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UNIQUENESS OF SOLUTIONS TO INITIAL VALUE PROBLEM OF FRACTIONAL DIFFERENTIAL EQUATIONS OF VARIABLE-ORDER

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ABSTRACT: In this paper, by using the Banach contraction mapping principle, we study the existence and uniqueness of solutions for initial value problem of fractional differential equations of variable-order

$$\begin{cases} D_{0+}^{q(t)}x(t) = f(t, x(t)), \quad 0 < t \leq T, \\ t^{2-q(t)}x(t)|_{t=0} = 0, \end{cases}$$

where $1 < q(t) < 2$, $0 < T < +\infty$, $D_{0+}^{q(t)}$ is the Riemann-Liouvile fractional derivative of variable-order $q(t)$.

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1. INTRODUCTION

The initial value and boundary value problem of fractional differential equations have become the subject of extensive research. As early as 1659 years, the integer differential is still in the development stage, and the discussion and conjecture of fractional calculus and fractional differential equations have begun. The fractional integral theory has experienced more than three hundred years of development, it is in fluid Mechanics, biomathematics, ecology, viscoelastic dynamics, aerodynamics, control theory and complex dielectric electro-dynamics have a wide range of applications(see 1). The researches have gained many beautify results for existence of solutions to differential equations of fractional order by the nonlinear functional analysis methods, such as some fixed point theorems, monotone iterative method(see 2-19).

In [2], the author considered the following initial value problem of fractional differential equation

$$\begin{cases} D_{0+}^{q}x(t) = f(t, x(t)), \quad 0 < t \leq T < +\infty, \\ x(0) = x_0, \end{cases}$$

where $0 < q < 1$, $f \in C([0,T], R)$, D_{0+}^{q} is the Riemann-Liouvile fractional derivative. The author obtained the basic theory of the above fractional differential equations by the classical approach.

On the other hand, the operators of variable-order, which are the derivatives and integrals whose order is a function of certain variables, attract attention due to their applied significance in various research areas. In recent years, more and more researchers have found that many variables in the dynamic process reflect the performance of fractional order, which can change with time and space. Many facts show that variable fractional calculations provide an effective mathematical framework for complex dynamic problems. However, having the formalism available to consider variable-order differential and integral operations opens a whole new field of mathematical analysis, with profound implications to many fields of science and technology(see 20-32). In [31], using the monotone iterative method, the author considered existence and uniqueness of solutions to initial value problems for fractional differential equation of variable-order

$$\begin{cases} D_{0+}^{p(x)}x(t) = f(t, x), \quad 0 < t \leq T, \quad 0 < T < +\infty \\ x(0) = 0, \end{cases}$$

Where $0 < p(x) < 1, 0 \leq t \leq T, x \in \mathbb{R}$, and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. $D_{0+}^{p(x)}$ denotes fractional derivative of variable-order defined by

$$D_{a+}^{p(t)}f(t) = \left(\frac{d}{dt}\right)^n I_{a+}^{n-p(t)}f(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-p(t)-1}}{\Gamma(n-p(t))}f(s)ds, \quad t > a.$$

In [32], by using the Banach contraction mapping principle, the author considered the uniqueness result of solutions to initial value problems of differential equations of variable-order

$$\begin{cases} D_{0+}^{q(t)}x(t) = f(t, x(t)), \quad 0 < t \leq T, \\ x(0) = 0, \end{cases}$$

where $0 < q(t) < 1$, $0 < T < +\infty$, $D_{0+}^{q(t)}$ denotes derivative of variable-order defined by

$$D_{0+}^{q(t)}x(t) = \frac{d}{dt}\int_0^t \frac{(t-s)^{-q(t)}}{\Gamma(1-q(t)),}x(s)ds, \quad t > 0.$$

Motivated by the aforementioned works, in this paper, by using the Banach contraction mapping principle, we shall consider the existence and uniqueness of solution for the following initial value problem for fractional differential equation of variable-order

$$\begin{cases} D_{0+}^{q(t)}x(t) = f(t, x(t)), \quad 0 < t \le T, \\ t^{2-q(t)}x(t)|_{t=0} = 0, \end{cases} \tag{1.1}$$

where $1 < q(t) < 2$, $0 < T < +\infty$, $D_{0+}^{q(t)}$ denotes derivative of variable-order defined by

$$D_{a+}^{p(t)}f(t) = \left(\frac{d}{dt}\right)^n I_{a+}^{n-p(t)}f(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-p(t)-1}}{\Gamma(n-p(t))}f(s)ds, \quad t > a.$$

2. PRELIMINARIES

Definition 1 ([30]). *Let $p(t) : [a, b] \to (0, +\infty)(-\infty < a < b < +\infty)$, the fractional integral of variable-order $p(t)$ for function $x(t)$ is defined as following*

$$I_{a+}^{p(t)}x(t) = \int_a^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))}x(s)ds, \quad t > a, \tag{2.1}$$

where $\Gamma(\cdot)$ denotes the Gamma function, provided that the right-hand side is pointwise defined.

