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DELAY-DEPENDENT EXPONENTIAL STABILITY CRITERIA FOR CERTAIN NEUTRAL TIME-DELAY EQUATION VIA MODEL TRANSFORMATIONS APPROACH

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ABSTRACT: This paper deals with the problem of delay-dependent exponential stability analysis for certain neutral differential equation with time-varying delays by using mixed model transformations. Based on new class of augmented Lyapunov-Krasovskii functional, Leibniz-Newton formula, utilization of zero equation, Wirtinger-based integral inequality and Peng-Park's integral inequality, improved delay-dependent exponential stability criteria are derived in terms of linear matrix inequalities (LMIs) for the proposed equation. Numerical examples are given to illustrate the effectiveness and improvement over some existing methods.

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Key Words: model transformation, neutral differential equation, exponential stability, Wirtinger-based integral inequality, Peng-Park's integral inequality

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1. INTRODUCTION

In the past several years, the problem of various stability analysis for uncertain neutral systems with delays has been intensively considered by several researchers [2]-[45].

Since neutral delayed systems (equations) have already been applied in many fields, such as population ecology, distributed networks containing lossless transmission lines, propagation and diffusion models, and partial element equivalent circuits in very large scale integration systems [25]. Furthermore, the delay-dependent stability criteria of certain neutral differential equations (CNDE) have been received considerable attention in recent years. Delay-dependent asymptotic stability criteria for CNDE with constant delays have greatly been discussed in [10, 22, 27, 29, 35, 41] by using several model transformation method and Lyapunov-Krasovskii functional approach, while the problem of exponential stability analysis has been studied with model transformation method in [35]. In [7, 8, 20], the authors investigated the problem of exponential stability analysis for CNDE with time-varying delays by several methods. In [7], the results are derived without the use of the model transformation methods and the bounding technique, but using model transformation method, radially unboundedness and Lyapunov-Krasovskii functional approach are presented in [20].

In the existing literatures, the CNDE with constant delays has extremely been considered in [10, 22, 27, 29, 35, 41] of the form

$$\frac{d}{dt}[x(t) + px(t - \tau_2)] = -ax(t) + b \tanh x(t - \sigma_2), \quad t \geq 0, \tag{1}$$

where a, b, τ_2, σ_2 are positive real constants and $|p| < 1$. For each solution $x(t)$ of (1), we assume the initial condition

$$x_0(t) = \phi(t), \quad t \in [-\omega, 0],$$

where $\phi \in C([-\omega, 0]; R)$; $C([-\omega, 0], R)$ denotes the space of all continuous vector functions mapping $[-\omega, 0]$ into R when $\omega = \max\{\sigma_2, \tau_2\} \in R^+$. The asymptotic stability of (1) has greatly been discussed in [10, 22, 27, 29, 35, 41].

More recently, the authors studied the problem of exponential stability analysis for CNDE with time-varying delays in [7, 8, 20]

$$\frac{d}{dt}[x(t) + px(t - \tau(t))] = -ax(t) + b \tanh x(t - \sigma(t)), \quad t \geq 0, \tag{2}$$

where a, b are positive real constants and $|p| < 1$. $\tau(t)$ and $\sigma(t)$ are neutral and discrete time-varying delays, respectively,

$$0 \leq \tau(t) \leq \tau_2, \quad \dot{\tau}(t) < \tau_d, \tag{3}$$

$$0 \leq \sigma(t) \leq \sigma_2, \quad \dot{\sigma}(t) < \sigma_d, \tag{4}$$

where τ_2, σ_2, τ_d and σ_d are given positive real constants. For each solution $x(t)$ of (2), we assume the initial condition

$$x_0(t) = \phi(t), \quad t \in [-\omega, 0], \tag{5}$$

where $\phi \in C([-\omega, 0]; R)$.

Motivated by above discussions, the main objective of this paper aim to study the delay-dependent exponential stability problem for CNDE with time-varying delays. Main contribution of our study is the following. By using the combinations of mixed model transformations, mixed integral inequalities, mixed utilization of zero equations and new Lyapunov-Krasovskii functional, improved delay-dependent exponential stability criteria are obtained and formulated in terms of LMIs for the equation [8, 20]. Finally, numerical examples are shown to illustrate the effectiveness and benifits of the proposed methods.

2. PRELIMINARIES

Definition 2.1. [21] The equation (2) with (3)-(5) is said to be *exponentially stable*, if there exist positive real constants α, β such that for each $\phi(t) \in C([-\omega, 0], R)$, the solution $x(t, \phi)$ of the equation satisfies

$$\|x(t,\phi)\| \leq \beta \|\phi\| e^{-\alpha t}, \quad t \geq 0.$$

Lemma 2.1. *[37] For any constant matrix $Q \in R^{n\times n}$, $Q = Q^T > 0$, positive real constants k_1, k_2 and a vector-valued function $\dot{x} : [-k_2, 0] \to R^n$ such that the following integrals are well-defined, we have*

$$\begin{aligned} -k_2 \int_{t-k_2}^{t} \dot{x}^T(s) Q \dot{x}(s) ds &\leq -\Big(\int_{t-k_2}^{t} \dot{x}(s) ds\Big)^T Q \Big(\int_{t-k_2}^{t} \dot{x}(s) ds\Big), \\ &- \frac{(k_2^2 - k_1^2)}{2} \int_{-k_2}^{-k_1} \int_{t+s}^{t} x^T(u) Q x(u) du ds \\ &\leq -\Big(\int_{-k_2}^{-k_1} \int_{t+s}^{t} x(u) du ds\Big)^T Q \Big(\int_{-k_2}^{-k_1} \int_{t+s}^{t} x(u) du ds\Big). \end{aligned}$$

Lemma 2.2. *[38] For any constant matrices $Q_1, Q_2, Q_3 \in R^{n\times n}$, $Q_1 \geq 0, Q_3 > 0$, $\begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \geq 0$, $k(t)$ is time-varying delay with $0 \leq k_1 \leq k(t) \leq k_2$, $k_1, k_2 \in R$, vector-valued functions x and $\dot{x} : [-k_2, -k_1] \to R^n$ such that the following integration is well-defined, we have*

$$- (k_2 - k_1) \int_{t-k_2}^{t-k_1} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds$$

$$\leq \begin{bmatrix} x(t-k_1) \\ x(t-k(t)) \\ x(t-k_2) \\ \int_{t-k(t)}^{t-k_1} x(s)ds \\ \int_{t-k_2}^{t-k(t)} x(s)ds \end{bmatrix}^T \begin{bmatrix} -Q_3 & Q_3 & 0 & -Q_2^T & 0 \\ * & -Q_3 - Q_3^T & Q_3 & Q_2^T & -Q_2^T \\ * & * & -Q_3 & 0 & Q_2^T \\ * & * & * & -Q_1 & 0 \\ * & * & * & * & -Q_1 \end{bmatrix} \begin{bmatrix} x(t-k_1) \\ x(t-k(t)) \\ x(t-k_2) \\ \int_{t-k(t)}^{t-k_1} x(s)ds \\ \int_{t-k_2}^{t-k(t)} x(s)ds \end{bmatrix}.$$

