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THE EXISTENCE AND HYERS-ULAM STABILITY FOR SECOND ORDER RANDOM IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT: In this paper, we investigate the existencce and Hyers-Ulam stability of solution second order random impulsive differential equations. Firstly, the solution for the equation is proved. Then, the existence results are obtained by fixed point which is a version of the topological transversality theorem. Finally, the Hyers-Ulam stability of the solution will be proved.

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1. INTRODUCTION

Many evolutionary processes in nature often undergoes changes in state at some point in time. However, compared with the duration of the process, the duration of the process can be neglected and can be regarded as a short-term disturbance. Therefore, we can naturally assume that these disturbances are instantaneous, i.e. in the form of impulses. The definition of impulsive system is a special continuous-time dynamic hybrid system with instantaneous jump, [1], [2]. The impulsive differential equation can describe the rapid change or jump of some motion states at fixed or random time. This can more truly reflect the natural development process. Through impulsive differential equation, we can provide a good model for many practical problems in many fields of science and technology, [3]. In recent years, more and more scholars have paid great attention to the application of impulse system in practical natural problems, such as food network system, [4], complex network system, [5], ecosystem, [6], [7], Gilpin-Ayala competition system, [8], sampling data control system, [9] and so on. Up to now, some literatures, [10], [11], [12], [13] have proved the existence, uniqueness and stability of impulsive differential equations. As far as we known, the earliest research on differential equation with impulses can been seen in [11], where the existence of solutions to impulsive differential equations have been investigated by Kaul. Wang et al. [10] discussed Ulam's type stability for the following impulsive ordinary differential equations

$$\begin{cases} x'(t) = f(t, x(t)), \quad t \in J' := J \backslash \{t_1, t_2, \ldots, t_m\}, \quad J = [0, T], \\ \triangle x(t_k) = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \end{cases}$$

where $x(t_k^-) = \lim s \to t_k^- x(s)$, $x(t_k^+) = \lim s \to t_k^+ x(s)$represent the left and right limits of $x(t)$ at $t = t_k$.

However, in many practical problems, the time when the pulse occurs is always generated randomly. Hence, the time when the pulse occurs is a random variable. Therefore, the solutions of differential equations with random impulsive time are stochastic processes. Because the solution of impulsive systems with fixed impulsive time is only a piecewise function, but the solution of stochastic impulsive systems is a stochastic process, which results in the great difference of their solution properties. By adding stochastic process to the mathematical model, our model can better match the corresponding practical problems. So stochastic impulsive differential equations are widely used in the modeling of various systems, [14], [15], [16]. So far, there have been some studies in this field, in which the existence, uniqueness and stability and other properties of first-order differential equations with random impulses have been obtained [17], [18], [19]. The random impulsive differential equations involving fractional derivative also have been discussed in [20], [21], [22], [23], [24]. As far as we known, the earliest research on differential equation with random impulses can been seen in [17]. The solutions to random impulsive differential equations have been investigated by Wu, [19]:

$$\begin{cases} x'(t) = f(t, x(t)) \quad a.e., \quad t > t_0, \quad t \neq \tau_k, \\ x(\tau_k) = I_k(\tau_k^-, x(\tau_k^-)) \quad a.e., \quad k = 1, 2, \cdots, \end{cases}$$

where $f : \mathbf{R}_\tau \times \mathbf{R}^n \to \mathbf{R}^n$. $\tau_k = \tau_{k-1} + w_k$ as $k = 1, 2, 3, \ldots$, here $t_0 \in \mathbf{R}_\tau$ and w_k denotes the waiting time that the solution to system jumps after the (k-1)th jump. $x(\tau_k^-) = \lim t \to \tau_k^- x(t)$. $I_k : \mathbf{R}_\tau \times \mathbf{R}^n \to \mathbf{R}^n$. "$a.e$" is the abbreviation of "$almost \quad everywher$".

The initial condition for system is given by $x(t_0) = x_0$, where $x_0 \in \mathbf{R}^n$.

In recent years, there have been many studies on the Hyers-Ulam stability of linear functional differential equations. And Wang and Zada et al, [25] have proved Ulams-Type stability of first-order impulsive differential equations with variable delay. Some scholars have proved the stability of first-order functional differential equations with multiple delays and general functional differential equations with delays, [26], [27], [28].

As far as we know, the existence and Hyers-Ulam stability of second order stochastic impulsive differential equations have not been studied so far. However, second-order differential equations, as a kind of differential equations with important applications, have been widely used in various fields. It is of theoretical and practical significance to study the Hyers-Ulam stability of second order stochastic impulsive stochastic functional differential equations. Motivated by the above mentioned works, the main purpose of this article is to study random impulsive second-order differential systems. We consider Second order differential equation with random impulses of the form

$$\begin{cases} x''(t) = f(t, x(t)), \quad t \in J, \quad t \neq \xi_k, \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-), \quad k = 1, 2, \cdots, \\ x(0) = x_0 \quad x'(0) = x_1, \end{cases} \tag{1.1}$$

