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IMPULSIVE MODELS: MAXIMUM YIELD OF SOME BIOLOGICAL SYSTEMS, I

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ABSTRACT: The laboratory cultivation of two or more populations is usually carried out when we need to extract some volume or size from the populations. In this case, it is natural to set the questions for maximum yield of biomass.

In this article, we use the impulsive differential equations as an adequate tool for modelling of such processes (i.e. processes with external impulsive influences as withdrawing or adding of some quantity of biomass) and to set and investigate the maximum yield problems.

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1. STATEMENT OF THE PROBLEM

In this article, we consider the classical predator-pray model:

$$\dot{x}_1 = x_1 h_1(x_1, x_2), \tag{1}$$

$$\dot{x}_2 = x_2 h_2(x_1, x_2), \tag{2}$$

where the functions h_1 and h_2 denote the growth rates of the two populations with sizes x_1 and x_2, respectively. We will always suppose that the functions h_1 and h_2 are continuously differentiable in $\mathbb{R}^2_+ = \{(x_1, x_2) : x_1 > 0,\ x_2 > 0\}$.

The system (1), (2) is called:

1. *Competitive* (see [11]-[16]) if for all $(x, y) \in \mathbb{R}^2_+$, we have
$$\frac{\partial h_1(x_1, x_2)}{\partial x_2} < 0 \text{ and } \frac{\partial h_2(x_1, x_2)}{\partial x_1} < 0,$$
i.e. competing populations affect each others growth rates negatively.
2. *Cooperative* (see [11]-[16]) if for all $(x, y) \in \mathbb{R}^2_+$, we have
$$\frac{\partial h_1(x_1, x_2)}{\partial x_2} > 0 \text{ and } \frac{\partial h_2(x_1, x_2)}{\partial x_1} > 0.$$
3. *Kolmogorov's predator-pray model* (see [9] and [2]) if for all $(x, y) \in \mathbb{R}^2_+$, we have
$$\frac{\partial h_1(x_1, x_2)}{\partial x_2} < 0 \text{ and } \frac{\partial h_2(x_1, x_2)}{\partial x_1} > 0.$$

One of the main questions, arises in analysis of the predator-prey models, is related to the laboratory cultivation of the two populations and the yield (output) of the predator and/or yield (output) of both communities. As a general example: How to plan the moments on withdrawal (in an given time interval $[0, T]$) such that the total yield (output) be maximal. Here we suppose that the time period of all withdrawals is zero.

The adequate mathematical model is the following impulsive system

$$\begin{aligned}
\dot{x}_1 &= x_1 h_1(x_1, x_2), && t \neq \tau_i,\ i = 1, \dots, p-1, && (3)\\
\dot{x}_2 &= x_2 h_2(x_1, x_2), && t \neq \tau_i,\ i = 1, \dots, p-1, && (4)\\
x_1(\tau_i + 0) &= x_1(\tau_i) + \phi_1(x_1, x_2), && i = 1, \dots, p-1, && (5)\\
x_2(\tau_i + 0) &= x_2(\tau_i) + \phi_2(x_1, x_2), && i = 1, \dots, p-1, && (6)\\
x_1(0) &= x_{10},\ x_2(0) = x_{20}. && && (7)
\end{aligned}$$

Here: $\boldsymbol{\tau} = \{\tau_0, \dots, \tau_p\} \in \mathcal{T}$, where $\mathcal{T}$ is the set of all increasing sequences in $[0, T]$, i.e.
$$\mathcal{T} = \{\boldsymbol{\tau} = \{\tau_0 = 0, \tau_1, \dots, \tau_{p-1}, \tau_p = T\} : \tau_i < \tau_{i+1},\ i = 0, \dots, p\};$$
$\phi_i \in C(\mathbb{R}^2_+, \mathbb{R})$, $i = 1, 2$; $x_{i0} \geq 0$, $i = 1, 2$.

Let us set $\Phi(x_1, x_2) = \left(\begin{smallmatrix} \phi_1(x_1, x_2) \\ \phi_2(x_1, x_2) \end{smallmatrix}\right)$ and let $\boldsymbol{x}(t; \boldsymbol{\tau}, \Phi)$ be the solution of (3)-(7).