Definition 2 ([30]). *Let $p(t) : [a, b] \to (n-1, n)(-\infty < a < b < +\infty$, n is a given natural number), the fractional derivative of variable-order $p(t)$ for function $x(t)$ is defined as following*

$$D_{a+}^{p(t)}x(t) = \left(\frac{d}{dt}\right)^n I_{a+}^{n-p(t)}x(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-p(t)-1}}{\Gamma(n-p(t))}x(s)ds, \quad t > a, \tag{2.2}$$

provided that the right-hand side is pointwise defined.

The variable-order fractional derivative is an extension of constant order fractional derivative. According to (2.1) and (2.2), it is clear that when $q(t)$ is a constant function, i.e. $q(t) \equiv q$(q is a finite positive constant), then $I_{a+}^{q(t)}$, $D_{a+}^{q(t)}$ are the usual Riemann-Liouvile fractional integral and fractional derivative [1]. It is well known that fractional calculus I_{a+}^{q}, D_{a+}^{q} have some very important properties, which play very important role in considering existence of solutions of fractional differential equation denoted by D^q_{a+}, by means of nonlinear analysis method. Such as, the following some properties, which can be founded in [1].

Proposition 3 ([1]). *The equality* $I^\gamma_{a+}I^\delta_{a+}f(t) = I^{\gamma+\delta}_{a+}f(t), \gamma > 0, \delta > 0$ *holds for* $f \in L(0, b), 0 < b < +\infty$.

Proposition 4 ([1]). *The equality* $I^\gamma_{a+}I^\delta_{a+}f(t) = f(t), \gamma > 0$ *holds for* $f \in L(0, b), 0 < b < +\infty$.

Proposition 5 ([1]). *Let* $\alpha > 0$, *Then the differential equation*

$$D^\alpha_{a+}u = 0$$

has unique solution

$$u(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots c_N(t-a)^{\alpha-N},$$

$c_i \in \mathbb{R}, i = 1, 2, \cdots, n$, *here* $n - 1 < \alpha \leq n$.

Proposition 6 ([1]). *Let* $a > 0$, $u \in L(a, b)$, $D^\alpha_{0+}u \in L(a, b)$. *Then the following equality holds*

$$I^\alpha_{a+}D^\alpha_{a+}u(t) = u(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots c_n(t-a)^{\alpha-n},$$

$c_i \in \mathbb{R}, i = 1, 2, \cdots, n$, *here* $n - 1 < \alpha \leq n$.

But, in general, these properties don't hold for fractional calculus of variable-order $I^{q(t)}_{a+}$, $D^{q(t)}_{a+}$ defined by (2.1) and (2.2). For example,

$$I^{p(t)}_{a+}I^{q(t)}_{a+}f(t) \neq I^{p(t)+q(t)}_{a+}f(t), p(t) > 0, q(t) > 0, f \in L(0, b), \tag{2.3}$$

where $I^{q(t)}_{a+}$ denote one of fractional integral defined by (2.1).

Example 7 Let $p(t) = t, 0 \leq t \leq 6$, $q(t) = \begin{cases} 2, & 0 \leq t \leq 2, \\ 1, & 2 < t \leq 3, \\ t, & 3 < t \leq 6, \end{cases}$ $f(t) = 1, 0 \leq t \leq 6$. We calculate $I^{p(t)}_{0+}f(t)$ and $I^{p(t)q(t)}_{0+}f(t)$ defined by (2.1).

$$\begin{aligned} I^{p(t)}_{0+}I^{q(t)}_{0+}f(t) &= \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))}\int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))}f(\tau)d\tau ds \\ &= \int_0^2 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))}\int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))}f(\tau)d\tau ds \end{aligned}$$

$$
\begin{aligned}
&+ \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds \\
&= \int_0^2 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{2-1}}{\Gamma(2))} f(\tau) d\tau ds \\
&+ \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds \\
&= \int_0^2 \frac{(t-s)^{p(t)-1} s^2}{2\Gamma(p(t))} ds + \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds,
\end{aligned}
$$

$$
I_{0+}^{p(t)q(t)} f(t) = \int_0^t \frac{(t-s)^{p(t)+q(t)-1}}{\Gamma(p(t)+q(t))} f(s) ds,
$$

we see that

$$
\begin{aligned}
I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) \mid_{t=3} &= \int_0^2 \frac{(3-s)^{3-1} s^2}{2\Gamma(3))} ds + \int_2^3 \frac{(3-s)^{3-1}}{\Gamma(3))} \int_0^s \frac{(s-\tau)^{1-1}}{\Gamma(1)} f(\tau) d\tau ds, \\
&= \frac{8}{5} + \int_2^3 \frac{(3-s)^{3-1}}{\Gamma(3))} ds = \frac{8}{5} + \frac{9}{24} = \frac{79}{40}, \\
I_{0+}^{p(t)q(t)} f(t) \mid_{t=3} &= \int_0^3 \frac{(3-s)^{p(3)+q(3)-1}}{\Gamma(p(3)+q(3))} f(s) ds = \int_0^3 \frac{(3-s)^{p(3)+1-1}}{\Gamma(3+1)} ds = \frac{27}{8},
\end{aligned}
$$

we see easily that

$$
I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) \mid_{t=3} \neq I_{0+}^{p(t)q(t)} f(t) \mid_{t=3} .
$$

According to (2.3), we could claim that variable-order calculus of non-constant function $p(t)$ for $x(t)$ defined by (2.1), (2.2) don't have the properties like Propositions 3-6. Therefor, Propositions 3-6 don't hold for $D_{a+}^{p(t)}$ and $I_{a+}^{p(t)}$ defined by (2.1) and (2.2).