where $f: \mathbf{R}_\tau \times \mathbf{R}^n \to \mathbf{R}^n$, is the set of piecewise continuous function mapping; $b_k : D_k \to \mathbf{R}$ for each k=1,2,....; τ_k is a random variable defined from Ω to $D_k = (0, d_k)$ for k=1,2,$\cdots$, where $0 < d_k < +\infty$, furthermore, assume that τ_i and τ_j are independent from each other as $i \neq j$ for $i, j = 1, 2,\cdots$. Obviously, $\xi_1 < \xi_2 < \cdots < \lim_{k\to\infty} \xi_k = \infty$ and $x(\xi_k^-) = \lim_{t\to\xi_k} x(t)$. We assume that $t_k = t_{k-1} + \tau_k$ for $k = 1, 2, \cdots$. Denote $\{B_t, t \geq 0\}$ the simple counting process generated by $\{t_n\}$, that is, $\{B_t \geq n\} = \{t_n \leq t\}$, and denote $\mathcal{F}$ the $\sigma-$algebra generated by$\{B_t, t \geq 0\}$.

The rest of the paper is organized as follows: in section 2, some basic definitions, notations and preliminary facts that are used throughout the paper are presented. In Section 3, we study the existence of solutions to second order random impulsive functional differential equations by fixed point which is a version of the topological transversality theorem and the Hyers-Ulam stability of second order random impulsive functional differential equations.

2. PRELIMINARIES

In this section, we mention notations, definitions, lemmas and preliminary facts needed to establish our main results.

We fix $(\Omega, \mathcal{F}, P)$ is a probability space. Let $L^p(\Omega, R^d)$ be the collection of all strongly measurable, pth integrable, $\mathcal{F}_t$-measurable, R^d-valued random variables x with norm $\|x\|_{L^p} = (E\|x\|^p)^{\frac{1}{p}}$, where the expectation E is defined by $Ex = \int_\Omega xdP$.

Let τ, $T \in \Re$ be two constants satisfying $\tau < T$. For the sake of simplicity, we denote $\Re^+ = [0, +\infty); \Re_\tau = [\tau, +\infty)$. Let J=$[0, +\infty]$.

We introduce the space $PC(J, \Re^+) : = \{u : u$ is a map from J into $\Re^+$ such that $u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $x(t_k^+)$ exist, $k = 1, 2, \ldots, m\}$ is a Banach space with norm

$$\|u\|_{PC} = \left(\sup_{t\epsilon J} E \parallel u(t) \parallel^2\right)^{1/2}.$$

Notice that $PC^1(J, \Re^+) : = \{u : u$ is a map from J into $\Re^+$ such that $u(t)$ is continuously differentiable at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+), u'(t_k^+)$, $u'(t_k^-)$ exist, $k = 1, 2 \ldots .\}$ is also a Banach space with norm

$$\|u\|_{PC_1} = \max\left\{\left(\sup_{t\epsilon J} E \parallel u(t) \parallel^2\right)^{1/2}, \left(\sup_{t\epsilon J} E \parallel u'(t) \parallel^2\right)^{1/2}\right\}.$$

A function $u \in PC^1(J, \Re^+) \cap C^2(J', \Re)$ is called a solution of problem (1.1) if it satisfied (1.1).

First, we consider the solution of the following problem,

$$\left\{\begin{array}{l} x''(t) = f(t), \quad t \in J, \quad t \neq \xi_k, \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-), \quad k = 1, 2, \cdots, \\ x(0) = x_0 \quad x'(0) = x_1. \end{array}\right. \tag{2.1}$$

Lemma 2.1: If $f \in L^1(J, \Re^+)$, a $\mathcal{R}^d$-valued stochastic process $x(t)$ in $(\Omega, P, \{\mathcal{F}_t\})$ is a solution to (2.1) with the initial data $x(0) = x_0$, $x'(0) = x_1$ if for every $t \in J$, $x(t)$ is $F_t - adapted$ and

$$\begin{aligned} x(t) \quad = \quad & \sum_{k=0}^{\infty} \left\{ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) x_1 t_i + \prod_{i=1}^{k} b_i(\tau_i) x_0 + x_1 t \right. \\ & + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \left[\int_{t_{i-1}}^{t_i} (t_i - s) f(s) ds + t_i \int_0^{t_{i-1}} f(s) ds \right] \\ & \left. + \int_{t_k}^{t} (t - s) f(s) ds + t \int_0^{t_k} f(s) ds \right\} I_{(t_k, t_{k+1}]}(t), \end{aligned}$$

where $\prod_{j=m}^{n}(\cdot) = 1$ as $m > n$, $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_K) b_k(\tau_K) ... b_i(\tau_i)$, and $I_A(.)$ is the index function, i.e.

$$I_A(t) = \left\{\begin{array}{ll} 1, & t \in A, \\ 0, & t \notin A. \end{array}\right.$$

proof: Suppose $t_1, t_2, \ldots$ is a sample orbit.

(i) First, we consider the interval $t \in [0, t_1)$.

Let us consider the Homogeneous equation of (2.1), given by $x^2(t) = 0$.

By characteristic equation method, we know all solution of the homogeneous equation are of the form :

$$x(t) = C_1 + C_2 t.$$

where C_1, C_2 represent constants, which needs to be determined by the problem.

And we could easily to know that

$$x(t) = \int_a^t (t-s)f(s)ds$$

is a special solution of the (2.1).

Determine the solution by initial value $x(0) = x_0$, $x'(0) = x_1$. We can get C_1, C_2 corresponding values.

