl} \int_{S_0} b(y)\Phi_1(x,y)\mathrm{d}s(y) \\ & + \operatorname{curl} \int_{\Gamma} c(y)\Phi_1(x,y)\mathrm{d}s(y) + \frac{\mathrm{i}}{k_1^2}\operatorname{curl}\operatorname{curl} \int_{\Gamma} \nu(y) \\ & \times (\widehat{S}^2 c)(y)\Phi_1(x,y)\mathrm{d}s(y), \end{aligned} \tag{13}$$

$$\begin{aligned} G(x) =& \frac{1}{\mathrm{i}k_1}\operatorname{curl}F(x) = \frac{1}{\mathrm{i}k_1}\operatorname{curl}\operatorname{curl} \int_{S_0} a(y)\Phi_1(x,y)\mathrm{d}s(y) \\ & - \mathrm{i}k_1\operatorname{curl} \int_{S_0} b(y)\Phi_1(x,y)\mathrm{d}s(y) + \frac{1}{\mathrm{i}k_1}\operatorname{curl}\operatorname{curl} \int_{S_1} \widetilde{c}(y) \\ & \Phi_1(x,y)\mathrm{d}s(y) + \frac{1}{k_1}\operatorname{curl} \int_{S_1} \nu(y) \times (\widehat{S}^2\widetilde{c})(y)\Phi_1(x,y)\mathrm{d}s(y) \\ =& \frac{1}{\mathrm{i}k_1}\operatorname{curl}\operatorname{curl} \int_{S_0} a(y)\Phi_1(x,y)\mathrm{d}s(y) - \mathrm{i}k_1\operatorname{curl} \int_{S_0} b(y) \\ & \Phi_1(x,y)\mathrm{d}s(y) + \frac{1}{\mathrm{i}k_1}\operatorname{curl}\operatorname{curl} \int_{\Gamma} c(y)\Phi_1(x,y)\mathrm{d}s(y) \\ & + \frac{1}{k_1}\operatorname{curl} \int_{\Gamma} \nu(y) \times (\widehat{S}^2 c)(y)\Phi_1(x,y)\mathrm{d}s(y) \end{aligned} \tag{14}$$

for $x \in \Omega_1$ and

$$\begin{aligned} F(x) =& \int_{S_1} \widetilde{d}(y)\Phi_1(x,y)\mathrm{d}s(y) + \mathrm{i}\lambda\operatorname{curl} \int_{S_1} \nu(y) \times (\widehat{S}^2\widetilde{d})(y) \\ & \Phi_1(x,y)\mathrm{d}s(y) + \operatorname{grad} \int_{S_1} \widetilde{\psi}(y)\Phi_1(x,y)\mathrm{d}s(y) \\ & + \mathrm{i}\lambda \int_{S_1} \nu(y)\widetilde{\psi}(y)\Phi_1(x,y)\mathrm{d}s(y) \\ =& \int_{\Gamma} d(y)\Phi_1(x,y)\mathrm{d}s(y) + \mathrm{i}\lambda\operatorname{curl} \int_{\Gamma} \nu(y) \times (\widehat{S}^2 d)(y) \\ & \Phi_1(x,y)\mathrm{d}s(y) + \operatorname{grad} \int_{\Gamma} \psi(y)\Phi_1(x,y)\mathrm{d}s(y) \\ & + \mathrm{i}\lambda \int_{\Gamma} \nu(y)\psi(y)\Phi_1(x,y)\mathrm{d}s(y), \end{aligned} \tag{15}$$

$$
\begin{aligned}
G(x) =& \frac{1}{\mathrm{i}k_1}\mathrm{curl}F(x) = \frac{1}{\mathrm{i}k_1}\mathrm{curl}\int_{S_1}\widetilde{d}(y)\Phi_1(x,y)\mathrm{d}s(y)\\
&+ \frac{\lambda}{k_1}\mathrm{curlcurl}\int_{S_1}\nu(y)\times(\widehat{S}^2\widetilde{d})(y)\Phi_1(x,y)\mathrm{d}s(y)\\
&+ \frac{\lambda}{k_1}\mathrm{curl}\int_{S_1}\nu(y)\widetilde{\psi}(y)\Phi_1(x,y)\mathrm{d}s(y)\\
=& \frac{1}{\mathrm{i}k_1}\mathrm{curl}\int_{\Gamma}d(y)\Phi_1(x,y)\mathrm{d}s(y)\\
&+ \frac{\lambda}{k_1}\mathrm{curlcurl}\int_{\Gamma}\nu(y)\times(\widehat{S}^2 d)(y)\Phi_1(x,y)\mathrm{d}s(y)\\
&+ \frac{\lambda}{k_1}\mathrm{curl}\int_{\Gamma}\nu(y)\psi(y)\Phi_1(x,y)\mathrm{d}s(y)
\end{aligned}
\tag{16}
$$

for $x \in \Omega_2$, where $\widehat{S}$ is the single-layer operator given by

$$
(\widehat{S}c)(x) := \frac{1}{2\pi}\int_{\Gamma}\frac{1}{|x-z|}c(z)\mathrm{d}s(z), \qquad x \in \Gamma.
$$