Therefor, without those properties like Propositions 3-6, ones can not transform a fractional differential equation of variable-order into an equivalent integral equation, so that one can consider existence of solutions of a fractional differential equation of variable-order, by means of nonlinear functional analysis method.

We introduce some definitions which are used throughout this paper. Let $-\infty < a < b < +\infty$.

Definition 8 ([32]). *A generalized interval is a subset I of $\mathbb{R}$ which is either an interval (i.e. a set of the form $[a, b], (a, b), [a, b)$ or $(a, b]$); a point $\{a\}$; or the empty set ϕ.*

Definition 9 ([32]). *If I is a generalized interval. A partition of I is a finite set P of generalized intervals contained in I, such that every x in I lies in exactly one of the generalized intervals J in P.*

Definition 10 ([32]). *Let I be a generalized interval, let $f : I \longrightarrow \mathbb{R}$ be a function, and let be P a partition of I. f is said to be piecewise constant with respect to P if for every $J \in P$, f is constant on J.*

Definition 11 ([32]). *Let I be a generalized interval. The function $f : I \longrightarrow \mathbb{R}$ is called piecewise constant on I, if there exists a partition P of I such that f is piecewise constant with respect to P.*

3. MAIN RESULTS

In this section, we present our main results. Now we make the following assumptions:
(H1) Let $P = \{[0, T_1], (T_1, T_2], (T_2, T_3], \cdots, (T_{N^*-1}, T]\}$($N^*$ is a given natural number) be a partition of the finite interval $[0, T]$, and let $q(t) : [0, T] \to (1, 2]$ be a piecewise constant function with respect to P, i.e.

$$q(t) = \sum_{k=1}^{N^*} q_k I_k(t), \quad t \in [0, T], \tag{3.1}$$

where $1 < q_k \leq 2, k = 1, 2, \cdots, N^*$ are constants, and I_k is the indicator of the interval $[T_{k-1}, T_k], k = 1, 2, \cdots, N^*$(*here* $T_0 = 0, T_{N^*} = T$), that is $I_k = 1$ for $t \in [T_{k-1}, T_k]$, $I_k = 0$ for elsewhere.
(H2) For $0 \leq r \leq q_i - 1, i = 1, 2, \cdots, N^*$, $t^r f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exists a positive constant L satisfying

$$\frac{LT_i^{q_i-r}\Gamma(q_i - r - 1)}{\Gamma(2q_i - r - 1)} < 1$$

such that

$$t^r|f(t, x) - f(t, y)| \leq L|x - y|, \quad 0 < t \leq T, \quad x, y \in \mathbb{R}.$$

(H3) For $0 \leq r \leq q_i - 1, i = 1, 2, \cdots, N^*$, $t^r f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exists a positive constant $M, \mu > 1$ satisfying

$$\frac{4 \max\limits_{0\leq t\leq T_i} t^r|f(t, 0)|\Gamma(q_i - r - 1)T_i^{q_i-r}}{\Gamma(2q_i - r - 1)} < \Big(\frac{\Gamma(2q_i - r - 1)}{2M\Gamma(q_i - r - 1)T_i^{q_i-r}}\Big)^{\frac{1}{\mu-1}}, (T_{N^*} = T)$$

such that

$$t^r|f(t, x) - f(t, y)| \leq M|x - y|^{\mu}, \quad 0 < t \leq T, \quad x, y \in \mathbb{R}.$$

In order to obtain our main results, we firstly carry on essential analysis to equation of (1.1).

According to (H1), we have

$$q(t)=\sum_{k=1}^{N^*} q_k I_k(t), \quad t\in[0,T].$$

Hence, we get

$$\int_0^t \frac{(t-s)^{1-q(t)}}{\Gamma(2-q(t))}x(s)ds=\sum_{k=1}^{N^*} I_k(t)\int_0^t \frac{(t-s)^{1-q_k}}{\Gamma(2-q_k)}x(s)ds. \tag{3.2}$$

So, equation of (1.1) can be written by

$$\left(\frac{d}{dt}\right)^2\sum_{k=1}^{N^*} I_k(t)\int_0^t \frac{(t-s)^{1-q_k}}{\Gamma(2-q_k)}x(s)ds=f(t,x(t)), \quad 0<t\le T. \tag{3.3}$$