Lemma 2.3. *[38] Let $x(t) \in R^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any constant matrices $Q, M_i \in R^{n\times n}, i = 1, 2, \ldots, 5$ and $k(t)$ is time-varying delays with $0 \leq k_1 \leq k(t) \leq k_2$, $k_1, k_2 \in R^+$,*

$$\begin{aligned} -\int_{t-k_2}^{t-k_1} \dot{x}^T(s)Q\dot{x}(s)ds \quad \leq \quad & \begin{bmatrix} x(t-k_1) \\ x(t-k(t)) \\ x(t-k_2) \end{bmatrix}^T \\ & \times \begin{bmatrix} M_1 + M_1^T & -M_1^T + M_2 & 0 \\ * & M_1 + M_1^T - M_2 - M_2^T & -M_1^T + M_2 \\ * & * & -M_2 - M_2^T \end{bmatrix} \\ & \times \begin{bmatrix} x(t-k_1) \\ x(t-k(t)) \\ x(t-k_2) \end{bmatrix} + [h_2 - h_1] \begin{bmatrix} x(t-k_1) \\ x(t-k(t)) \\ x(t-k_2) \end{bmatrix}^T \\ & \times \begin{bmatrix} M_3 & M_4 & 0 \\ * & M_3 + M_5 & M_4 \\ * & * & M_5 \end{bmatrix} \begin{bmatrix} x(t-k_1) \\ x(t-k(t)) \\ x(t-k_2) \end{bmatrix}, \end{aligned}$$

where

$$\begin{bmatrix} Q & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0.$$

Lemma 2.4. *[36] For any constant matrix $Q \in R^{n\times n}$, $Q = Q^T > 0$, nonnegative real constants k_1, k_2 and a vector-valued function $\dot{x} : [-k_2, -k_1] \to R^n$ such that the following integrals are well-defined, we have*

$$-(k_2 - k_1) \int_{t-k_2}^{t-k_1} \dot{x}^T(s)Q\dot{x}(s)ds \leq \omega^T \ominus \omega,$$

where $\omega = \left[x^T(t-k_1), x^T(t-k_2), \frac{1}{k_2-k_1} \int_{t-k_2}^{t-k_1} x^T(s)ds\right]^T$ and

$$\ominus = \begin{bmatrix} -4Q & -2Q & 6Q \\ * & -4Q & 6Q \\ * & * & -12Q \end{bmatrix}.$$

Lemma 2.5. *[30, 32] For any constant matrices $Q, S \in R^{n\times n}$, $Q \geq 0$, $\begin{bmatrix} Q & S \\ * & Q \end{bmatrix} \geq 0$, $h(t)$ is time-varying delay with $0 \leq k(t) \leq k_2$, $k_2 \in R^+$, a vector-valued function $\dot{x} : [-k_2, 0] \to R^n$ such that the concerned integrations are well-defined, we have*

$$-k_2 \int_{t-k_2}^{t} \dot{x}^T(s) Q \dot{x}(s) ds \leq \omega^T(t) \ominus \omega(t),$$

where $\omega(t) = \left[x^T(t), x^T(t-k(t)), x^T(t-k_2)\right]^T$ and

$$\ominus = \begin{bmatrix} -Q & Q-S & S \\ * & -2Q+S+S^T & Q-S \\ * & * & -Q \end{bmatrix}.$$

3. MAIN RESULTS

In this section, the following two theorems present the improved delay-dependent exponential stability criteria for Equation (2) with (3)-(5) via two model transformations. Firstly, we consider the Leibniz-Newton formula of the form

$$0 = x(t) - x(t-\sigma(t)) - \textstyle\int_{t-\sigma(t)}^{t} \dot{x}(s) ds. \tag{6}$$

By utilizing the following zero equation, we obtain

$$0 = v_1 x(t) - v_1 x(t-\sigma(t)) - v_1 \textstyle\int_{t-\sigma(t)}^{t} \dot{x}(s) ds, \tag{7}$$

where $v_1 \in R$ will be chosen to guarantee the exponential stability of Equation (2). By descriptor system and (7), the CNDE (2) can be represented by the form

$$\dot{x}(t) = y(t) + v_1 x(t) - v_1 x(t-\sigma(t)) - v_1 \int_{t-\sigma(t)}^{t} y(s) ds, \tag{8}$$

$$0 = -y(t) - ax(t) + b \tanh x(t-\sigma(t)) - py(t-\tau(t)). \tag{9}$$

We introduce the following notation for later use:

$$\sum = \left[\Lambda_{(i,j)}\right]_{15\times 15}, \tag{10}$$

where $\Lambda_{(i,j)} = \Lambda_{(j,i)}$,

$$\begin{aligned} \Lambda_{(1,1)} = {} & 2k_1 v_1 - 2q_2 a + 2q_5 + k_2 + k_3 + k_4 + k_5 + k_6 \sigma_2^2 + 2m_1 \\ & + \sigma_2 m_3 - 4k_8 e^{-2\alpha\sigma_2} - k_9 e^{-2\alpha\sigma_2} + r_1 \sigma_2^2 - r_3 e^{-2\alpha\sigma_2} \end{aligned}$$