The vector field F given by (13), (15) clearly satisfies the vector Helmholtz equation and its cartesian components satisfy the Sommerfeld radiation condition. Hence, if we insist that $\mathrm{div}F = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$, then by Theorems 6.4 and 6.8 (see [7]) we have that F, G satisfy the Maxwell equations (2) and the Silver-Müller radiation condition (5). Since $\mathrm{div}F$ satisfies the scalar Helmholtz equation and the Sommerfeld radiation condition, by the uniqueness for the exterior Dirichlet problem (see theorem 3.9 in [7]) and the continuity on Γ, it suffices to impose $\mathrm{div}F = 0$ only on the boundary Γ. From the jump and regularity conditions of Theorems 3.1, 3.3, 6.12 and 6.13 (see [7]), we can now conclude that E, H, F, G defined in (11)-(16) satisfy the following system of integral equations (17),

$$
\begin{cases}
\frac{\lambda_E k_1+\lambda_H k_0}{k_1}a + L_1 a + M_1 b + N_1 c = 2T_1 & \text{on} \quad S_0,\\
\frac{\lambda_E k_0+\lambda_H k_1}{\mathrm{i}}b + L_2 a + M_2 b + N_2 c = 2T_2 & \text{on} \quad S_0,\\
c + L_3 a + M_3 b + N_3 c = 2T_3 & \text{on} \quad \Gamma,\\
-d + P_1 d + Q_1\psi = 2T_4 & \text{on} \quad \Gamma,\\
\mathrm{i}\lambda\psi + P_2 d + Q_2\psi = 0 & \text{on} \quad \Gamma,
\end{cases}
\tag{17}
$$

where

$$
\begin{aligned}
&L_1 := \frac{\lambda_H k_0}{k_1}M_{0,0} - \lambda_E M_{0,1}, \quad M_1 := \lambda_E(N_{0,0} - N_{0,1})R,\\
&N_1 := -\lambda_E(\widetilde{M_0} + \frac{\mathrm{i}}{k_1^2}\widetilde{N}_0 P\widehat{S}^2), \quad L_2 := \frac{\lambda_H}{\mathrm{i}k_1}(N_{0,0} - N_{0,1})R,\\
&M_2 := \frac{\lambda_E k_0}{\mathrm{i}}M_{0,0} - \frac{\lambda_H k_1}{\mathrm{i}}M_{0,1}, \quad N_2 := -\frac{\lambda_H}{\mathrm{i}k_1}\widetilde{N}_0 R + \frac{\lambda_H}{k_1}\widetilde{M_0}R\widehat{S}^2,\\
&L_3 := M_{1,1}, \quad M_3 := N_{1,1}R, \quad N_3 := \widetilde{M_1} + \frac{\mathrm{i}}{k_1^2}\widetilde{N}_1 P\widehat{S}^2,\\
&P_1 := \widetilde{M_1} + \mathrm{i}\lambda\widetilde{N}_1 P\widehat{S}^2 - \mathrm{i}\lambda P\widetilde{S} - \lambda^2 R\widetilde{M_1}R\widehat{S}^2 + \lambda^2 P\widehat{S}^2,
\end{aligned}
$$

$$Q_1\psi := 2\mathrm{i}\lambda\nu(x) \times \int_\Gamma \mathrm{grad}_x\Phi_1(x,y) \times \{\nu(y)-\nu(x)\}\psi(y)\mathrm{d}s(y)$$
$$+ \lambda^2 P\widetilde{S}(\nu\psi),$$
$$P_2 d := -2\int_\Gamma \mathrm{grad}_x\Phi_1(x,y)\cdot d(y)\mathrm{d}s(y), \quad Q_2 := k_1^2\widetilde{S} + \mathrm{i}\lambda\widetilde{K}.$$

Here R, P are defined by $Ra := a\times\nu$ and $Pa := (\nu\times a)\times\nu$, respectively.