Hence, the solution of (2.1) is

$$x(t) = \int_0^t (t-s)f(s)ds + x_0 + x_1 t.$$

Thus, we obtain

$$x(t_1^-) = \int_0^{t_1} (t_1 - s)f(s)ds + x_0 + x_1 t.$$

$$x'(t_1) = \int_0^{t_1} f(s)ds + x_1.$$

Since

$$x(t_1) = b_1(\tau_1)x(t_1^-).$$

We obtain

$$x(t_1) = b_1(\tau_1)\int_0^{t_1} (t_1 - s)f(s)ds + b_1(\tau_1)(x_0 + x_1 t).$$

(ii) Next, we consider the interval $t \in [t_1, t_2)$. We change initial values

$$\begin{cases} x''(t) = f(t) \\ x(t_1) = b_1(\tau_1)\int_0^{t_1}(t_1 - s)f(s)ds + b_1(\tau_1)(x_0 + x_1 t) \\ x'(t_1) = \int_0^{t_1} f(s)ds + x_1. \end{cases} \tag{2.2}$$

By initial value, we obtain the solution of (2.2)

$$\begin{aligned} x(t) \quad &= \quad \int_{t_1}^t (t-s)f(s)ds + b_1(\tau_1)\int_0^{t_1}(t_1 - s)f(s)ds \\ &\quad + b_1(\tau_1)(x_0 + x_1 t_1) + t\left(\int_0^{t_1} f(s)ds + x_1\right). \end{aligned}$$

Thus, we obtain

$$x(t_2^-) \quad = \quad \int_{t_1}^{t_2} (t_2 - s)f(s)ds + b_1(\tau_1)\int_0^{t_1}(t_1 - s)f(s)ds$$

$$
\begin{aligned}
&+b_1(\tau_1)(x_0+x_1t_1)+t_2\left(\int_0^{t_1} f(s)ds+x_1\right)\\
x'(t_2) &= \int_0^{t_2} f(s)ds+x_1.
\end{aligned}
$$

Since

$$x(t_2)=b_2(\tau_2)x(t_2^-).$$

We obtain

$$
\begin{aligned}
x(t_2) &= b_2(\tau_2)\int_{t_1}^{t_2}(t_2-s)f(s)ds+b_2(\tau_2)b_1(\tau_1)\int_0^{t_1}(t_1-s)f(s)ds\\
&\quad +b_2(\tau_2)b_1(\tau_1)(x_0+x_1t_1)+t_2b_2(\tau_2)\left(\int_0^{t_1} f(s)ds+x_1\right).
\end{aligned}
$$

(iii) For $t\in[t_2,t_3)$, we change initial values

$$
\begin{aligned}
x(t_2) &= b_2(\tau_2)\int_{t_1}^{t}(t-s)f(s)ds+b_2(\tau_2)b_1(\tau_1)\int_0^{t_1}(t_1-s)f(s)ds\\
&\quad +b_2(\tau_2)b_1(\tau_1)(x_0+x_1t)+tb_2(\tau_2)\left(\int_0^{t_1} f(s)ds+x_1\right)\\
x'(t_2) &= \int_0^{t_1} f(s)ds+x_1.
\end{aligned}
$$

Then, by initial value, we obtain the solution of (2.2)

$$
\begin{aligned}
x(t) &= \int_{t_2}^{t}(t-s)f(s)ds+b_2(\tau_2)\int_{t_1}^{t_2}(t_2-s)f(s)ds\\
&\quad +b_2(\tau_2)b_1(\tau_1)\int_0^{t_1}(t_1-s)f(s)ds+b_2(\tau_2)b_1(\tau_1)(x_0+x_1t_1)\\
&\quad +b_2(\tau_2)t_2\left(\int_0^{t_1} f(s)ds+x_1\right)+t\left(\int_0^{t_2} f(s)ds+x_1\right).
\end{aligned}
$$

The next interval is as same as above. By mathematical induction, we obtain the solution.

Theorem 2.1 For a given $T>0$, a $\mathcal{R}^d$-valued stochastic process $x(t)$ on $[0,T]$ which is $\mathcal{F}_t-adapted$ for every $t\in J$ is called a mild solution to (1.1) in $(\Omega,P,\{\mathcal{F}_t\})$, if $x(t)$ is a solution of the following the integral equation

$$
\begin{aligned}
x(t) &= \sum_{k=0}^{\infty}\left\{\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)x_1t_i+\prod_{i=1}^{k} b_i(\tau_i)x_0+x_1t\right.\\
&\quad +\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\left[\int_{t_{i-1}}^{t_i}(t_i-s)f(s,x(s))ds\right.
\end{aligned}
$$

$$+t_i \int_0^{t_{i-1}} f(s,x(s))ds\Big] + \int_{t_k}^{t}(t-s)f(s,x(s))ds$$
$$+t\int_0^{t_k} f(s,x(s))ds\Big\} I_{(t_k,t_{k+1}]}(t).$$

where $\prod_{j=m}^{n}(\cdot) = 1$ as $m > n$, $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_K)b_k(\tau_K)...b_i(\tau_i)$, and $I_A(.)$ is the index function, i.e.

$$I_A(t) = \begin{cases} 1, & t \in A. \\ 0, & t \notin A, \end{cases}$$

proof:

From the above definition, we can prove one side, and then we just need to prove the other side.