$$
\begin{aligned}
& & +k_{10}\sigma_2^2 - k_{12}\sigma_2^2 e^{-4\alpha\sigma_2} + k_{13}\frac{\sigma_2^4}{4}, \\
\Lambda_{(1,2)} &=& k_1 - q_1 a - q_2 + r_2\sigma_2^2, \\
\Lambda_{(1,3)} &=& -k_1 v_1 - q_5 + q_6 - m_1 + m_2 + m_4\sigma_2 + k_9 e^{-2\alpha\sigma_2} - s + r_3 e^{-2\alpha\sigma_2}, \\
\Lambda_{(1,4)} &=& -k_1 v_1 + q_7 - q_5, \\
\Lambda_{(1,5)} &=& bq_2 - q_3 a, \\
\Lambda_{(1,6)} &=& -q_2 p - q_4 a, \\
\Lambda_{(1,7)} &=& -2k_8 e^{-2\alpha\sigma_2} + s, \\
\Lambda_{(1,9)} &=& 6k_8 e^{-2\alpha\sigma_2}, \\
\Lambda_{(1,10)} &=& -r_2 e^{-2\alpha\sigma_2}, \\
\Lambda_{(1,14)} &=& k_{12}\sigma_2 e^{-4\alpha\sigma_2}, \\
\Lambda_{(2,1)} &=& k_1 - q_1 a - q_2 + r_2\sigma_2^2, \\
\Lambda_{(2,2)} &=& -2q_1 + r_3\sigma_2^2 + k_{11}\frac{\sigma_2^4}{4} + k_{12}\frac{\sigma_2^4}{4}, \\
\Lambda_{(2,5)} &=& q_1 b - q_3, \\
\Lambda_{(2,6)} &=& -q_1 p - q_4, \\
\Lambda_{(3,1)} &=& -k_1 r_1 - q_5 + q_6 - m_1 + m_2 + m_4\sigma_2 + k_9 e^{-2\alpha\sigma_2} - s + r_3 e^{-2\alpha\sigma_2}, \\
\Lambda_{(3,3)} &=& -2q_6 - k_2 e^{-2\alpha\sigma_2} + k_2\sigma_d + 2m_1 - 2m_2 + m_5\sigma_2 \\
& & -2k_9 e^{-2\alpha\sigma_2} + 2s - 2r_3 e^{-2\alpha\sigma_2} \\
\Lambda_{(3,4)} &=& -q_6 - q_7, \\
\Lambda_{(3,7)} &=& -m_1 + m_2 + m_4\sigma_2 + k_9 e^{-2\alpha\sigma_2} - s + r_3 e^{-2\alpha\sigma_2}, \\
\Lambda_{(3,10)} &=& r_2 e^{-2\alpha\sigma_2}, \\
\Lambda_{(3,11)} &=& -r_2 e^{-2\alpha\sigma_2}, \\
\Lambda_{(4,1)} &=& -k_1 r_1 + q_7 - q_5, \\
\Lambda_{(4,3)} &=& -q_6 - q_7, \\
\Lambda_{(4,4)} &=& -2q_7, \\
\Lambda_{(5,1)} &=& q_2 b - q_3 a, \\
\Lambda_{(5,2)} &=& q_1 b - q_3, \\
\Lambda_{(5,5)} &=& 2q3b - k4e^{-2\alpha\sigma_2} + k4\sigma_d, \\
\Lambda_{(5,6)} &=& -q_3 p + q_4 b, \\
\Lambda_{(6,1)} &=& -q_2 p - q_4 a, \\
\Lambda_{(6,2)} &=& -q_1 p - q_4, \\
\Lambda_{(6,5)} &=& -q_3 p + q_4 b, \\
\Lambda_{(6,6)} &=& -2q_4 p - k_{14} e^{-2\alpha\tau} + k_{14}\tau_d,
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{(7,1)} &= -2k_8e^{-2\alpha\sigma_2} + s,\\
\Lambda_{(7,3)} &= -m_1 + m_2 + m_4\sigma_2 + k_9e^{-2\alpha\sigma_2} - s + r_3e^{-2\alpha\sigma_2},\\
\Lambda_{(7,7)} &= -k_3e^{-2\alpha\sigma_2} - 2m_2 + m_5\sigma_2 - 4k_8e^{-2\alpha\sigma_2} - k_9e^{-2\alpha\sigma_2} - r_3e^{-2\alpha\sigma_2},\\
\Lambda_{(7,9)} &= 6k_8e^{-2\alpha\sigma_2},\\
\Lambda_{(7,11)} &= r_2e^{-2\alpha\sigma_2},\\
\Lambda_{(8,8)} &= -k_5e^{-2\alpha\sigma_2},\\
\Lambda_{(9,1)} &= 6k_8e^{-2\alpha\sigma_2},\\
\Lambda_{(9,7)} &= 6k_8e^{-2\alpha\sigma_2},\\
\Lambda_{(9,10)} &= -k_6\sigma_2^2e^{-2\alpha\sigma_2} - 12k_8e^{-2\alpha\sigma_2},\\
\Lambda_{(10,1)} &= -r_2e^{-2\alpha\sigma_2}, \qquad \Lambda_{(10,3)} = r_2e^{-2\alpha\sigma_2},\\
\Lambda_{(10,10)} &= -r_1e^{-2\alpha\sigma_2}, \qquad \Lambda_{(11,3)} = -r_2e^{-2\alpha\sigma_2},\\
\Lambda_{(11,7)} &= r_2e^{-2\alpha\sigma_2}, \qquad \Lambda_{(11,11)} = -r_1e^{-2\alpha\sigma_2},\\
\Lambda_{(12,12)} &= -k_{10}e^{-2\alpha\sigma_2}, \qquad \Lambda_{(13,13)} = -k_{11}e^{-4\alpha\sigma_2},\\
\Lambda_{(14,1)} &= k_{12}\sigma_2e^{-4\alpha\sigma_2}, \qquad \Lambda_{(14,14)} = -k_{12}e^{-4\alpha\sigma_2}, \qquad \Lambda_{(15,15)} = -k_{13}e^{-4\alpha\sigma_2},
\end{aligned}
$$

and the other terms are 0.

Theorem 3.1. *For given positive real constants $a, b, \sigma_2, \sigma_d, \tau_2$ and τ_d, Equation (2) with (3)-(5) is exponentially stable with a decay rate α, if $|p| < 1$ and there exists positive real constants k_i where $i = 1, 2, \ldots, 14$, and real constants $s, v_1, r_1, r_2, r_3, q_j, m_k$ where $j = 1, 2, \ldots, 7, k = 1, 2, \ldots, 5$ such that the following symmetric linear matrix inequalities hold*

$$\sum < 0, \tag{11}$$

$$\begin{bmatrix} r_1 & r_2 \\ * & r_3 \end{bmatrix} \geq 0, \tag{12}$$

$$\begin{bmatrix} k_9e^{-2\alpha\sigma_2} & s \\ * & k_9e^{-2\alpha\sigma_2} \end{bmatrix} \geq 0, \tag{13}$$

$$\begin{bmatrix} k_7e^{-2\alpha\sigma_2} & m_1 & m_2 \\ * & m_3 & m_4 \\ * & * & m_5 \end{bmatrix} \geq 0. \tag{14}$$