Writing the system of integral equations (17) in the matrix form

$$\begin{pmatrix} \lambda_a & 0 & 0 & 0 & 0 \\ 0 & \lambda_b & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & P_2 & \mathrm{i}\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ \psi \end{pmatrix} +$$

$$\begin{pmatrix} L_1 & M_1 & N_1 & 0 & 0 \\ L_2 & M_2 & N_2 & 0 & 0 \\ L_3 & M_3 & N_3 & 0 & 0 \\ 0 & 0 & 0 & P_1 & Q_1 \\ 0 & 0 & 0 & 0 & Q_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ \psi \end{pmatrix} = \begin{pmatrix} 2T_1 \\ 2T_2 \\ 2T_3 \\ 2T_4 \\ 0 \end{pmatrix},$$

where $\lambda_a = \lambda_E + \lambda_H k_0/k_1$, $\lambda_b = -\mathrm{i}\lambda_E k_0 - \mathrm{i}\lambda_H k_1$. It is easy to see that the first matrix operator has a bounded inverse because of its triangular form. A similar argument in the proof of Theorems 6.21 and Theorems 9.11 (see [7]), we can see the second matrix is compact by Theorems 6.16 - 6.19 (see [7]). Hence, we can apply the Riesz-Fredholm theory to (17).

For this purpose, suppose a, b, c, d and ψ are a solution to the homogeneous equation corresponding to (17) (i.e., $T_1 = T_2 = 0$ on S_0, $T_3 = T_4 = 0$ on Γ). We then need to show that $a = b = c = d = \psi = 0$. Firstly, by Theorem 2.1 we conclude that $E = H = 0$ in Ω_0, $F = G = 0$ in $\Omega \setminus \overline{\Gamma}$.

Now, by the jump relations we have

$$c = \nu \times F_+ = T_3 = 0 \quad \text{on} \quad \Gamma, \tag{18}$$

$$-\nu \times F_- = \mathrm{i}\lambda\nu \times \widehat{S}^2 d, \qquad -\nu\times G_- = \frac{1}{\mathrm{i}k_1}d \quad \text{on} \quad \Gamma, \tag{19}$$

$$-\mathrm{div}F_- = -\mathrm{i}\lambda\psi, \qquad -\nu\cdot F_- = -\psi \quad \text{on} \quad \Gamma. \tag{20}$$

Hence, using the Greens vector theorem, we derive from (19), (20) that

$$\mathrm{i}\lambda\int_\Gamma |\widehat{S}d|^2\mathrm{d}s + \mathrm{i}\lambda\int_\Gamma |\psi|^2\mathrm{d}s$$
$$= \int_{S_1}\{\nu\times\overline{F_-}\cdot\mathrm{curl}F + \nu\cdot\overline{F_-}\mathrm{div}F\}\mathrm{d}s$$
$$= \int_{\Omega_2}\{|\mathrm{curl}F|^2 + |\mathrm{div}F|^2 - [(\mathrm{Re}k_1)^2$$
$$- (\mathrm{Im}k_1)^2]|F|^2 - 2\mathrm{i}\mathrm{Re}k_1\mathrm{Im}k_1|F|^2\}\mathrm{d}x.$$

Taking the imaginary part of the last equation shows that $\widehat{S}d = 0$ and $\psi = 0$ on Γ. Since $\widehat{S}$ is injective (see the proof of Theorem 3.12 in [7]), we have that $d = 0$ on Γ. By (18), we can conclude $c = d = \psi = 0$ on Γ.

On the other hand, define

$$\begin{aligned}\widetilde{E}(x) =& - m\mathrm{curl}\int_{S_0} a(y)\Phi_0(x,y)\mathrm{d}s(y)\\ &- \frac{m\lambda_E k_1}{\lambda_H k_0}\mathrm{curlcurl}\int_{S_0} b(y)\Phi_0(x,y)\mathrm{d}s(y),\end{aligned} \tag{21}$$

$$\begin{aligned}\widetilde{H}(x) =&\frac{1}{\mathrm{i}k_0}\mathrm{curl}\widetilde{E}(x) = \frac{\mathrm{i}m}{k_0}\mathrm{curlcurl}\int_{S_0} a(y)\Phi_0(x,y)\mathrm{d}s(y)\\ &- \frac{m\lambda_E k_1}{\mathrm{i}\lambda_H}\mathrm{curl}\int_{S_0} b(y)\Phi_0(x,y)\mathrm{d}s(y)\end{aligned} \tag{22}$$