The derivative of the integral equation is

$$x'(t) = x_1 + \int_0^t f(s,x(s))ds.$$

After that, it's easy to get the second order differential

$$x''(t) = f(t,x(t)).$$

And satisfy the initial condition

$$x(0) = x_0, \quad x'(0) = x_1.$$

Therefore, it is fully and necessary to prove on both sides.

Our existence are based on the following theorem, which is a version of the topological transversality theorem.

Theorem 2.2 (see [17]) Let B be a convex subset of a Banach space E and assume that $0 \in B$. Let F: $B \to B$ be a completely continuous operator and let

$$U(F) = \{x \in B : x = \lambda Fx \quad for \quad some \quad 0 < \lambda < 1\};$$

then either $U(F)$ is unbounded or F has a fixed point.

Next, we consider the Hyers-Ulam stability for the equation.

Consider the following inequality

$$E\|x''(t) - f(t,x(t))\|^2 < \varepsilon.$$

Theorem 2.3 (see [18]) Equation (1.1) is Hyers-Ulam stable if, for any $\varepsilon > 0$, there exists a solution $y(t)$ which satisfies the above inequality and has the same initial value as $x(t)$, where $x(t)$ is a solution to (1.1). Then, $y(t)$ satisfies

$$E\|y(t) - x(t)\|^2 < K\varepsilon,$$

in which K is a constant.

3. MAIN RESULT

In the 3.1, we study the existence of solutions to second order random impulsive functional differential equations by fixed point which is a version of the topological transversality theorem. The Hyers-Ulam stability of second order random impulsive functional differential equations are given in 3.2. We list the following basic assumptions of this paper and prove our main results.

3.1 EXISTENCE

(H_1) : $max_{i,k}\{\prod_{j=i}^{k}\|b_j(\tau_j)\|\}$ is uniformy bounded, that is to say, there is a constant $B > 0$ such that

$$\max_{i,k}\left\{\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\} \le B$$

for all $\tau_j \in D_j$, $j = 1, 2, \ldots$
(H_2) : There exists $\eta \in L^1([0,T], \Re^+)$ and a continuous increasing function K : $\Re^+ \to (0, \infty)$

$$E\|f(t, \psi(t))\|^2 \le \eta(t)K(E\|\psi(t)\|^2)$$

for any piecewise continuous function $\psi \in R^n$ and for $t \in [0, T]$.
(H_3) : Let $M = 32\max\{1, B^2\}T^3$, $c_1 = 18\max\{1, B^2\}M_1^2M_2^2$, $M_1 = max\{x_0, x_1\}$ and $M_2 = max\{1, T\}$.

Theorem 3.1 Suppose the conditions $(H_1) - (H_3)$ are satisfied, then system(2.1) has at least one solution $x(t)$ on J provided that :

$$M\int_{t_0}^{T}\eta(s)ds < \int_{c_1}^{\infty}\frac{ds}{K(s)}$$

proof: Assume T be a fixed any number in $0 < T < +\infty$ and satisfied condition of Theorem. First, We define an operator $F : \Gamma \to \Gamma$ by

$$\begin{aligned}(Fy)(t) \quad = \quad & \sum_{k=0}^{\infty}\left\{\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)x_1t_i + \prod_{i=1}^{k}b_i(\tau_i)x_0 + x_1t\right. \\ & \sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\left[\int_{t_{i-1}}^{t_i}(t_i - s)f(s, x(s))ds\right. \\ & \left. +t_i\int_0^{t_{i-1}}f(s, x(s))ds\right] + \int_{t_k}^{t}(t-s)f(s, x(s))ds\end{aligned}$$

$$+t\int_0^{t_k} f(s,x(s))ds\Bigg\} I_{(t_k,t_{k+1}]}(t).$$

Then the problem of finding solutions for problem (2.1) is reduced to finding the fixed point of F.

Step 1: We prove $U(F)$ is bounded. First, we give an estimate for the following solution in $\lambda \in (0,1)$. Then we apply the transversality theorem

$$\begin{aligned} x(t) &= \lambda \sum_{k=0}^{\infty}\Bigg\{\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)x_1t_i + \prod_{i=1}^{k} b_i(\tau_i)x_0 + x_1 t \\ &\quad + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\Bigg[\int_{t_{i-1}}^{t_i}(t_i-s)f(s,x(s))ds \\ &\quad + t_i\int_0^{t_{i-1}} f(s,x(s))ds\Bigg] + \int_{t_k}^{t}(t-s)f(s,x(s))ds \\ &\quad + t\int_0^{t_k} f(s,x(s))ds\Bigg\} I_{(t_k,t_{k+1}]}(t). \end{aligned}$$