Then, Eq. (3.3) in the interval $[0,T_1]$ can be written by

$$\left(\frac{d}{dt}\right)^2\int_0^t \frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)}x(s)ds=D_{0+}^{q_1}x(t))=f(t,x(t)), \quad 0<t\le T_1. \tag{3.4}$$

Again, Eq. (3.3) in the interval $(T_1,T_2]$ can be written by

$$\left(\frac{d}{dt}\right)^2\int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds=f(t,x(t)), \quad T_1<t\le T_2. \tag{3.5}$$

As well, Eq. (3.3) in the interval $(T_2,T_3]$ can be written by

$$\left(\frac{d}{dt}\right)^2\int_0^t \frac{(t-s)^{1-q_3}}{\Gamma(2-q_3)}x(s)ds=f(t,x(t)), \quad T_2<t\le T_3. \tag{3.6}$$

In the same way, Eq. (3.3) in the interval $(T_{i-1},T_i], i=4,5,\cdots,N^*(T_{N^*}=T)$ can be written by

$$\left(\frac{d}{dt}\right)^2\int_0^t \frac{(t-s)^{1-q_i}}{\Gamma(2-q_i)}x(s)ds=f(t,x(t)), \quad T_{i-1}<t\le T_i. \tag{3.7}$$

Now we present definition of solution to problem (1.1), which is fundamental in our work.

Definition 12 *We say problem (1.1) has a solution, if there exist functions $u_i(t), i=1,2,\cdots,N^*$, such that $u_1\in[0,T_1]$ satisfying Eq.(3.4) and $t^{2-q_1}u_1(0)=0$; $u_2\in[0,T_2]$ satisfying Eq.(3.5) and $t^{2-q_2}u_2(0)=0$; $u_3\in[0,T_3]$ satisfying Eq.(3.6) and $t^{2-q_3}u_3(0)=0$; $u_i\in[0,T_i]$ satisfying Eq.(3.7) and $t^{2-q_i}u_i(0)=0, i=4,5,\cdots,N^*(T_{N^*}=T)$.*

Remark 13 We say problem (1.1) has one unique solution, if functions $u_i(t)$ of Definition 12 are unique, $i=1,2,\cdots,N^*$.

Based on the previous arguments, we have the following results.

Theorem 14. *Assume that conditions (H1) and (H2) hold, then problem (1.1) has one unique solution.*

Proof. According the above analysis, equation of problem (1.1) can be written as equation (3.3). Equation (3.3) in the interval $[0, T_1]$ can be written as (3.4). Applying operator $I_{0+}^{q_1}$ to both sides of (3.4), by Propositions 6, we have

$$x(t) = c_1 t^{q_1-1} + c_2 t^{q_1-2} + \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} f(s, x(s))ds, \quad 0 < t \leq T_1.$$

By $t^{2-q_1}x(0) = 0$ and the assumption of function f, we could get $c_1 = 0, c_2 = 0$. Define operator $T : C[0, T_1] \to C[0, T_1]$ by

$$Tx(t) = \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} f(s, x(s))ds, \quad 0 < t \leq T_1. \tag{3.8}$$

It follows from the continuity of function $t^r f(t, x(t))$ that operator $T : C[0, T_1] \to C[0, T_1]$ is well defined.

In fact, let $g(t, x(t)) = t^r f(t, x(t))$, by (H2), one has $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. For $x(t) \in C[0, T_1], t_0 \in C[0, T_1]$ we have

$$\begin{aligned}
& |Tx(t) - Tx(t_0)| \\
= \ & |\frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} f(s, x(s))ds - \frac{1}{\Gamma(q_1)} \int_0^{t_0} (t_0-s)^{q_1-1} f(s, x(s))ds| \\
\leq \ & |\frac{t^{q_1-r}}{\Gamma(q_1)} \int_0^1 (1-\tau)^{q_1-1} g(t\tau, x(t\tau))d\tau - \frac{t_0^{q_1-r}}{\Gamma(q_1)} \int_0^1 (1-\tau)^{q_1-1} g(t_0\tau, x(t_0\tau))d\tau| \\
\leq \ & \frac{|t^{q_1-r} - t_0^{q_1-r}|}{\Gamma(q_1)} \int_0^1 (1-\tau)^{q_1-1} |g(t\tau, x(t\tau))|d\tau \\
+ \ & \frac{t_0^{q_1-r}}{\Gamma(q_1)} \int_0^1 (1-\tau)^{q_1-1} |g(t\tau, x(t\tau) - g(t_0\tau, x(t_0\tau))|d\tau.
\end{aligned}$$

Together with continuity of functions g and t^{q_1-r}, we could easily obtain $Tx(t) \in C[0, T_1]$.