The stated above extremal problem is: Find the moments of impulsive effect $\boldsymbol{\tau}^* \in \mathcal{T}$ and function Φ^* such that

$$\begin{aligned}
&\sum_{i=1}^{p} \|\boldsymbol{x}(\tau_i^* + 0; \boldsymbol{\tau}^*, \Phi^*) - \boldsymbol{x}(\tau_i^*; \boldsymbol{\tau}^*, \Phi^*)\|^2 \\
&\quad = \sup\left\{\sum_{i=1}^{p} \|\boldsymbol{x}(\tau_i + 0; \boldsymbol{\tau}, \Phi) - \boldsymbol{x}(\tau_i; \boldsymbol{\tau}, \Phi)\|^2 : \boldsymbol{\tau} \in \mathcal{T},\ \mathrm{Id} + \Phi \in C(\mathbb{R}^2_+, \mathbb{R}^2_+)\right\}. \quad (8)
\end{aligned}$$

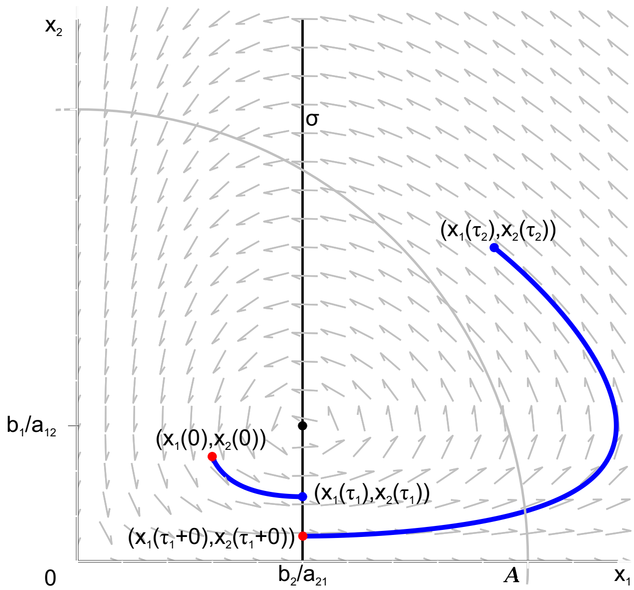

Figure 1

Remark 1. We suppose that $\tau_p \leq T$ and $x_1(\tau_p+0) = x_2(\tau_p+0) = 0$, i.e. at the final impulsive moment τ_p we collect the total biomass of both populations.

Remark 2 (Global extremum)**.** In the general case, the problem (8) does not have a global solution even Φ is a bounded map.

Indeed, consider the Lotka-Volterra system: $h_1(x_1,x_2) = b_1 - a_{12}x_2$, $h_2(x_1,x_2) = -b_2 + a_{21}x_1$; $b_1, a_{12}, b_2, a_{21} > 0$.

Let $A > 1$ be any sufficiently large number.

For any initial conditions, we will define two moments of impulsive effects τ_1 and τ_2 such that $\tau_1 < \tau_2 < \infty$ and

$$\sum_{i=1}^{2} \|\boldsymbol{x}(\tau_i+0) - \boldsymbol{x}(\tau_i)\|^2 \geq A^2 > A.$$

Let σ be the half-line perpendicular to the x_1-axis and containing the stationary point $\left(\frac{b_2}{a_{21}}, \frac{b_1}{a_{12}}\right)$, see Figure 1.