By $(H_1),(H_3)$, we obtain

$$\begin{aligned} \|x(t)\|^2 &\leq \lambda^2\left(\sum_{k=0}^{\infty}\Bigg\{\sum_{i=1}^{k}\left\|\prod_{j=i}^{k} b_j(\tau_j)\right\|\|x_1\|\|t_i\| + \left\|\prod_{i=1}^{k} b_i(\tau_i)\right\|\|x_0\|\right. \\ &\quad + \sum_{i=1}^{k}\left\|\prod_{j=i}^{k} b_j(\tau_j)\right\|\Bigg[\int_{t_{i-1}}^{t_i}\|(t_i-s)f(s,x(s))\|ds \\ &\quad + \|t_i\|\int_0^{t_{i-1}}\|f(s,x(s))\|ds\Bigg] + \int_{t_k}^{t}\|(t-s)f(s,x(s))\|ds \\ &\quad \left. + \|t\|\int_0^{t_k}\|f(s,x(s))\|ds\Bigg\} I_{(t_k,t_{k+1}]}(t)\right)^2 \\ &\leq 2\Bigg[\left(\sum_{k=0}^{\infty}\Bigg\{\sum_{i=1}^{k}\left\|\prod_{j=i}^{k} b_j(\tau_j)\right\|\|x_1\|\|t_i\|\right. \\ &\quad \left. + \left\|\prod_{i=1}^{k} b_i(\tau_i)\right\|\|x_0\| + \|x_1\|\|t\|\Bigg\} I_{(t_k,t_{k+1}]}(t)\right)^2 \\ &\quad + \left(\sum_{k=0}^{\infty}\Bigg\{\sum_{i=1}^{k}\left\|\prod_{j=i}^{k} b_j(\tau_j)\right\|\Bigg[\int_{t_{i-1}}^{t_i}\|(t_i-s)f(s,x(s))\|ds\right. \\ &\quad + \|t_i\|\int_0^{t_{i-1}}\|f(s,x(s))\|ds\Bigg] + \int_{t_k}^{t}\|(t-s)f(s,x(s))\|ds \end{aligned}$$

$$
\begin{aligned}
& +\|t\| \int_0^{t_k} \|f(s,x(s))\|ds\Big\} I_{(t_k,t_{k+1}]}(t)\Big)^2\Bigg] \\
\leq\ & 18\left[\max_{i,k}\left\{1, \prod_{j=i}^{k}\|b_j(\tau_j)\|, \prod_{i=1}^{k}\|b_i(\tau_i)\|\right\}\right]^2 M_1^2 M_2^2 \\
& +32\left[\max_{i,k}\left\{1, \prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}\right]^2 T^2 \left(\int_0^t \|f(s,x(s))\|ds\right)^2 \\
\leq\ & 18\max\{1,B^2\}M_1^2M_2^2 + 32\max\{1,B^2\}T^3\int_0^t \|f(s,x(s))\|^2 ds.
\end{aligned}
$$

Therefore, by (H_2), we have

$$
\begin{aligned}
E\|x(t)\|^2 \ \leq\ & 18\max\{1,B^2\}M_1^2M_2^2 \\
& +32\max\{1,B^2\}T^3\int_0^t E[\|f(s,x(s))\|^2]ds \\
\leq\ & 18\max\{1,B^2\}M_1^2M_2^2 \\
& +32\max\{1,B^2\}T^3\int_0^t \eta(s)K(E[\|x(s)\|^2])ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sup_{0\leq v\leq t} E\|x(v)\|^2 \ \leq\ & 18\max\{1,B^2\}M_1^2M_2^2 \\
& +32\max\{1,B^2\}T^3\int_0^t \eta(s)K(E[\|x(s)\|^2])ds \\
\leq\ & 18\max\{1,B^2\}M_1^2M_2^2 \\
& +32\max\{1,B^2\}T^3\int_0^t \eta(s)K(\sup_{0\leq v\leq s} E[\|x(v)\|^2])ds.
\end{aligned}
$$

Next, we define the function $\vartheta(t)$

$$
\vartheta(t) = \sup_{t_0\leq v\leq t} E\|x(v)\|^2, \qquad t\in[0,T].
$$

It also satisfy the following inequality for any $t\in[0,T]$,

$$
\vartheta(t) \leq 18\max\{1,B^2\}M_1^2M_2^2 + 32\max\{1,B^2\}T^3\int_0^t \eta(s)K(\vartheta(s))ds. \tag{3.2}
$$

Defined the inequality(3.2) as $m(t)$, we easily get that

$$
\begin{aligned}
\vartheta(t) &\leq m(t), \qquad t\in[0,T], \\
m(0) &= 18\max\{1,B^2\}M_1^2M_2^2 = c_1,
\end{aligned}
$$

and

$$\begin{aligned} m'(t) &= 32\max\{1,B^2\}T^3\eta(t)K(\vartheta(t)) \\ &\leq 32\max\{1,B^2\}T^3\eta(t)K(m(t)), \qquad t\in[0,T]. \end{aligned}$$

Hence, we obtain

$$\frac{m'(t)}{K(m(t))}\leq 32\max\{1,B^2\}T^3\eta(t), \qquad t\in[0,T]. \tag{3.3}$$

To integrate (3.3), we have

$$\begin{aligned} \int_{m(0)}^{m(t)}\frac{ds}{K(s)} &\leq 32\max\{1,B^2\}T^3\int_0^t\eta(s)ds \\ &\leq 32\max\{1,B^2\}T^3\int_0^T\eta(s)ds \\ &\leq \int_{m(0)}^{\infty}\frac{ds}{K(s)}, \qquad t\in[0,T], \end{aligned}$$

Thus, $m(t)$ is obviously bounded by the mean value theorem. Because $\vartheta(t)\leq m(t)$, we can get $\vartheta(t)\leq\eta_1$ where η_1 is a constant that is decided by T, K and η. Because $sup_{0\leq v\leq t}E\|x(v)\|^2=\vartheta(t)$ for every $t\in[0,T]$, we obtain $sup_{0\leq v\leq t}E\|x(v)\|^2\leq\eta_1$

$$E\|x\|_\Gamma^2=\sup_{0\leq v\leq t}E\|x(v)\|^2\leq\eta_1.$$

Hence, we know that $U(F)=\{x\in\Gamma: x=\lambda Fx, for \quad some \quad 0<\lambda<1\}$ is bounded.