Proof. For k_i are positive real constants where $i = 1, 2, \ldots, 14$, consider the Lyapunov-Krasovskii functional candidate for Equations (8)-(9) with (3)-(5) of the form

$$V(t) = \sum_{i=1}^{9} V_i(t), \tag{15}$$

where

$$\begin{aligned}
V_1(t) &= k_1 x^2(t), \\
V_2(t) &= k_2 \int_{t-\sigma(t)}^{t} e^{2\alpha(s-t)} x^2(s)ds + k_3 \int_{t-\sigma_2}^{t} e^{2\alpha(s-t)} x^2(s)ds, \\
V_3(t) &= k_4 \int_{t-\sigma(t)}^{t} e^{2\alpha(s-t)} \tanh^2 x(s)ds \\
&\quad + k_5 \int_{t-\sigma_2}^{t} e^{2\alpha(s-t)} \tanh^2 x(s)ds, \\
V_4(t) &= k_6\sigma_2 \int_{-\sigma_2}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} x^2(\theta)d\theta ds, \\
V_5(t) &= k_7\sigma_2 \int_{-\sigma_2}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} y^2(\theta)d\theta ds \\
&\quad + k_8\sigma_2 \int_{-\sigma_2}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} y^2(\theta)d\theta ds \\
&\quad + k_9\sigma_2 \int_{-\sigma_2}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} y^2(\theta)d\theta ds, \\
V_6(t) &= \sigma_2 \int_{-\sigma_2}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} \begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix}^T \begin{bmatrix} r_1 & r_2 \\ * & r_3 \end{bmatrix} \begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix} d\theta ds, \\
V_7(t) &= k_{10}\sigma_2 \int_{-\sigma_2}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} \tanh^2 x(\theta)d\theta ds, \\
V_8(t) &= k_{11}\Big(\frac{\sigma_2^2}{2}\Big) \int_{-\sigma_2}^{0} \int_{\zeta}^{0} \int_{t+s}^{t} e^{2\alpha(\theta+s-t)} y^2(\theta)d\theta ds d\zeta \\
&\quad + k_{12}\Big(\frac{\sigma_2^2}{2}\Big) \int_{-\sigma_2}^{0} \int_{\zeta}^{0} \int_{t+s}^{t} e^{2\alpha(\theta+s-t)} y^2(\theta)d\theta ds d\zeta \\
&\quad + k_{13}\Big(\frac{\sigma_2^2}{2}\Big) \int_{-\sigma_2}^{0} \int_{\zeta}^{0} \int_{t+s}^{t} e^{2\alpha(\theta+s-t)} \tanh^2 x(\theta)d\theta ds d\zeta, \\
V_9(t) &= k_{14} \int_{t-\tau(t)}^{t} e^{2\alpha(s-t)} y^2(s)ds.
\end{aligned}$$

Calculating the time derivative of $V(t)$ along the solution of Equations (8)-(9) with (3)-(5) yields

$$\dot{V}(t) = \sum_{i=1}^{9} \dot{V}_i(t). \tag{16}$$

The time derivative of $V_1(t)$ is calculated as

$$\dot{V}_1(t) = 2k_1 x(t)\dot{x}(t)$$

$$
\begin{aligned}
&= 2k_1x(t)\Big[y(t)+v_1x(t)-v_1x(t-\sigma(t))-v_1\int_{t-\sigma}^{t}y(s)ds\Big]\\
&\quad +2q_1y(t)\Big[-y(t)-ax(t)+b\tanh x(t-\sigma(t))\\
&\quad -py(t-\tau(t))\Big]+2q_2x(t)\Big[-y(t)-ax(t)\\
&\quad +b\tanh x(t-\sigma(t))-py(t-\tau(t))\Big]+2q_3\tanh x(t-\sigma(t))\\
&\quad \times\Big[-y(t)-ax(t)+b\tanh x(t-\sigma(t))-py(t-\tau(t))\Big]\\
&\quad +2q_4y(t-\tau(t))\Big[-y(t)-ax(t)+b\tanh x(t-\sigma(t))\\
&\quad -py(t-\tau(t))\Big]+2q_5x(t)\Big[x(t)-x(t-\sigma(t))\\
&\quad -\int_{t-\sigma(t)}^{t}y(s)ds\Big]+2q_6x(t-\sigma(t))\Big[x(t)-x(t-\sigma(t))\\
&\quad -\int_{t-\sigma(t)}^{t}y(s)ds\Big]+2q_7\int_{t-\sigma(t)}^{t}y(s)ds\Big[x(t)-x(t-\sigma(t))\\
&\quad -\int_{t-\sigma(t)}^{t}y(s)ds\Big], \qquad (17)
\end{aligned}
$$

where $q_j,\ j=1,2,\ldots,7$ are real constants. For the time derivatives calculation of $V_2(t)$ and $V_3(t)$, we obtain

$$
\begin{aligned}
\dot{V}_2(t) &= k_2x^2(t)-k_2e^{-2\alpha\sigma(t)}x^2(t-\sigma(t))\\
&\quad +k_2\dot{\sigma}(t)e^{-2\alpha\sigma(t)}x^2(t-\sigma(t))\\
&\quad +k_3x^2(t)-k_3e^{-2\alpha\sigma_2}x^2(t-\sigma_2)-2\alpha V_2(t)\\
&\le k_2x^2(t)-k_2e^{-2\alpha\sigma_2}x^2(t-\sigma(t))+k_2\sigma_dx^2(t-\sigma(t))\\
&\quad +k_3x^2(t)-k_3e^{-2\alpha\sigma_2}x^2(t-\sigma_2)-2\alpha V_2(t), \qquad (18)\\
\dot{V}_3(t) &= k_4\tanh x^2(t)-k_4e^{-2\alpha\sigma(t)}\tanh^2x(t-\sigma(t))\\
&\quad +k_4\dot{\sigma}(t)e^{-2\alpha\sigma(t)}\tanh^2x(t-\sigma(t))\\
&\quad +k_5\tanh^2x(t)-k_5e^{-2\alpha\sigma_2}\tanh^2x(t-\sigma_2)-2\alpha V_3(t)\\
&\le k_4x^2(t)-k_4e^{-2\alpha\sigma_2}\tanh^2x(t-\sigma(t))\\
&\quad +k_4\sigma_d\tanh^2x(t-\sigma(t))\\
&\quad +k_5x^2(t)-k_5e^{-2\alpha\sigma_2}\tanh^2x(t-\sigma_2)-2\alpha V_3(t). \qquad (19)
\end{aligned}
$$