For $x(t), y(t) \in C[0, T_1]$, we obtain that

$$\begin{aligned}
t^{2-q_1}|Tx(t) - Ty(t| &= t^{2-q_1} \mid \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1}(f(s, x(s)) - f(s, y(s)))ds \mid \\
&\leq t^{2-q_1} \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} \mid f(s, x(s)) - f(s, y(s) \mid ds \\
&\leq t^{2-q_1} \frac{L}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} s^{-r} \mid x(s) - y(s) \mid ds
\end{aligned}$$

$$\begin{aligned}
&\leq t^{2-q_1}\frac{L}{\Gamma(q_1)}\int_0^t (t-s)^{q_1-1}s^{q_1-r-2} \parallel x-y \parallel ds\\
&= L\frac{\Gamma(q_1-r-1)}{\Gamma(2q_1-r-1)}t^{q_1-r} \parallel x-y \parallel\\
&\leq L\frac{\Gamma(q_1-r-1)}{\Gamma(2q_1-r-1)}T_1^{q_1-r} \parallel x-y \parallel .
\end{aligned}$$

According to $\frac{LT_1^{q_1-r}\Gamma(q_1-r-1)}{\Gamma(2q_1-r-1)} < 1$, the Banach contraction mapping principle assures that operator T has one unique fixed point $x_1(t) \in C[0, T_1]$. Obviously, we could get $t^{2-q_1}x_1(0) = 0$, So, $x_1(t)$ is one unique solution of Eq.(3.4) with initial value condition $t^{2-q_1}x(0) = 0$.

Also we have obtained that Eq.(3.3) in the interval $(T_1, T_2]$ can be written by (3.5). In order to consider the existence result of solution to (3.5), we may discuss the following equation defined on interval $(T_1, T_2]$

$$\left(\frac{d}{dt}\right)^2 \int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds = D_{0+}^{q_2}x(t)) = f(t, x(t)), \quad 0 < t \leq T_2. \tag{3.9}$$

It is clear that if function $x \in C[0, T_2]$ satisfies Eq.(3.9), then $x(t)$ must satisfy Eq.(3.5). In fact, if $x^* \in C[0, T_2]$ with $t^{2-q_2}x^*(0) = 0$ is a solution of Eq.(3.9) with initial value condition $t^{2-q_2}x(0) = 0$, that is

$$D_{0+}^{q_2}x^*(t)) = \left(\frac{d}{dt}\right)^2 \int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x^*(s)ds = f(t, x^*(t)), \quad 0 < t \leq T_2;\ t^{2-q_2}x^*(0) = 0.$$

Hence, form the above equality, it holds that

$$\left(\frac{d}{dt}\right)^2 \int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x^*(s)ds = f(t, x^*(t)), \quad T_1 < t \leq T_2.$$

As a result, we have that $x^* \in C[0, T_2]$ with $t^{2-q_2}x^*(0) = 0$ satisfies equation

$$\left(\frac{d}{dt}\right)^2 \int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds = f(t, x(t)), \quad T_1 < t \leq T_2,$$

which means the the function $x^* \in C[0, T_2]$ with $t^{2-q_2}x^*(0) = 0$ is a solution of Eq.(3.5).

Based on this fact, we will consider existence of solution to Eq.(3.9) with initial value condition $t^{2-q_2}x(0) = 0$.

Now, applying operator $I_{0+}^{q_2}$ to both sides of (3.9), by Propositions 6, we have that

$$x(t) = c_1t^{q_2-1} + c_2t^{q_2-2} + \frac{1}{\Gamma(q_2)}\int_0^t (t-s)^{q_2-1}f(s, x(s))ds, \quad 0 < t \leq T_2.$$

By initial value condition $t^{2-q_2}x(0) = 0$, we have $c_1 = 0, c_2 = 0$.

Define operator $T : C[0,T_2] \to C[0,T_2]$ by

$$Tx(t) = \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} f(s,x(s))ds, \quad 0 < t \le T_2.$$

From the previous arguments, it follows from the continuity of function $t^r f(t,x(t))$ that operator $T : C[0,T_2] \to C[0,T_2]$ is well defined.
For $u(t), v(t) \in C[0,T_2]$, we obtain that

$$\begin{aligned}
t^{2-q_2}|Tu(t) - Tv(t| &= t^{2-q_2} \mid \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1}(f(s,u(s)) - f(s,v(s)))ds \mid \\
&\le t^{2-q_2} \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} \mid f(s,u(s)) - f(s,v(s) \mid ds \\
&\le t^{2-q_2} \frac{L}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} s^{-r} \mid u(s) - v(s) \mid ds \\
&\le t^{2-q_2} \frac{L}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} s^{q_2-r-2} \parallel u - v \parallel ds \\
&= L \frac{\Gamma(q_2 - r - 1)}{\Gamma(2q_2 - r - 1)} t^{q_2-r} \parallel u - v \parallel \\
&\le L \frac{\Gamma(q_2 - r - 1)}{\Gamma(2q_2 - r - 1)} T_2^{q_2-r} \parallel u - v \parallel .
\end{aligned}$$

By condition (H2), the the Banach contraction mapping principle assures that operator T has one unique fixed point $x_2(t) \in [0,T_2]$, that is

$$x_2(t) = \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} f(s,x(s))ds, \quad 0 < t \le T_2. \tag{3.10}$$