Next, by steps 2 to 4, we prove that F is completely continuous.

Step 2: We give a proof that F is continuous.

Assume $\{x_n\}$ be a convergent sequence and converges to X in Γ. Hence, for each $t\in[0,T]$, we obtain

$$\begin{aligned} (Fx_n)(t) &= \sum_{k=0}^{\infty}\Bigg\{\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)x_1t_i+\prod_{i=1}^{k}b_i(\tau_i)x_0+x_1t \\ &\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\Bigg[\int_{t_{i-1}}^{t_i}(t_i-s)f(s,x_n(s))ds \\ &\quad+t_i\int_0^{t_{i-1}}f(s,x_n(s))ds\Bigg]+\int_{t_k}^{t}(t-s)f(s,x_n(s))ds \\ &\quad+t\int_0^{t_k}f(s,x_n(s))ds\Bigg\}I_{(t_k,t_{k+1}]}(t). \end{aligned}$$

Hence,

$$(Fx_n)(t)-(Fx)(t) = \sum_{k=0}^{\infty}\Bigg\{\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\Bigg[\int_{t_{i-1}}^{t_i}(t_i-s)(f(s,x_n(s))-f(s,x(s)))ds$$

$$+t_i\int_0^{t_{i-1}}(f(s,x_n(s))-f(s,x(s)))ds\Big]$$
$$+\int_{t_k}^{t}(t-s)(f(s,x_n(s))-f(s,x(s)))ds$$
$$+t\int_0^{t_k}(f(s,x_n(s))-f(s,x(s)))ds\Big\}I_{(t_k,t_{k+1}]}(t)$$

and

$$\begin{aligned}E\|(Fx_n)(t)-(Fx)(t)\|^2 &\le 16\max\{1,B^2\}T^3\int_0^t E\|f(s,x_n(s))-f(s,x(s))\|^2ds\\ &\longrightarrow 0 \quad as \quad n\longrightarrow\infty.\end{aligned}$$

Thus F is obviously continuous.

Step 3: We prove F maps B_m to an equicontinuous family.

Define

$$B_m=\{x\in\Gamma\mid \|x\|_\Gamma^2\le m\}$$

for any $m\ge 0$.

Let $y\in B_m$ and $t_1,t_2\in[0,T]$. If $t_0<t_1<t_2<T$, thus by hypotheses $(H_1)-(H_3)$ and the condition of the theorem, we obtain

$$\begin{aligned}(Fx)(t_1)-(Fx)(t_2)=&\sum_{k=0}^{\infty}\Big\{\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)x_1t_i+\prod_{i=1}^{k}b_i(\tau_i)x_0+x_1t_1\\ &+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\Big[\int_{t_{i-1}}^{t_i}(t_i-s)f(s,x(s))ds+t_i\int_0^{t_{i-1}}f(s,x(s))ds\Big]\\ &+\int_{t_k}^{t_1}(t_1-s)f(s,x(s))ds+t_1\int_0^{t_k}f(s,x(s))ds\Big\}I_{(t_k,t_{k+1}]}(t_1)\\ &-\sum_{k=0}^{\infty}\Big\{\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)x_1t_i+\prod_{i=1}^{k}b_i(\tau_i)x_0+x_1t_2\\ &+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\Big[\int_{t_{i-1}}^{t_i}(t_i-s)f(s,x(s))ds+t_i\int_0^{t_{i-1}}f(s,x(s))ds\Big]\\ &+\int_{t_k}^{t_2}(t_2-s)f(s,x(s))ds+t_2\int_0^{t_k}f(s,x(s))ds\Big\}I_{(t_k,t_{k+1}]}(t_2).\end{aligned}$$

Hence,

$$\begin{aligned}(Fx)(t_1)-(Fx)(t_2) &= \sum_{k=0}^{\infty}\Big\{\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)x_1t_i+\prod_{i=1}^{k}b_i(\tau_i)x_0+x_1t_1\\ &+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\Big[\int_{t_{i-1}}^{t_i}(t_i-s)f(s,x(s))ds+t_i\int_0^{t_{i-1}}f(s,x(s))ds\Big]\end{aligned}$$

$$+\int_{t_k}^{t_1}(t_1-s)f(s,x(s))ds+t_1\int_0^{t_k}f(s,x(s))ds\Big\}$$
$$\big(I_{(t_k,t_{k+1}]}(t_1)-I_{(t_k,t_{k+1}]}(t_2)\big)$$
$$+\sum_{k=0}^{\infty}\Big\{x_1(t_1-t_2)+\int_0^{t_1}(t_1-t_2)f(s,x(s))ds$$
$$+\int_{t_2}^{t_1}(t_2-s)f(s,x(s))ds\Big\}I_{(t_k,t_{k+1}]}(t_2).$$