Obviously, for any scalar $s\in[t-\sigma_2,t]$, we get $e^{-2\alpha\sigma_2}\le e^{2\alpha(s-t)}\le 1$. Calculating the time derivatives of $V_4(t)$ and $V_5(t)$ with Lemma 2.1, Lemma 2.3, Lemma 2.4 and Lemma 2.5, we get

$$
\dot{V}_4(t) = k_6\sigma_2\int_{-\sigma_2}^{0}\Big[x^2(t)-e^{2\alpha s}x^2(t+s)
$$

$$
\begin{aligned}
&\quad -2\alpha \int_{t+s}^{t} e^{2\alpha(\theta-t)} x^2(\theta) d\theta \Big] ds \\
&\leq \quad k_6\sigma_2^2 x^2(t) - k_6\sigma_2^2 e^{-2\alpha\sigma_2} \Big[\frac{1}{\sigma_2}\int_{t-\sigma_2}^{t} x(s)ds\Big]^2 - 2\alpha V_4(t), \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_5(t) \quad = \quad & k_7\sigma_2 \int_{-\sigma_2}^{0} y^2(t) - e^{2\alpha s} y^2(t+s)\ ds \\
& +k_8\sigma_2 \int_{-\sigma_2}^{0} y^2(t) - e^{2\alpha s} y^2(t+s)\ ds \\
& +k_9\sigma_2 \int_{-\sigma_2}^{0} y^2(t) - e^{2\alpha s} y^2(t+s)\ ds - 2\alpha V_5(t)
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & (k_7\sigma_2^2 + k_8\sigma_2^2 + k_9\sigma_2^2) y^2(t) + \begin{bmatrix} x(t) \\ x(t-\sigma(t)) \\ x(t-\sigma_2) \end{bmatrix}^T \\
& \times \begin{bmatrix} m_1 + m_1^T & -m_1^T + m_2 & 0 \\ * & m_1 + m_1^T - m_2 - m_2^T & -m_1^T + m_2 \\ * & * & -m_2 - m_2^T \end{bmatrix} \\
& \times \begin{bmatrix} x(t) \\ x(t-\sigma(t)) \\ x(t-\sigma_2) \end{bmatrix} + \begin{bmatrix} x(t) \\ x(t-\sigma(t)) \\ x(t-\sigma_2) \end{bmatrix}^T \begin{bmatrix} m_3 & m_4 & 0 \\ * & m_3 + m_5 & m_4 \\ * & * & m_5 \end{bmatrix} \\
& \times \begin{bmatrix} x(t) \\ x(t-\sigma(t)) \\ x(t-\sigma_2) \end{bmatrix} + \begin{bmatrix} x(t) \\ x(t-\sigma_2) \\ \frac{1}{\sigma_2}\int_{t-\sigma_2}^{t} x(s)\ ds \end{bmatrix}^T \\
& \times \begin{bmatrix} -4k_8 e^{-2\alpha\sigma_2} & -2k_8 e^{-2\alpha\sigma_2} & 6k_8 e^{-2\alpha\sigma_2} \\ * & -4k_8 e^{-2\alpha\sigma_2} & 6k_8 e^{-2\alpha\sigma_2} \\ * & * & -12k_8 e^{-2\alpha\sigma_2} \end{bmatrix} \\
& \times \begin{bmatrix} x(t) \\ x(t-\sigma_2) \\ \frac{1}{\sigma_2}\int_{t-\sigma_2}^{t} x(s)\ ds \end{bmatrix} + \begin{bmatrix} x(t) \\ x(t-\sigma(t)) \\ x(t-\sigma_2) \end{bmatrix}^T
\end{aligned}
$$

$$
\begin{aligned}
& \times \begin{bmatrix} -k_9 e^{-2\alpha\sigma_2} & k_9 e^{-2\alpha\sigma_2} - s & s \\ * & -2k_9 e^{-2\alpha\sigma_2} + s + s^T & k_9 e^{-2\alpha\sigma_2} - s \\ * & * & -k_9 e^{-2\alpha\sigma_2} \end{bmatrix} \\
& \times \begin{bmatrix} x(t) \\ x(t-\sigma(t)) \\ x(t-\sigma_2) \end{bmatrix} - 2\alpha V_5(t). \qquad (21)
\end{aligned}
$$

From Lemma 2.2, the time derivative of $V_6(t)$ is given by

$$\begin{aligned}
\dot{V}_6(t) &= \sigma_2^2 \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} r_1 & r_2 \\ * & r_3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\
&\quad -\sigma_2 \int_{-\sigma_2}^{0} e^{2\alpha s} \begin{bmatrix} x(t+s) \\ y(t+s) \end{bmatrix}^T \begin{bmatrix} r_1 & r_2 \\ * & r_3 \end{bmatrix} \begin{bmatrix} x(t+s) \\ y(t+s) \end{bmatrix} ds \\
&\quad -2\alpha V_6(t) \\
&\leq \sigma_2^2 \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} r_1 & r_2 \\ * & r_3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\
&\quad +e^{-2\alpha\sigma_2} \begin{bmatrix} x(t) \\ x(t-\sigma(t)) \\ x(t-\sigma_2) \\ \int_{t-\sigma(t)}^{t} x(s)\, ds \\ \int_{t-\sigma_2}^{t-\sigma(t)} x(s)\, ds \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} -r_3 & r_3 & 0 & -r_2^T & 0 \\ * & -2r_3 & r_3 & r_2^T & -r_2^T \\ * & * & -r_3 & 0 & r_2^T \\ * & * & * & -r_1 & 0 \\ * & * & * & * & -r_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\sigma(t)) \\ x(t-\sigma_2) \\ \int_{t-\sigma(t)}^{t} x(s)\, ds \\ \int_{t-\sigma_2}^{t-\sigma(t)} x(s)\, ds \end{bmatrix} \\
&\quad -2\alpha V_6(t).
\end{aligned} \tag{22}$$

By applying $\tanh^2 x(t) \leq x^2(t)$ and Lemma 2.1, the time derivatives of $V_7(t)$ and $V_8(t)$ are calculated as

$$\begin{aligned}
\dot{V}_7(t) &= k_{10}\sigma_2^2 \tanh^2 x(t) - k_{10}\sigma_2 \int_{-\sigma_2}^{0} e^{2\alpha s} \tanh^2 x(t+s)\, ds \\
&\quad -2\alpha V_7(t) \\
&\leq k_{10}\sigma_2^2 x(t)^2 - k_{10} e^{-2\alpha\sigma_2} \Big[\int_{t-\sigma_2}^{t} \tanh x(s)\, ds \Big]^2 \\
&\quad -2\alpha V_7(t),
\end{aligned} \tag{23}$$

$$\begin{aligned}
\dot{V}_8(t) &= k_{11} \Big(\frac{\sigma_2^2}{2}\Big) \int_{-\sigma_2}^{0} \int_{\zeta}^{0} e^{2\alpha s} y^2(t) - e^{4\alpha s} y^2(t+s)\, ds d\zeta \\
&\quad k_{12} \Big(\frac{\sigma_2^2}{2}\Big) \int_{-\sigma_2}^{0} \int_{\zeta}^{0} e^{2\alpha s} y^2(t) - e^{4\alpha s} y^2(t+s)\, ds d\zeta \\
&\quad k_{13} \Big(\frac{\sigma_2^2}{2}\Big) \int_{-\sigma_2}^{0} \int_{\zeta}^{0} e^{2\alpha s} \tanh x^2(t) \\
&\quad -e^{4\alpha s} \tanh x^2(t+s)\, ds d\zeta - 2\alpha V_8(t)
\end{aligned}$$