Now, let us define: τ_1 to be the first moment when the orbit through the initial condition $(x_1(0), x_2(0))$ crosses σ such that $x_1(\tau_1) = \frac{b_2}{a_{21}}$ and $x_2(\tau_1) \in (0, \frac{b_1}{a_{12}})$. We set $\phi_1(x_1,x_2) = 0$ and

$$\begin{pmatrix} x_1(\tau_1+0) \\ x_2(\tau_1+0) \end{pmatrix} = \begin{pmatrix} x_1(\tau_1) \\ x_2(\tau_1) \end{pmatrix} - \begin{pmatrix} 0 \\ \phi_2(x_2(\tau_1)) \end{pmatrix} = \begin{pmatrix} \frac{b_2}{a_{21}} \\ x_2(\tau_1) - \phi_2(x_2(\tau_1)) \end{pmatrix}.$$

Moreover, let the point $(\frac{b_2}{a_{21}}, x_2(\tau_1) - \phi_2(x_2(\tau_1))$, where $x_2(\tau_1) - \phi_2(x_2(\tau_1) > 0$, be chosen such that the orbit of system (3), (4), through it, crosses the circle with center (0,0) and radius A. On the end let τ_2 be the second impulsive moment such that the point $(x_1(\tau_2), x_2(\tau_2))$ lies outside the central disk with radius A.

Then, we have only to define the second impulse at $(x_1(\tau_2), x_2(\tau_2))$:

$$(x_1(\tau_2+0), x_2(\tau_2+0)) = (0,0).$$

The global extrema of the same problem exists under additional constraints. For example:

1. The time interval $[0,T]$ is fixed.
2. The functions ϕ_i are bounded, etc.

2. MAXIMUM YIELD AT THE FINAL MOMENT: NECESSARY CONDITIONS

Let (3)-(7) be a solution of the following extreme problem:

$$\begin{aligned} &\|\boldsymbol{x}(\tau_p^*+0;\boldsymbol{\tau}^*,\Phi^*) - \boldsymbol{x}(\tau_p^*;\boldsymbol{\tau}^*,\Phi^*)\|^2 = \|\boldsymbol{x}(\tau_p^*;\boldsymbol{\tau}^*,\Phi^*)\|^2 \\ &\quad = \sup\left\{\|\boldsymbol{x}(\tau_p;\boldsymbol{\tau},\Phi)\|^2 : \boldsymbol{\tau}\in\mathcal{T},\ \|\Phi(x_1,x_2)\|\le M,\ 0\le\phi_i(x_1,x_2),\ i=1,2\right\}. \end{aligned} \tag{9}$$

Let Ψ be a first integral of the system without impulses (1), (2), defined in $\mathbb{R}^2_+$.

Obviously the points $\boldsymbol{x}_0 = \boldsymbol{x}_0^\pm$, $\boldsymbol{x}_i^+ = \boldsymbol{x}(\tau_i+0,\boldsymbol{x}_0)$, $\boldsymbol{x}_i^- = \boldsymbol{x}(\tau_i,\boldsymbol{x}_0)$, $i=1,\dots,p$ (here $\boldsymbol{x}_p^- = \boldsymbol{x}_p(\tau_p-0,\boldsymbol{x}_0)$, $\boldsymbol{x}_p^+ = \mathbf{0}$) satisfy

$$\Psi(\boldsymbol{x}_i^+) = \Psi(\boldsymbol{x}_{i+1}^-), \quad i=0,\dots,p-1.$$

Hence, the stated max-problem is equivalent to the following Lagrange problem: Find the extremum of

$$\|\boldsymbol{x}_p^-\|^2, \tag{10}$$

subject to the following constraints

$$\begin{aligned} &\Psi(\boldsymbol{x}_i^+) = \Psi(\boldsymbol{x}_{i+1}^-), \quad i=0,\dots,p-1. \\ &\|\boldsymbol{x}_i^+ - \boldsymbol{x}_i^-\|^2 \le M, \quad i=1,\dots,p-1, \end{aligned} \tag{11}$$

and

$$\boldsymbol{x}(\tau_i+0,\boldsymbol{x}_0)\in\mathbb{R}^2_+, \quad i=1,\dots,p-1. \tag{12}$$

Let us note that inclusions (12) follow from the condition $\phi_i(x_1,x_2) \geq 0$ and $x_{i0} > 0$, $i = 1, 2$.