Thus,

$$E\|(Fx)(t_1)-(Fx)(t_2)\|^2\le 2E\|H_1\|^2+2E\|H_2\|^2 \tag{3.1}$$

where

$$\begin{aligned} H_1 &= \sum_{k=0}^{\infty}\Big\{\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)x_1t_i+\prod_{i=1}^{k}b_i(\tau_i)x_0+x_1t_1 \\ &\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\left[\int_{t_{i-1}}^{t_i}(t_i-s)f(s,x(s))ds\right. \\ &\quad\left.+t_i\int_0^{t_{i-1}}f(s,x(s))ds\right]+\int_{t_k}^{t_1}(t_1-s)f(s,x(s))ds \\ &\quad+t_1\int_0^{t_k}f(s,x(s))ds\Big\}\big(I_{(t_k,t_{k+1}]}(t_1)-I_{(t_k,t_{k+1}]}(t_2)\big)\end{aligned}$$

and

$$\begin{aligned} H_2 &= \sum_{k=0}^{\infty}\Big\{x_1(t_1-t_2)+\int_0^{t_1}(t_1-t_2)f(s,x(s))ds \\ &\quad+\int_{t_2}^{t_1}(t_2-s)f(s,x(s))ds\Big\}I_{(t_k,t_{k+1}]}(t_2).\end{aligned}$$

Therefore,

$$\begin{aligned} E\|H_1\|^2 &\le E\Bigg(\sum_{k=0}^{\infty}\Big\{\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\|x_1\|\|t_i\|+\prod_{i=1}^{k}\|b_i(\tau_i)\|x_0+x_1t_1 \\ &\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\left[\int_{t_{i-1}}^{t_i}\|t_i-s\|\|f(s,x(s))\|ds\right. \\ &\quad\left.+\|t_i\|\int_0^{t_{i-1}}\|f(s,x(s))\|ds\right]+\int_{t_k}^{t_1}\|t_1-s\|\|f(s,x(s))\|ds \\ &\quad+t_1\int_0^{t_k}\|f(s,x(s))\|ds\Big\}H_{(t_k,t_{k+1}]}(t_1)-H_{(t_k,t_{k+1}]}(t_2)\Bigg)^2 \\ &\le 18\max\{1,B^2\}M_1^2M_2^2E\left(H_{(t_k,t_{k+1}]}(t_1)-H_{(t_k,t_{k+1}]}(t_2)\right)\end{aligned}$$

$$
\begin{aligned}
& +32\max\{1,B^2\}T^2t_1E\int_0^{t_1}\|f(s,x(s))\|^2ds\\
& E\left(H_{(t_k,t_{k+1}]}(t_1)-H_{(t_k,t_{k+1}]}(t_2)\right)\\
\leq\ & 18\max\{1,B^2\}M_1^2M_2^2E\left(H_{(t_k,t_{k+1}]}(t_1)-H_{(t_k,t_{k+1}]}(t_2)\right)\\
& +32\max\{1,B^2\}T^2t_1\int_0^{t_1}\eta(s)K(E\|x(s)\|^2)ds\\
& E\left(H_{(t_k,t_{k+1}]}(t_1)-H_{(t_k,t_{k+1}]}(t_2)\right)\\
\leq\ & 18\max\{1,B^2\}M_1^2M_2^2E\left(H_{(t_k,t_{k+1}]}(t_1)-H_{(t_k,t_{k+1}]}(t_2)\right)\\
& +32\max\{1,B^2\}T^2t_1\int_0^{t_1}P^*K(E(m))ds\\
& E\left(H_{(t_k,t_{k+1}]}(t_1)-H_{(t_k,t_{k+1}]}(t_2)\right)\\
& \longrightarrow 0 \quad as \quad t_2\longrightarrow t_1,
\end{aligned}
$$

where $P^*=sup\{\eta(t):t\in[0,T]\}$, thus

$$
\begin{aligned}
E\|H_2\|^2 \leq\ & E\left(\sum_{k=0}^{\infty}\left\{x_1\|t_1-t_2\|+\int_0^{t_1}\|t_1-t_2\|\|f(s,x(s))\|ds\right.\right.\\
& \left.\left.-\int_{t_1}^{t_2}\|t_2-s\|\|f(s,x(s))\|ds\right\}I_{(t_k,t_{k+1}]}(t_2)\right)^2\\
\leq\ & 3x_1^2\|t_1-t_2\|^2+3t_1E\int_0^{t_1}\|t_1-t_2\|^2\|f(s,x(s))\|^2ds\\
& +3(t_2-t_1)E\int_{t_1}^{t_2}\|t_2-s\|^2\|f(s,x(s))\|^2ds\\
\leq\ & 3x_1^2\|t_1-t_2\|^2+3t_1E\int_0^{t_1}\|t_1-t_2\|^2P^*K(m)ds\\
& +3(t_2-t_1)E\int_{t_1}^{t_2}\|t_2-s\|^2P^*K(m)ds\\
& \longrightarrow 0 \quad as \quad t_2\longrightarrow t_1.
\end{aligned}
$$