From (3.10), we know $t^{2-q_2}x_2(0) = 0$. Applying operator $D_{0+}^{q_2}$ on both sides of (3.10), by Propositions 4, we can obtain that

$$D_{0+}^{q_2} x(t)) = f(t,x(t)), \quad 0 < t \le T_2,$$

that is, $x_2(t)$ satisfies equation as following

$$\left(\frac{d}{dt}\right)^2 \int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)} x_2(s)ds = f(t,x_2(t)), \quad 0 < t \le T_2, \quad t^{2-q_2}x_2(0) = 0.$$

Form the previous arguments, we obtain $x_2(t) \in [0,T_2]$ with $t^{2-q_2}x_2(0) = 0$ satisfies Eq.(3.5).

By the similar way, for $i = 3, 4, \cdots, N^*$, we could get that Eq.(3.3) defined on $(T_{i-1}, T_i]$ has one unique solutions $x_i(t) \in [0,T_i]$ with $t^{2-q_i}x_i(0) = 0$ ($T_{N^*} = T$).

As a result, we obtain that problem (1.1) has one unique solution. The proof is completed. $\square$

Theorem 15. *Assume that conditions (H1) and (H3) hold, then problem (1.1) has one unique solution.*

Proof. The proof is similar to Theorem 14. By (H1), we have obtained that equation of problem (1.1) can be written by (3.3). And (3.3) in the interval $[0, T_1]$ can be written by (3.4). Applying the operator $I_{0+}^{q_1}$ to both sides of (3.4), by Propositions 6, we have

$$x(t) = c_1 t^{q_1-1} + c_2 t^{q_1-2} + \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} f(s, x(s)) ds, \quad 0 < t \le T_1.$$

By initial value condition $t^{2-q_1}x(0) = 0$ and the assumption of function f, we could get $c_1 = 0, c_2 = 0$.

Define operator $T : C[0, T_1] \to C[0, T_1]$ by

$$Tx(t) = \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} f(s, x(s)) ds, \quad 0 < t \le T_1.$$

From the previous arguments, it follows from the continuity of function $t^r f(t, x(t))$ that operator $T : C[0, T_1] \to C[0, T_1]$ is well defined.

Let $\Omega_1 = \{x \in C[0, T_1 : \|x\| \le R_1]\}$ be a close subset of $C[0, T_1]$, where R_1 is a positive constant satisfying

$$\frac{4 \max\limits_{0 \le t \le T_1} t^r |f(t, 0)| \Gamma(q_1 - r - 1) T_1^{q_1 - r}}{\Gamma(2q_1 - r - 1)} < 2R_1 < \Big(\frac{\Gamma(2q_1 - r - 1)}{2M\Gamma(q_1 - r - 1) T_1^{q_1 - r}} \Big)^{\frac{1}{\mu - 1}}.$$

Obviously, Ω_1 is a Banach space with the metric in $C[0, T_1]$.
For $x(t) \in \Omega_1$, by (H3), we have that

$$\begin{aligned}
t^{2-q_1}|Tx(t)| &= t^{2-q_1} \mid \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} f(s, x(s)) ds \mid \\
&\le t^{2-q_1} \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} \mid f(s, x(s)) \mid ds \\
&\le t^{2-q_1} \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} (\mid f(s, x(s)) - f(s, 0) \mid + \mid f(s, 0) \mid) ds \\
&\le t^{2-q_1} \frac{M}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} s^{-r} \mid x(s) \mid^{\mu} ds \\
&+ t^{2-q_1} \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} s^{-r} \mid f(s, 0) \mid) ds \\
&\le t^{2-q_1} \frac{M}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} s^{q_1-r-2} \mid x \mid^{\mu} ds \\
&+ t^{2-q_1} \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} s^{q_1-r-2} \mid f(s, 0) \mid) ds
\end{aligned}$$

$$
\begin{aligned}
&\leq M\frac{\Gamma(q_1-r-1)}{\Gamma(2q_1-r-1)}T_1^{q_1-r}R_1R_1^{\mu-1}\\
&+\frac{\max\limits_{0\leq t\leq T_1} t^r|f(t,0)|\Gamma(q_1-r-1)T_1^{q_1-r}}{\Gamma(2q_1-r-1)}\\
&\leq M\frac{\Gamma(q_1-r-1)}{\Gamma(2q_1-r-1)}T_1^{q_1-r}2^{1-\mu}R_1(2R_1)^{\mu-1}\\
&+\frac{\max\limits_{0\leq t\leq T_1} t^r|f(t,0)|\Gamma(q_1-r-1)T_1^{q_1-r}}{\Gamma(2q_1-r-1)}\\
&\leq 2^{1-\mu}\frac{R_1}{2}+\frac{R_1}{2}\\
&\leq \frac{R_1}{2}+\frac{R_1}{2}\\
&= R_1,
\end{aligned}
$$

which implies $T:\Omega_1\to\Omega_1$ is well defined.
For $x(t),y(t)\in\Omega_1$, by (H3), we get