for $x \in \Omega$ and

$$\widetilde{F}(x) = \mathrm{curl}\int_{S_0} a(y)\Phi_0(x,y)\mathrm{d}s(y) + \mathrm{curlcurl}\int_{S_0} b(y)\Phi_0(x,y)\mathrm{d}s(y), \tag{23}$$

$$\begin{aligned}\widetilde{G}(x) =&\frac{1}{\mathrm{i}k_1}\mathrm{curl}\widetilde{F}(x) = \frac{1}{\mathrm{i}k_1}\mathrm{curlcurl}\int_{S_0} a(y)\Phi_0(x,y)\mathrm{d}s(y)\\ &- \mathrm{i}k_1\mathrm{curl}\int_{S_0} b(y)\Phi_0(x,y)\mathrm{d}s(y)\end{aligned} \tag{24}$$

for $x \in \Omega_0$, where m is a constant given by $m := \sqrt{\lambda_H/\lambda_E}$. Thus, $\widetilde{E}$, $\widetilde{H}$, $\widetilde{F}$, $\widetilde{G}$ given by (21)-(24) solve the homogeneous transmission problem

$$\begin{cases} \mathrm{curl}\widetilde{F} - \mathrm{i}k_0\widetilde{G} = 0, \qquad \mathrm{curl}\widetilde{G} + \mathrm{i}k_0\widetilde{F} = 0 & \text{in} \quad \Omega_0,\\ \mathrm{curl}\widetilde{E} - \mathrm{i}k_1\widetilde{H} = 0, \qquad \mathrm{curl}\widetilde{H} + \mathrm{i}k_1\widetilde{E} = 0 & \text{in} \quad \Omega,\\ \nu\times\widetilde{F} - \frac{1}{m}\nu\times\widetilde{E} = 0, \quad \nu\times\widetilde{G} - \frac{\lambda_H}{m\lambda_E}\nu\times\widetilde{H} = 0 & \text{on} \quad S_0,\\ \lim_{|x|\to\infty}(\widetilde{G}\times x - |x|\widetilde{F}) = 0, \end{cases} \tag{25}$$

where the limit holds uniformly in all directions $x/|x|$. Similar to the argument as in the proof of Theorem 1, we will use the Green's vector theorem and by the definition of m that

$$\begin{aligned}\int_{S_0}\nu\times\widetilde{F}\cdot\overline{\widetilde{G}}\mathrm{d}s =& \int_{S_0}\nu\times\widetilde{F}\cdot[(\nu\times\overline{\widetilde{G}})\times\nu]\mathrm{d}s\\ =& \int_{S_0}\nu\times\widetilde{E}\cdot[(\nu\times\overline{\widetilde{H}})\times\nu]\mathrm{d}s\\ =& \int_{\Omega}(\mathrm{curl}\widetilde{E}\cdot\overline{\widetilde{H}} - \widetilde{E}\cdot\mathrm{curl}\overline{\widetilde{H}})\mathrm{d}x\\ =&\mathrm{i}k_0\int_{\Omega}(|\widetilde{H}|^2 - |\widetilde{E}|^2)\mathrm{d}x.\end{aligned} \tag{26}$$

Taking the real part of (26), we have

$$\mathrm{Re}\int_{S_0} \nu \times \widetilde{F} \cdot \overline{\widetilde{G}} \mathrm{d}s \le 0.$$

Therefore, by Rellich's lemma or Theorem 6.11 (see [7]), it follows that $\widetilde{F} = \widetilde{G} = 0$ in Ω_0. The transmission boundary conditions in (25) and the Holmgren's uniqueness theorem or Theorem 6.5 (see [7]) imply that $\widetilde{E} = \widetilde{H} = 0$ in Ω.

Besides, by the jump relations we have

$$\frac{k_1}{\lambda_H k_0} \nu \times E + \frac{1}{m} \nu \times \widetilde{E} = a \quad \text{on} \quad S_0, \tag{27}$$

$$\frac{k_1}{\lambda_E k_0} \nu \times H + \frac{\lambda_H}{\lambda_E m} \nu \times \widetilde{H} = -\mathrm{i} k_1 b \quad \text{on} \quad S_0. \tag{28}$$

We have known $E = H = 0$ in Ω_0, $\widetilde{E} = \widetilde{H} = 0$ in Ω. Hence, by relations (27) and (28), we conclude that $a = b = 0$ on S_0 and we have proved $c = d = \psi = 0$. So we complete the proof of the Theorem 2. □

Remark 3. The study of direct scattering problem (6) is the basis of the corresponding inverse scattering problem. How to recover the shape of the arc Γ (or S_0) by using some numerical methods, it is my future research work.

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