Therefore, we may rewrite the problem (10)-(12) in the form: Find the extremum of

$$\|\boldsymbol{x}_p^-\|^2, \tag{13}$$

subject to the following constraints

$$\begin{aligned} \Psi(\boldsymbol{x}_i^+) &= \Psi(\boldsymbol{x}_{i+1}^-), \quad i = 0, \ldots, p-1. \\ \|\boldsymbol{x}_i^+ - \boldsymbol{x}_i^-\|^2 &\leq M, \quad i = 1, \ldots, p-1, \end{aligned} \tag{14}$$

Remark 3. The condition (14) does not imply that the points $\boldsymbol{x}_i^+$ and $\boldsymbol{x}_{i+1}^-$ lie on one and the same orbit of system (1), (2). Indeed the level set of first integral, passing through $\boldsymbol{x}_i^+$ contains $\boldsymbol{x}_{i+1}^-$, but in the general case the level set may contain more than one solution of the system (1), (2).

Therefore, the constrained problems (10)-(12) and (13), (14) are not equivalent. But if the problem (10)-(12) has a solution, then the "autonomous" problem (13), (14) has a solution too.

Let us also mark that the two considered problems are equivalent if any level set of the first integral Ψ of system (1), (2) contains only one solution of the system. Here a simple example is the Lotka-Volterra system.

Using the introduced notations, the Lagrangian expression for the constrained problem (13), (14) (see also Remark 1: $\boldsymbol{x}_p^+ = \mathbf{0}$) is

$$\begin{aligned} &L(\boldsymbol{x}_1^-, \ldots, \boldsymbol{x}_p^-; \boldsymbol{x}_1^+, \ldots, \boldsymbol{x}_{p-1}^+; \lambda_0, \ldots, \lambda_{p-1}; \mu_1, \ldots, \mu_{p-1}) \\ &\qquad = \|\boldsymbol{x}_p^-\|^2 - \sum_{i=0}^{p-1} \lambda_i \left(\Psi(\boldsymbol{x}_i^+) - \Psi(\boldsymbol{x}_{i+1}^-)\right) - \sum_{i=1}^{p-1} \mu_i \|\boldsymbol{x}_i^+ - \boldsymbol{x}_i^-\|^2. \end{aligned}$$

Hence, applying the classical necessary condition for constrained extremum, if the vectors $\boldsymbol{x}_1^-, \ldots, \boldsymbol{x}_p^-$, $\boldsymbol{x}_1^+, \ldots, \boldsymbol{x}_{p-1}^+$ are a solution of (10)-(11), then there exist numbers $\lambda_0, \ldots, \lambda_{p-1}$ and $\mu_1, \ldots, \mu_{p-1}$ satisfying the system

$$\begin{aligned} \mathbf{0} =& \nabla_{\boldsymbol{x}_i^-} \sum_{i=0}^{p-1} \lambda_i \left(\Psi(\boldsymbol{x}_i^+) - \Psi(\boldsymbol{x}_{i+1}^-)\right) \\ &+ \nabla_{\boldsymbol{x}_i^-} \sum_{i=1}^{p-1} \mu_i \|\boldsymbol{x}_i^+ - \boldsymbol{x}_i^-\|^2, \; i = 1, \ldots, p-1, \\ \mathbf{0} =& \nabla_{\boldsymbol{x}_p^-} \|\boldsymbol{x}_p^-\|^2 - \nabla_{\boldsymbol{x}_p^-} \sum_{i=0}^{p-1} \lambda_i \left(\Psi(\boldsymbol{x}_i^+) - \Psi(\boldsymbol{x}_{i+1}^-)\right), \end{aligned}$$

$$\begin{aligned}\mathbf{0} =&\nabla_{\boldsymbol{x}_i^+}\sum_{i=0}^{p-1}\lambda_i\left(\Psi(\boldsymbol{x}_i^+)-\Psi(\boldsymbol{x}_{i+1}^-)\right)\\&+\nabla_{\boldsymbol{x}_i^+}\sum_{i=1}^{p-1}\mu_i\|\boldsymbol{x}_i^+-\boldsymbol{x}_i^-\|^2,\ i=0,\dots,p-1.\end{aligned}$$