The two inequalities we prove above do not depend on $x\in B_m$. For (3.1), by these two inequalities, we know that (3.1) converges to zero as $t_2\to t_1$.
Step 4: We prove FB_m is uniformly bounded.
By the condition of the theorem, we get $\| y \|_\Gamma^2\leq m$ and

$$
\begin{aligned}
& \|(Fx)(t)\|^2\\
\leq\ & \left(\sum_{k=0}^{\infty}\left\{\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\|x_1\|\|t_i\|+\prod_{i=1}^{k}\|b_i(\tau_i)\|\|x_0\|+\|x_1\|\|t\|\right.\right.\\
& +\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\left[\int_{t_{i-1}}^{t_i}\|t_i-s\|\|f(s,x(s))\|ds\right.
\end{aligned}
$$

$$
\begin{aligned}
& +t_i \int_0^{t_{i-1}} \|f(s,x(s))\|ds \Big] + \int_{t_k}^{t} \|t-s\|\|f(s,x(s))\|ds \\
& +t \int_0^{t_k} \|f(s,x(s))\|ds \Big\} I_{(t_k,t_{k+1}]}(t) \Big) \\
\leq \quad & 18 \left[\max_{i,k} \left\{ 1, \prod_{j=i}^{k} \|b_j(\tau_j)\|, \prod_{i=1}^{k} \|b_i(\tau_i)\| \right\} \right]^2 \\
& [max\{x_0,x_1\}]^2 [max\{1,T\}]^2 \\
& +32 \left[\max_{i,k} \left\{ 1, \prod_{j=i}^{k} \|b_j(\tau_j)\| \right\} \right]^2 T^2 \left(\int_0^t \|f(s,x(s))\|ds \right)^2 .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E\|(Fx)(t)\|^2 & \leq 18 \max\{1,C^2\} D_1^2 D_2^2 + 32 \max\{1,C^2\} T^3 \int_0^t E\|f(s,x(s))\|^2 ds, \\
E\|(Fx)(t)\|^2 & \leq 18 \max\{1,C^2\} D_1^2 D_2^2 + 32 \max\{1,C^2\} T^3 \|\alpha_m\|_{L^1}.
\end{aligned}
$$

Therefore, the set $\{(Fx)(t), \|y\|_\Gamma^2 \leq m\}$ and $\{FB\}$ is not uniformly unbounded. And we have proved FB_m is an equicontinuous set. Thus, according to Arzela-Ascoli theorem, we deuce that $\{FB\}$ is compact, and F is completely continuous. It follows from the fixed point theorem 2.2 that F has a fixed point on Γ. Hence, the system(1.1) has a solution. This completes the proof of the theorem.

3.2 HYERS-ULAM STABILITY

Theorem 3.2 Suppose $(H_1)-(H_3)$ hold. Then system (1.1) has at least one solution on J and this solution is Hyers-Ulam stable.

proof:

By Theorem 3.1, we can prove the existence of this solution. Therefore, we consider the inequality

$$E\|y''(t) - f(t,y(t))\|^2 < \varepsilon.$$

Assume there are a function $f_1(t,y(t))$ satisfies

$$\|f(t,x(t)) - f_1(t,y(t))\| < \varepsilon.$$

Thus,

$$
\begin{cases}
y''(t) = f_1(t,y(t)), \quad t \in J, \quad t \neq t_k, \\
y(t_k) = b_k(\tau_k) y(t_k^-), \quad k = 1,2,\cdots \\
y(0) = x_0 \quad y'(0) = x_1.
\end{cases}
\tag{4.1}
$$

To equation (4.1), by the fundamental solution, we get

$$\begin{aligned} y(t) &= \sum_{k=0}^{\infty}\left\{\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)x_1 t_i + \prod_{i=1}^{k} b_i(\tau_i)x_0 + x_1 t\right. \\ &\quad + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\left[\int_{t_{i-1}}^{t_i}(t_i - s)f_1(s, y(s))ds\right. \\ &\quad \left. + t_i\int_0^{t_{i-1}} f_1(s, y(s))ds\right] + \int_{t_k}^{t}(t-s)f_1(s, y(s))ds \\ &\quad \left. + t\int_0^{t_k} f_1(s, y(s))ds\right\} I_{(t_k, t_{k+1}]}(t). \end{aligned}$$

Assume $\varepsilon < 1$, we obtain

$$\begin{aligned} E\|x(t) - y(t)\|^2 &\le 4B^2T^2E\|\int_{t_{i-1}}^{t_i}(f(s, x(s)) - f_1(s, y(s)))\|^2 \\ &\quad + 4B^2E\|t_i\int_0^{t_{i-1}}(f(s, x(s)) - f_1(s, y(s)))ds\|^2 \\ &\quad + 4T^2E\|\int_{t_k}^{t}(f(s, x(s)) - f_1(s, y(s)))ds\|^2 \\ &\quad + 4T^2E\|\int_0^{t_k}(f(s, x(s)) - f_1(s, y(s)))ds\|^2 \\ &\le (8B^2T^4 + 8T^4)\varepsilon. \end{aligned}$$

Here $K = 8B^2T^4 + 8T^4$ satisfies Theorem 2.3.

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