$$
\begin{aligned}
&\leq k_{11}(\frac{\sigma_2^4}{4})y^2(t) - k_{11}e^{-4\alpha\sigma_2}\Big(\int_{-\sigma_2}^{0}\int_{t+\zeta}^{t} y(s)dsd\zeta\Big)^2 \\
&\quad +k_{12}(\frac{\sigma_2^4}{4})y^2(t) - k_{12}e^{-4\alpha\sigma_2}\Big(\sigma_2^2 x^2(t) \\
&\quad -2\sigma_2 x(t)\int_{t-\sigma_2}^{t} x(\zeta)d\zeta + (\int_{t-\zeta}^{t} x(\zeta)d\zeta)^2\Big) \\
&\quad +k_{13}(\frac{\sigma_2^4}{4})\tanh^2 x(t) \\
&\quad -k_{13}e^{-4\alpha\sigma_2}\Big(\int_{-\sigma_2}^{0}\int_{t+\zeta}^{t} \tanh x(s)dsd\zeta\Big)^2 \\
&\quad -2\alpha V_8(t).
\end{aligned}
\tag{24}
$$

From (3), the time derivative of $V_9(t)$ is given by

$$
\begin{aligned}
\dot{V}_9(t) &= k_{14}\Big[y^2(t) - (1-\dot{\tau}(t))e^{2\alpha(-\tau(t))}y^2(t-\tau(t)) \\
&\quad -2\int_{t-\tau(t)}^{t} \alpha e^{2\alpha(s-t)}y^2(t)ds\Big] \\
&\leq k_{14}y^2(t) - k_{14}e^{-2\alpha\tau_2}y^2(t-\tau(t)) + k_{14}\tau_d y^2(t-\tau(t)) \\
&\quad -2\alpha V_9(t).
\end{aligned}
\tag{25}
$$

According to (16)-(25), it is straightforward to see that

$$
\dot{V}(t) + 2\alpha V(t) \leq \xi^T(t)\sum \xi(t), \tag{26}
$$

where

$$
\begin{aligned}
\xi(t) &= \Big[x(t), y(t), x(t-\sigma(t)), \int_{t-\sigma(t)}^{t} y(s)ds, \tanh x(t-\sigma(t)), \\
&\quad y(t-\tau(t)), x(t-\sigma_2)), \tanh x(t-\sigma_2), \frac{1}{\sigma_2}\int_{t-\sigma_2}^{t} x(s)ds, \\
&\quad \int_{t-\sigma(t)}^{t} x(s)ds, \int_{t-\sigma_2}^{t-\sigma(t)} x(s)ds, \int_{t-\sigma_2}^{t} \tanh x(s), \\
&\quad \int_{-\sigma_2}^{0}\int_{t+\zeta}^{t} y(\zeta)dsd\zeta, \int_{t-\sigma_2}^{t} x(\zeta)d\zeta, \int_{-\sigma_2}^{0}\int_{t+\zeta}^{t} (\zeta)dsd\zeta\Big]^T,
\end{aligned}
$$

and $\sum$ is defined in (10). It is true that if Conditions (11)-(14) hold, then

$$
\dot{V}(t) + 2\alpha V(t) \leq 0, \quad \forall t \in R^+. \tag{27}
$$

From (27), it is easy to see that

$$
\|x(t,\phi)\| \leq \beta\|\phi\|e^{-\alpha t}, \quad \forall t \in R^+.
$$

This means that Equation (2) with (3)-(5) is exponentially stable. The proof of the theorem is complete. □

Then, we consider the CNDE (2) with (3)-(5). By model transformation and (7), system (2) can be represented by the form

$$\begin{aligned}\dot{x}(t) &= -ax(t) + b\tanh x(t-\sigma(t)) - p\dot{x}(t-\tau(t)) \\ &\quad + v_1 x(t) - v_1 x(t-\sigma(t)) - v_1 \int_{t-\sigma(t)}^{t} \dot{x}(s)ds. \end{aligned} \tag{28}$$

We introduce the following notation for later use:

$$\widehat{\sum} = \left[\Gamma_{(i,j)}\right]_{16\times 16}, \tag{29}$$

where $\Gamma_{(i,j)} = \Gamma_{(j,i)}$,

$$\begin{aligned}
\Gamma_{(1,1)} &= -2ak_1 + 2j - 2q_2 a + 2q_5 + k_2 + k_3 + k_6\sigma_2^2 + 2m_1 \\
&\quad + \sigma_2 m_3 - 4k_8 e^{-2\alpha\sigma_2} - k_9 e^{-2\alpha\sigma_2} + r_1\sigma_2^2 - r_3 e^{-2\alpha\sigma_2} \\
&\quad - k_{12}\sigma_2^2 e^{-4\alpha\sigma_2} + \varepsilon, \\
\Gamma_{(1,2)} &= -j - q_5 + q_6 - m_1 + m_2 + \sigma_2 m_4 + k_9 e^{-2\alpha\sigma_2} - s_1 \\
&\quad + r_3 e^{-2\alpha\sigma_2}, \\
\Gamma_{(1,3)} &= -2k_8 e^{-2\alpha\sigma_2} + s_1, \\
\Gamma_{(1,5)} &= +bk_1 + q_2 b - q_3 a, \\
\Gamma_{(1,7)} &= -q_1 a - q_2 + r_2\sigma_2^2, \\
\Gamma_{(1,8)} &= -k_1 p - q_2 p - q_4 a, \\
\Gamma_{(1,10)} &= +6k_8 e^{-2\alpha\sigma_2}, \\
\Gamma_{(1,11)} &= -r_2 e^{-2\alpha\sigma_2}, \\
\Gamma_{(1,13)} &= -j - q_5 + q_7, \\
\Gamma_{(1,16)} &= +k_{12}\sigma_2 e^{-4\alpha\sigma_2}, \\
\Gamma_{(2,2)} &= -2q_6 - k_2 e^{-2\alpha\sigma_2} + k_2\sigma_d + 2m_1 - 2m_2 + \sigma_2 m_3 \\
&\quad + \sigma_2 m_5 - 2k_9 e^{-2\alpha\sigma_2} + 2s_1 - 2r_3 e^{-2\alpha\sigma_2}, \\
\Gamma_{(2,3)} &= -m_1 + m_2 + \sigma_2 m_4 + k_9 e^{-2\alpha\sigma_2} - s_1 + r_3 e^{-2\alpha\sigma_2}, \\
\Gamma_{(2,9)} &= -r_2 e^{-2\alpha\sigma_2}, \\
\Gamma_{(2,11)} &= +r_2 e^{-2\alpha\sigma_2}, \\
\Gamma_{(2,13)} &= -q_6 - q_7, \\
\Gamma_{(3,3)} &= -k_3 e^{-2\alpha\sigma_2} - 2m_2 + \sigma_2 m_5 - 4k_8 e^{-2\alpha\sigma_2} - k_9 e^{-2\alpha\sigma_2} \\
&\quad - r_3 e^{-2\alpha\sigma_2}, \\
\Gamma_{(3,9)} &= +r_2 e^{-2\alpha\sigma_2}, \\
\Gamma_{(3,10)} &= +6k_8 e^{-2\alpha\sigma_2},
\end{aligned}$$