Primal feasibility:

$$\begin{aligned}\Psi(\boldsymbol{x}_i^+) =&\Psi(\boldsymbol{x}_{i+1}^-),\quad i=0,\dots,p-1,\\ \|\boldsymbol{x}_i^+-\boldsymbol{x}_i^-\|^2 \le& M,\quad i=1,\dots,p-1.\end{aligned}$$

Dual feasibility:

$$\mu_i \ge 0,\ i=1,\dots,p-1.$$

Therefore

$$2\mu_i\left(\boldsymbol{x}_i^+-\boldsymbol{x}_i^-\right) =-\lambda_{i-1}\nabla\Psi(\boldsymbol{x}_i^-),\ i=1,\dots,p-1, \tag{15}$$

$$2\boldsymbol{x}_p^- =-\lambda_{p-1}\nabla\Psi(\boldsymbol{x}_p^-), \tag{16}$$

$$2\mu_i\left(\boldsymbol{x}_i^+-\boldsymbol{x}_i^-\right) =-\lambda_i\nabla\Psi(\boldsymbol{x}_i^+),\quad i=1,\dots,p-1, \tag{17}$$

$$\Psi(\boldsymbol{x}_i^+) =\Psi(\boldsymbol{x}_{i+1}^-),\qquad i=0,\dots,p-1. \tag{18}$$

Theorem 4. *Let the vectors* $\boldsymbol{x}_1^-,\dots,\boldsymbol{x}_p^-$, $\boldsymbol{x}_1^+,\dots,\boldsymbol{x}_{p-1}^+$ *are a local extremum of* (13) *subject to constraints* (14)*. Moreover, let*

$$\det\begin{pmatrix}\nabla\Psi(\boldsymbol{x}_i^+) & \nabla\Psi(\boldsymbol{x}_{i+1}^-)\end{pmatrix}\neq 0,\quad i=1,\dots,p-1.$$

Then there exist numbers $\lambda_0,\dots,\lambda_{p-1}$ *such that equations* (15)-(18) *hold true.*

If Ψ *is first integral with nonzero gradient,* $\lambda_0\neq 0$*, and* $\boldsymbol{x}_i^+\neq\boldsymbol{x}_i^-$, $i=1,\dots,p-1$, *then* $\lambda_i\neq 0$ *and* $\mu_i\neq 0$, $i=1,\dots,p-1$.

Proof. Let us set ($(\cdot)^t$ is the transpose operator)

$$\begin{aligned}\bar{\boldsymbol{x}} =&\begin{pmatrix}\boldsymbol{x}_1^- & \cdots & \boldsymbol{x}_p^- & \boldsymbol{x}_1^+ & \cdots & \boldsymbol{x}_{p-1}^+\end{pmatrix}^t,\\ F_i(\bar{\boldsymbol{x}}) =&\Psi(\boldsymbol{x}_i^+)-\Psi(\boldsymbol{x}_{i+1}^-),\ i=0,\dots,p-1,\\ F(\bar{\boldsymbol{x}}) =&\begin{pmatrix}F_0(\bar{\boldsymbol{x}}) & F_1(\bar{\boldsymbol{x}}) & \cdots & F_{p-1}(\bar{\boldsymbol{x}})\end{pmatrix}^t.\end{aligned}$$

We have to prove that the vectors $\nabla_{\bar{\boldsymbol{x}}}F_i(\boldsymbol{x}_1^-,\cdots,\boldsymbol{x}_p^-,\boldsymbol{x}_1^+,\cdots,\boldsymbol{x}_{p-1}^+)$ are linearly independent, i.e. $(\boldsymbol{x}_1^-,\cdots,\boldsymbol{x}_p^-,\boldsymbol{x}_1^+,\cdots,\boldsymbol{x}_{p-1}^+)$ is a regular point of $F(\bar{\boldsymbol{x}})=\mathbf{0}$.

The rows of matrix $\nabla_{\bar{\boldsymbol{x}}}F(\bar{\boldsymbol{x}})$ are the gradient vectors $\nabla_{\bar{\boldsymbol{x}}}F_i$, $i=0,\dots,p-1$. Hence, the rank of matrix $\nabla_{\bar{\boldsymbol{x}}}F(\bar{\boldsymbol{x}})$ is maximal if the conditions of theorem hold true.