$$
\begin{aligned}
t^{2-q_1}|Tx(t)-Ty(t| &= t^{2-q_1}\mid\frac{1}{\Gamma(q_1)}\int_0^t(t-s)^{q_1-1}(f(s,x(s))-f(s,y(s)))ds\mid\\
&\leq t^{2-q_1}\frac{1}{\Gamma(q_1)}\int_0^t(t-s)^{q_1-1}\mid f(s,x(s))-f(s,y(s)\mid ds\\
&\leq t^{2-q_1}\frac{M}{\Gamma(q_1)}\int_0^t(t-s)^{q_1-1}s^{-r}\mid x(s)-y(s)\mid^{\mu} ds\\
&\leq t^{2-q_1}\frac{M}{\Gamma(q_1)}\int_0^t(t-s)^{q_1-1}s^{q_1-r-2}\parallel x-y\parallel^{\mu} ds\\
&\leq M\frac{\Gamma(q_1-r-1)}{\Gamma(2q_1-r-1)}T_1^{q_1-r}\parallel x-y\parallel\parallel x-y\parallel^{\mu-1}\\
&\leq M\frac{\Gamma(q_1-r-1)}{\Gamma(2q_1-r-1)}T_1^{q_1-r}(\parallel x\parallel-\parallel y\parallel)^{\mu-1}\parallel x-y\parallel\\
&\leq M\frac{\Gamma(q_1-r-1)}{\Gamma(2q_1-r-1)}T_1^{q_1-r}(2R_1)^{\mu-1}\parallel x-y\parallel\\
&< \frac{1}{2}\parallel x-y\parallel.
\end{aligned}
$$

Hence T is contraction operator. Therefore, the Banach contraction mapping principle assures that T has one unique fixed point $x_1(t)\in C[0,T_1]$. Obviously, we could get $t^{2-q_1}x_1(0)=0$, So, $x_1(t)$ is one unique solution of Eq.(3.4) with initial value condition $t^{2-q_1}x(0)=0$.

Equation (3.3) in the interval $(T_1,T_2]$ can be written by (3.5) defined by

$$\left(\frac{d}{dt}\right)^2\int_0^t\frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds=f(t,x(t)),\quad T_1<t\leq T_2.$$

In order to consider the existence result of solution to Eq.(3.5), we may consider the following equation defined on interval $(0, T_2]$

$$\left(\frac{d}{dt}\right)^2 \int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)} x(s)ds = D_{0+}^{q_2} x(t)) = f(t, x(t)), \quad 0 < t \leq T_2. \tag{3.11}$$

By the same arguments as done in Theorem 14, we see that, if function $x \in C[0, T_2]$ satisfies Eq.(3.11), then $x(t)$ must satisfy Eq.(3.5). So, we will consider the existence of solution of Eq.(3.11) with initial value condition $t^{2-q_2}x(0) = 0$.
Now, applying operator $I_{0+}^{q_2}$ to both sides of (3.11), by Propositions 6, we have

$$x(t) = c_1 t^{q_2-1} + c_2 t^{q_2-2} + \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} f(s, x(s))ds, \quad 0 < t \leq T_2.$$

By $t^{2-q_2}x(0) = 0$, we have $c_1 = 0, c_2 = 0$.

Define operator $T : C[0, T_2] \to C[0, T_2]$ by

$$Tx(t) = \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} f(s, x(s))ds, \quad 0 < t \leq T_2.$$

Also, it follows from the continuity of function $t^r f(t, x(t))$ that operator $T : C[0, T_2] \to C[0, T_2]$ is well defined.

Let $\Omega_2 = \{x \in C[0, T_2 : \|x\| \leq R_2]\}$ be a close subset of $C[0, T_2]$, where R_2 is a positive constant satisfying

$$\frac{4 \max\limits_{0 \leq t \leq T_2} t^r |f(t, 0)| \Gamma(q_2 - r - 1) T_2^{q_2-r}}{\Gamma(2q_2 - r - 1)} < 2R_2 < \left(\frac{\Gamma(2q_2 - r - 1)}{2M\Gamma(q_2 - r - 1)T_2^{q_2-r}}\right)^{\frac{1}{\mu-1}}.$$

It is clear that Ω_2 is a Banach space with the metric in $C[0, T_2]$.
For $x(t) \in \Omega_2$, by (H3), we have For $x(t), y(t) \in \Omega_1$, by (H3), we get

$$\begin{aligned}
t^{2-q_2}|Tx(t)| &= t^{2-q_2} \mid \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} f(s, x(s))ds \mid \\
&\leq t^{2-q_2} \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} \mid f(s, x(s)) \mid ds \\
&\leq t^{2-q_2} \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} (\mid f(s, x(s)) - f(s, 0) \mid + \mid f(s, 0) \mid)ds \\
&\leq t^{2-q_2} \frac{M}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} s^{-r} \mid x(s) \mid^{\mu} ds \\
&+ t^{2-q_2} \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} s^{-r} \mid f(s, 0) \mid)ds \\
&\leq t^{2-q_2} \frac{M}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} s^{q_2-r-2} \mid x \mid^{\mu} ds \\
&+ t^{2-q_2} \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} s^{q_2-r-2} \mid f(s, 0) \mid)ds
\end{aligned}$$