$$\Gamma_{(4,4)} = +k_4 + k_5 + k_{10}\sigma_2^2 + k_{13}\left(\frac{\sigma_2^4}{4}\right) - \varepsilon,$$

$$
\begin{aligned}
\Gamma_{(5,5)} &= +2q_3b - k_4e^{-2\alpha\sigma_2} + k_4\sigma_d,\\
\Gamma_{(5,7)} &= +q_1b - q_3,\\
\Gamma_{(5,8)} &= -q_3p + q_4b,\\
\Gamma_{(6,6)} &= -k_5e^{-2\alpha\sigma_2},\\
\Gamma_{(7,7)} &= -2q_1 + k_{14} + r_3\sigma_2^2 + k_{11}\Big(\frac{\sigma_2^4}{4}\Big) + k_{12}\Big(\frac{\sigma_2^4}{4}\Big),\\
\Gamma_{(7,8)} &= -q_1p - q_4,\\
\Gamma_{(8,8)} &= -2q_4p - k_{14}e^{-2\alpha\tau_2} + k_{14}\tau_d,\\
\Gamma_{(9,9)} &= -r_1e^{-2\alpha\sigma_2},\\
\Gamma_{(10,10)} &= -k_6\sigma_2^2e^{-2\alpha\sigma_2} - 12k_8e^{-2\alpha\sigma_2},\\
\Gamma_{(11,11)} &= -r_1e^{-2\alpha\sigma_2},\\
\Gamma_{(12,12)} &= -k_{10}e^{-2\alpha\sigma_2},\\
\Gamma_{(13,13)} &= -2q_7,\\
\Gamma_{(14,14)} &= -k_{13}e^{-4\alpha\sigma_2},\\
\Gamma_{(15,15)} &= -k_{11}e^{-4\alpha\sigma_2},\\
\Gamma_{(16,16)} &= -k_{12}e^{-4\alpha\sigma_2},\\
j &= k_1v_1,
\end{aligned}
$$

and the other terms are 0.

Theorem 3.2. *For given positive real constants $a, b, \sigma_2, \sigma_d, \tau_2$ and τ_d, Equation* (2) *with* (3)-(5) *is exponentially stable with a decay rate α, if $|p| < 1$ and there exist positive real constants k_i where $i = 1, 2, \ldots, 14$, and real constants $s_1, v_1, r_1, r_2, r_3, q_j, m_k$ where $j = 1, 2, \ldots, 7$ and $k = 1, 2, \ldots, 5$ such that the following symmetric linear matrix inequalities hold*

$$\widehat{\sum} < 0, \tag{30}$$

$$\begin{bmatrix} r_1 & r_2 \\ * & r_3 \end{bmatrix} \geq 0, \tag{31}$$

$$\begin{bmatrix} k_9e^{-2\alpha\sigma_2} & s_1 \\ * & k_9e^{-2\alpha\sigma_2} \end{bmatrix} \geq 0, \tag{32}$$

$$\begin{bmatrix} k_7e^{-2\alpha\sigma_2} & m_1 & m_2 \\ * & m_3 & m_4 \\ * & * & m_5 \end{bmatrix} \geq 0. \tag{33}$$

Proof. For k_i are positive real constants where $i = 1, 2, \ldots, 14$, consider the Lyapunov-

Krasovskii functional candidate for Equation (28) with (3)-(5) of the form

$$V(t)=\sum_{i=1}^{9} V_i(t), \tag{34}$$

where $V_1(t)$ to $V_9(t)$ are defined in Theorem 3.1. Calculating the time derivatives of $V(t)$ along the solution of Equation (28) with (3)-(4) yields

$$\dot{V}(t)=\sum_{i=1}^{9} \dot{V}_i(t). \tag{35}$$

The time derivatives of $V_1(t)$ is calculated as

$$\begin{aligned}
\dot{V}_1(t) &= 2k_1x(t)\dot{x}(t)\\
&= 2k_1x(t)\Big[-ax(t)+b\tanh x(t-\sigma(t))-p\dot{x}(t-\tau(t))\\
&\quad +v_1x(t)-v_1x(t-\sigma(t))-v_1\int_{t-\sigma(t)}^{t}\dot{x}(s)ds\Big]\\
&= -2ak_1x^2(t)+2bk_1x(t)\tanh x(t-\sigma(t))\\
&\quad -2pk_1x(t)\dot{x}(t-\tau(t))+2k_1v_1x^2(t)\\
&\quad -2k_1v_1x(t)x(t-\sigma(t))-2k_1v_1x(t)\int_{t-\sigma(t)}^{t}\dot{x}(s)ds\\
&\quad +2q_1\dot{x}(t)\Big[-\dot{x}(t)-ax(t)+b\tanh x(t-\sigma(t))\\
&\quad -p\dot{x}(t-\tau(t))\Big]\\
&\quad +2q_2x(t)\Big[-\dot{x}(t)-ax(t)+b\tanh x(t-\sigma(t))\\
&\quad -p\dot{x}(t-\tau(t))\Big]\\
&\quad +2q_3\tanh x(t-\sigma(t))\Big[-\dot{x}(t)-ax(t)\\
&\quad +b\tanh x(t-\sigma(t))-p\dot{x}(t-\tau(t))\Big]
\end{aligned}$$

$$\begin{aligned}
&+2q_4\dot{x}(t-\tau(t))\Big[-\dot{x}(t)-ax(t)+b\tanh x(t-\sigma(t))\\
&-p\dot{x}(t-\tau(t))\Big]\\
&+2q_5x(t)\Big[x(t)-x(t-\sigma(t))-\int_{t-\sigma(t)}^{t}\dot{x}(s)ds\Big]\\
&+2q_6x(t-\sigma(t))\Big[x(t)-x(t-\sigma(t))-\int_{t-\sigma(t)}^{t}\dot{x}(s)ds\Big]\\
&+2q_7\int_{t-\sigma(t)}^{t}\dot{x}(s)ds\Big[x(t)-x(t-\sigma(t))
\end{aligned}$$