Assume Ψ is a first integral with nonzero gradient.

Obviously, $\lambda_p \neq 0$.

Let us suppose there exists an index $i_0 = 1, \dots, p-1$ such that $\mu_{i_0} = 0$. Hence, by (17) and assumptions of the theorem, $\lambda_{i_0} = 0$. Analogously, by (15), $\lambda_{i_0-1} = 0$. Using (17), again, we receive $\mu_{i_0-1} = 0$, and so on. Therefore, $\lambda_0 = \mu_0 = 0$. The obtained contradiction proves that $\lambda_i \neq 0$ and $\mu_i \neq 0$ for all $i = 0, \dots, p-1$. $\square$

Let $f(\boldsymbol{x}) = (x_1 h_1(\boldsymbol{x})\ x_2 h_2(\boldsymbol{x}))^t$, $\boldsymbol{x} = (x_1\ x_2)^t$.

Corollary 5. *Let the vectors $\boldsymbol{x}_1^-, \dots, \boldsymbol{x}_p^-$, $\boldsymbol{x}_1^+, \dots, \boldsymbol{x}_{p-1}^+$ are a solution of* (14) *and*

$$\det\Big(\nabla\Psi(\boldsymbol{x}_i^+) \quad \nabla\Psi(\boldsymbol{x}_{i+1}^-)\Big) \neq 0, \quad i = 1, \dots, p-1.$$

Also, let Ψ is first integral with nonzero gradient, $\lambda_0 \neq 0$, and $\boldsymbol{x}_i^+ \neq \boldsymbol{x}_i^-$, $i = 1, \dots, p-1$.

Then, for all $i = 1, \dots, p-1$, the vector $\boldsymbol{x}_i^+ - \boldsymbol{x}_i^-$ is perpendicular to the vectors $f(\boldsymbol{x}_i^-)$ and $f(\boldsymbol{x}_i^+)$. Moreover, there exist numbers $\Lambda_1, \dots, \Lambda_p$ such that

$$\begin{aligned} \boldsymbol{x}(\tau_i + 0) =& \boldsymbol{x}(\tau_i) + \Lambda_i \nabla\Psi(\boldsymbol{x}(\tau_i)), \quad i = 1, \dots, p-1, \\ \boldsymbol{x}_p^- =& \Lambda_p \nabla\Psi(\boldsymbol{x}_p^-). \end{aligned}$$

Proof. The function $\Psi(\boldsymbol{x})$ is a first integral of (1), (2). Then

$$0 = \dot{\Psi}(\boldsymbol{x}) = \langle \nabla\Psi(\boldsymbol{x}), \dot{\boldsymbol{x}} \rangle = \langle \nabla\Psi(\boldsymbol{x}), f(\boldsymbol{x}) \rangle.$$

Hence (considering the dot product of (15) and (17)) we obtain

$$\begin{aligned} \langle \boldsymbol{x}_i^+ - \boldsymbol{x}_i^-, f(\boldsymbol{x}_{i+1}^-) \rangle =& -\frac{\lambda_{i-1}}{2\mu_{i-1}} \langle \nabla\Psi(\boldsymbol{x}_{i+1}^-), f(\boldsymbol{x}_{i+1}^-) \rangle = 0, \\ \langle \boldsymbol{x}_i^+ - \boldsymbol{x}_i^-, f(\boldsymbol{x}_i^+) \rangle =& -\frac{\lambda_i}{2\mu_i} \langle \nabla\Psi(\boldsymbol{x}_i^+), f(\boldsymbol{x}_i^+) \rangle = 0. \end{aligned}$$

To prove the second part of corollary it is sufficient to set

$$\Lambda_i = \frac{\lambda_{i-1}}{2\mu_i}, \ i = 1, \dots, p-1; \quad \Lambda_p = \frac{\lambda_{p-1}}{2}.$$