$$\begin{aligned}
&\leq M\frac{\Gamma(q_2-r-1)}{\Gamma(2q_2-r-1)}T_2^{q_2-r}R_2R_2^{\mu-1}\\
&+\frac{\max\limits_{0\leq t\leq T_2} t^r|f(t,0)|\Gamma(q_2-r-1)T_2^{q_2-r}}{\Gamma(2q_2-r-1)}\\
&\leq M\frac{\Gamma(q_2-r-1)}{\Gamma(2q_2-r-1)}T_2^{q_2-r}2^{1-\mu}R_2(2R_2)^{\mu-1}\\
&+\frac{\max\limits_{0\leq t\leq T_2} t^r|f(t,0)|\Gamma(q_2-r-1)T_2^{q_2-r}}{\Gamma(2q_2-r-1)}\\
&\leq 2^{1-\mu}\frac{R_2}{2}+\frac{R_2}{2}\\
&\leq \frac{R_2}{2}+\frac{R_2}{2}\\
&= R_2,
\end{aligned}$$

which implies that $T:\Omega_2\to\Omega_2$ is well defined.

For $x(t),y(t)\in\Omega_2$, by (H3), we have

$$\begin{aligned}
t^{2-q_2}|Tx(t)-Ty(t| &= t^{2-q_2}\mid\frac{1}{\Gamma(q_2)}\int_0^t(t-s)^{q_2-1}(f(s,x(s))-f(s,y(s)))ds\mid\\
&\leq t^{2-q_2}\frac{1}{\Gamma(q_2)}\int_0^t(t-s)^{q_2-1}\mid f(s,x(s))-f(s,y(s)\mid ds\\
&\leq t^{2-q_2}\frac{M}{\Gamma(q_2)}\int_0^t(t-s)^{q_2-1}s^{-r}\mid x(s)-y(s)\mid^{\mu} ds\\
&\leq t^{2-q_2}\frac{M}{\Gamma(q_2)}\int_0^t(t-s)^{q_2-1}s^{q_2-r-2}\parallel x-y\parallel^{\mu} ds\\
&\leq M\frac{\Gamma(q_2-r-1)}{\Gamma(2q_2-r-1)}T_2^{q_2-r}\parallel x-y\parallel\parallel x-y\parallel^{\mu-1}\\
&\leq M\frac{\Gamma(q_2-r-1)}{\Gamma(2q_2-r-1)}T_2^{q_2-r}(\parallel x\parallel-\parallel y\parallel)^{\mu-1}\parallel x-y\parallel\\
&\leq M\frac{\Gamma(q_2-r-1)}{\Gamma(2q_2-r-1)}T_2^{q_2-r}(2R_2)^{\mu-1}\parallel x-y\parallel\\
&< \frac{1}{2}\parallel x-y\parallel.
\end{aligned}$$

Hence, T is contraction operator. Thus, the Banach contraction mapping principle assures that T has one unique fixed point $x_2(t)\in\Omega_2$, that is

$$x_2(t)=\frac{1}{\Gamma(q_2)}\int_0^t(t-s)^{q_2-1}f(s,x(s))ds,\quad 0<t\leq T_2. \tag{3.12}$$

By (3.12), we get $t^{2-q_2}x_2(0)=0$. Applying operator $D_{0+}^{q_2}$ on both sides of (3.12), by Propositions 4, we obtain

$$D_{0+}^{q_2}x(t))=f(t,x(t)),\quad 0<t\leq T_2.$$

that is, $x_2(t)$ satisfies equation as following

$$\Big(\frac{d}{dt}\Big)^2\int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x_2(s)ds = f(t,x_2(t)),\quad 0<t\le T_2,\quad t^{2-q_2}x_2(0)=0.$$

Form the previous arguments, we obtain $x_2(t)\in\Omega$ satisfies Eq.(3.5).

By the similar way, we get that Eq.(3.3) defined on $(T_{i-1},T_i]$ has the solutions $x_i\in\Omega_i=\{x\in C[0,T_i:\|x\|\le R_i]\}$ with $t^{2-q_i}x_i(0)=0$, where R_i is a positive constant satisfying

$$\frac{4\max\limits_{0\le t\le T_i} t^r|f(t,0)|\Gamma(q_i-r-1)T_i^{q_i-r}}{\Gamma(2q_i-r-1)} < 2R_i < \Big(\frac{\Gamma(2q_i-r-1)}{2M\Gamma(q_i-r-1)T_i^{q_i-r}}\Big)^{\frac{1}{\mu-1}},$$

$i=3,4,\cdots,N^*(T_{N^*}=T)$. As a result, we obtain that problem (1.1) has one unique solution. The we completed the proof. □

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