$$- \int_{t-\sigma(t)}^{t} \dot{x}(s)ds \Big], \tag{36}$$

where q_j, $j = 1, 2, \ldots, 7$ are real constants. The time derivatives of $V_2(t)$-$V_9(t)$ are defined by Theorem 3.1. According to the proof of Theorem 3.1 and (35)-(36), it is straightforward to see that

$$\dot{V}(t) + 2\alpha V(t, x_t) \leq \eta^T(t)\widehat{\sum}\eta(t), \tag{37}$$

where

$$\begin{aligned} \eta^T(t) \quad = \quad & \Big[x(t), x(t-\sigma(t)), x(t-\sigma_2), \\ & \tanh x(t), \tanh x(t-\sigma(t)), \tanh x(t-\sigma_2), \\ & \dot{x}(t), \dot{x}(t-\tau(t)), \int_{t-\sigma_2}^{t-\sigma(t)} x(s)ds, \\ & \frac{1}{\sigma_2}\int_{t-\sigma_2}^{t} x(s)ds, \int_{t-\sigma(t)}^{t} x(s)ds, \\ & \int_{t-\sigma_2}^{t} \tanh x(s)ds, \int_{t-\sigma(t)}^{t} \dot{x}(s)ds, \\ & \int_{-\sigma_2}^{0}\int_{t+\zeta}^{t} \tanh x(s)dsd\zeta, \\ & \int_{-\sigma_2}^{0}\int_{t+\zeta}^{t} \dot{x}(s)dsd\zeta, \int_{t-\sigma_2}^{t} x(\zeta)d\zeta\Big], \end{aligned}$$

and $\sum$ is defined in (29). It is true that if Conditions (30)-(33) hold, then

$$\dot{V}(t) + 2\alpha V(t) \leq 0, \quad \forall t \in R^+. \tag{38}$$

From (38), it is easy to see that

$$\|x(t, \phi)\| \leq \beta\|\phi\|e^{-\alpha t}, \quad t \in R^+.$$

This means that Equation (2) with (3)-(5) is exponentially stable. The proof of the theorem is complete. □

4. NUMERICAL EXAMPLES

Three numerical examples are given to present the effectiveness of our main results by comparing the upper bounds of the delays σ and the parameter b as well as investigating the rate of convergence.

Example 4.1. Consider the following equation studied in [8, 20]:

$$\frac{d}{dt}[x(t) + 0.2x(t-\tau(t))] = -0.6x(t) + 0.5\tanh x(t-\sigma(t)). \tag{39}$$

when $\tau(t) = \frac{\sin^2(t)}{10}$ and $\sigma_d = 0.2$. By solving the linear matrix inequalities (11)-(14), the maximum upper bounds σ_2 for exponential stability of this example are listed in the comparison in Table 1, for different values of α. We obtain that our results in Theorem 3.1-3.2 are much less conservative than those obtained in [8, 20].

Table 1: The upper bounds of time delay $\sigma(t)$ for Example 4.1

Methods	$\alpha = 0.0038$	$\alpha = 0.02$	$\alpha = 0.028$
Chen, et al. (2011) [8]	infeasible	infeasible	infeasible
Keadnarmol, et al. (2014) [20]	7.5231	0.5234	0.0321
Theorem 3.2	10.0061	1.8978	1.3544
Theorem 3.1	3.5×10^3	564.9979	448.9991

Example 4.2. Consider the following equation, which is considered in [2, 7, 8, 10, 13, 22, 20, 23, 27, 31, 35]:

$$\frac{d}{dt}[x(t) + 0.35x(t-0.5)] = -1.5x(t) + b\tanh x(t-0.5). \tag{40}$$

Table 2 lists the comparison of the upper bounds b for asymptotic stability ($\alpha = 0$) and exponential stability ($\alpha = 0.177$) of Equation (40) by different methods. We get from Table 2 that our result (Theorem 3.1-3.2) is better than other existing works.

Example 4.3. Consider the following equation in [7, 8, 22, 20, 23, 27, 29, 31, 35]:

$$\frac{d}{dt}[x(t) + 0.2x(t-0.1)] = -0.6x(t) + 0.3\tanh x(t-\sigma_2). \tag{41}$$

Table 3 lists the comparison of the upper bounds delay for asymptotic stability ($\alpha = 0$) and exponential stability ($\alpha = 0.0038$) of (41) by different methods. It is clear that our result (Theorem 3.1) are significantly better than some existing criteria.

5. CONCLUSIONS

Two model transformations were constructed to study the delay-dependent exponential stability criteria for CNDE with time-varying delays in this paper. By employing mixed integral inequalities, mixed utilization of zero equations and new Lyapunov-Krasovskii functional, the proposed exponential stability criteria have been formulated in the form of LMIs. Finally, three numerical examples are given to show that the proposed criteria are less conservative than some existing stability criteria.

Table 2: Upper bounds of b for Example 4.2 when $\gamma = 0.5$.

Methods	A.S. ($\alpha = 0$)	E.S. ($\alpha = 0.177$)
$\sigma_2 = \tau_2 = 0.5$	b	b
Agarwal, et al. (2000) [2]	0.318	-
El-Morshedy, et al. (2000) [13]	0.424	-
Park, et al. (2008) [31]	0.422	-
Kwon, et al. (2008) [22]	1.49	-
Li (2009) [23]	0.699	0.722
Deng et al. (2009) [10]	0.889	-
Nam, et al. (2009) [27]	1.405	-
Rojsiraphisal, et al. (2010) [35]	1.405	0.478
Chen, et al. (2011) [8]	1.346	-
Chen (2012) [7]	1.405	1.092
Keadnarmol, et al. (2014) [20]	1.405	1.1089
Theorem 3.1	1.4051	1.2686
Theorem 3.2	2.5624	1.9744

Table 3: Upper bounds of σ_2 for Example 4.3 when $\gamma = 0.05$.

Methods	A.S. ($\alpha = 0$)	E.S. ($\alpha = 0.0038$)
$\tau_2 = 0.1$	σ_2	σ_2
Park (2004) [29]	0.444	-
Park, et al. (2008) [31]	1.90	-
Kwon, et al. (2008) [22]	10^7	-
Li (2009) [23]	2.07	-
Nam, et al.t (2009) [27]	2.32	-
Rojsiraphisal, et al. (2010) [35]	2.32	1.947
Chen, et al. (2011) [8]	10^{21}	-
Chen (2012) [7]	1.34×10^{21}	175.289
Theorem 3.1	139.7466	132.7331

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