Hence, the proof is complete. $\square$

Example 6. Consider the Lotka-Volterra initial value problem

$$\dot{x}_1 = 0.9x_1 - 1.5x_1x_2, \tag{19}$$

$$\dot{x}_2 = -1.5x_2 + 3x_1x_2, \tag{20}$$

$$x_1(0) = 0.15, \quad x_2(0) = 0.15. \tag{21}$$

Let $p = 3$. The stated extremal problem is: Find the moments of impulsive effect $\boldsymbol{\tau}^* = \{\tau_0 = 0, \tau_1^*, \tau_2^*, \tau_3^*\} \in \mathcal{T}$ and function Φ^* such that

$$\begin{aligned}&\|\boldsymbol{x}(\tau_p^*; \boldsymbol{\tau}^*, \Phi^*)\|^2 \\ &\qquad = \sup\left\{\|\boldsymbol{x}(\tau_p; \boldsymbol{\tau}, \Phi)\|^2 : \boldsymbol{\tau} \in \mathcal{T},\ \|\Phi_i(x_1, x_2)\| \le 0.6,\ 0 < \phi_i(x_1, x_2)\right\}. \end{aligned} \tag{22}$$

Using a simple code on CAS Maple (more precisely the procedure `GlobalSolve` to obtain numerically all results, see [1]), it is not hard to calculate the coordinates of all points $\boldsymbol{x}_i^{\pm}$, $i = 1, 2, 3$ (we round down the results to three decimal places):

$$\boldsymbol{x}_1^- = (1.364\ 1.153)^t, \quad \boldsymbol{x}_1^+ = (1.925\ 1.365)^t; \tag{23}$$
$$\boldsymbol{x}_2^- = (1.859\ 1.53)^t, \quad \boldsymbol{x}_2^+ = (2.413\ 1.76)^t; \tag{24}$$
$$\boldsymbol{x}_3^- = (0.526\ 4.587)^t, \quad \boldsymbol{x}_3^+ = (0\ 0)^t. \tag{25}$$

The corresponding impulsive moments are

$$\tau_1^* = 3.417, \quad \tau_2^* = 3.444, \quad \tau_3^* = 3.793.$$

The maximum yield is 4.617, see Figure 2.

Now, let us calculate the gradients of the first integral

$$\Psi(\boldsymbol{x}) = 3x_1 - 1.5\ln(x_1) + 1.5x_2 - 0.9\ln(x_2)$$

at the obtained impulsive points:

$$\begin{aligned}\nabla\Psi(\boldsymbol{x}_1^-) &= (1.9\ 0.719)^t,\\ \nabla\Psi(\boldsymbol{x}_2^-) &= (2.193\ 0.911)^t.\end{aligned}$$

Therefore, using

$$\begin{aligned}\boldsymbol{x}_1^+ - \boldsymbol{x}_1^- &= (0.561\ 0.212)^t,\\ \boldsymbol{x}_2^+ - \boldsymbol{x}_2^- &= (0.554\ 0.23)^t,\end{aligned}$$

we receive

$$\begin{aligned}\boldsymbol{x}_1^+ - \boldsymbol{x}_1^- &= (0.561\ 0.212)^t \approx 0.295(1.9\ 0.719)^t = \Lambda_1\nabla\Psi(\boldsymbol{x}_1^-),\\ \boldsymbol{x}_2^+ - \boldsymbol{x}_2^- &= (0.554\ 0.23)^t \approx 0.252(2.193\ 0.911)^t = \Lambda_2\nabla\Psi(\boldsymbol{x}_2^-),\end{aligned}$$

i.e. $\Lambda_1 = 0.295$ and $\Lambda_2 = 0.252$, see the conclusion of Corollary 5.

Moreover, at the final impulsive point $\boldsymbol{x}_3^- = (0.526\ 4.587)^t$ we have $\nabla\Psi(\boldsymbol{x}_3^-) = (0.149\ 1.303)^t$, and

$$\boldsymbol{x}_3^- = (0.526\ 4.587)^t = 3.518(0.149\ 1.303)^t = \nabla\Psi(\boldsymbol{x}_3^-),$$

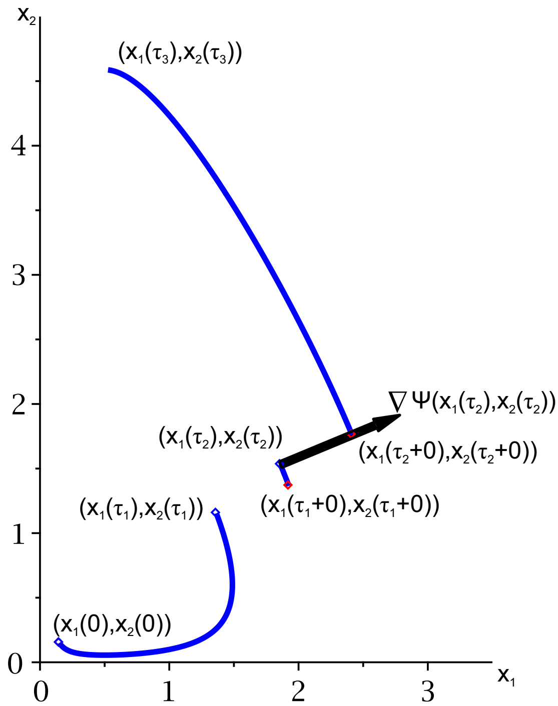

Figure 2: Example 6

i.e. $\Lambda_3 = 2 \times 3.518 = 7.036$.

Therefore, the solution of extreme problem (22) is the following impulsive system

$$\begin{aligned}
\dot{x}_1 =& 0.9x_1 - 1.5x_1x_2, && t \neq \tau_i,\ i = 1, \ldots, 2,\\
\dot{x}_2 =& -1.5x_2 + 3x_1x_2, && t \neq \tau_i,\ i = 1, \ldots, 2,\\
x_1(\tau_i + 0) =& x_1(\tau_i) + \frac{3x_1 - 1.5}{x_1}, && i = 1, \ldots, 2,\\
x_2(\tau_i + 0) =& x_2(\tau_i) + \frac{1.5x_2 - 0.9}{x_2}, && i = 1, \ldots, 2,\\
x_1(0) =& 0.15,\ x_2(0) = 0.15.
\end{aligned}$$

Let us summarize: At initial moment: $\|(0.15\ 0.15)^t\| = 0.212$. Without any impulsive influence, the maximum yield is 2.348.

In our example: adding two times biomass of total size 1.2, at the final moment $\tau_3^* = 3.783$, we receive total biomass of size 4.617 (here, we associate the size of biomass with the norm of solution).

The similar construction is plotted on Figure 3 with initial assumption $p = 5$. In this case the maximal yield is 7.022.

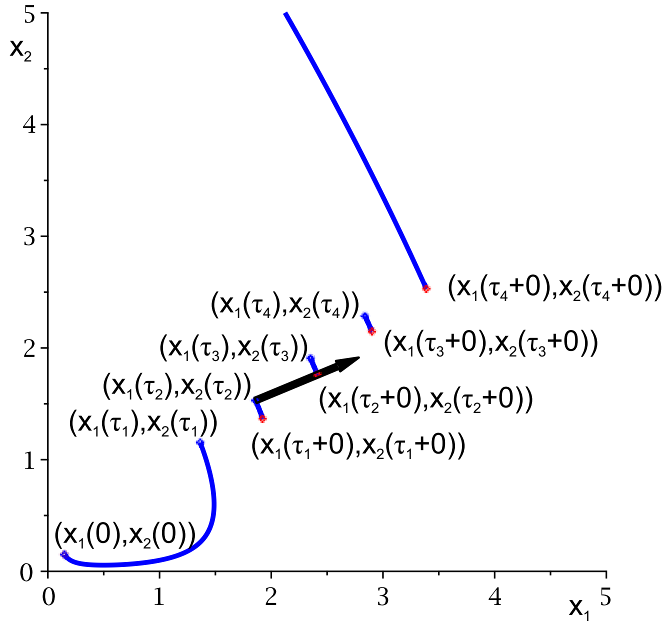

Figure 3: Example 6

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