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COMMENTS ON SOME NEW CLASSES OF SIGMOIDAL AND ACTIVATION FUNCTIONS. APPLICATIONS

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ABSTRACT: In this paper we study a new class of sigmoidal functions. We will consider the possibility of approximating the function

$$\kappa(t) = \begin{cases} 0, \ 0 \le t < 1 \\ [0,1], \ t = 1 \end{cases}$$

by new family with respect to Hausdorff distance. Some applications in the theory of impulse technics, filter synthesis and debugging theory are given. We analyze also the "real wealth data" and "actual data to estimate the number of software residual faults" by the new sigmoid. Some visualizations of the typical emitting charts are also given. We will consider also the possibility of approximating the function

$$v(t) = \begin{cases} 0, \ -1 \le t \le -\frac{2\beta}{m} \\ \frac{1}{2}, \ -\frac{2\beta}{m} < t < \frac{2\beta}{m} \\ 1, \ \frac{2\beta}{m} \le t \le 1 \end{cases}$$

by the new generalized Yun's activation function. Approximating of this function is related to the analysis of electric steps and chains. Numerical examples using *CAS Mathematica*, illustrating our results are presented.

AMS Subject Classification: 41A46

Key Words: new family of sigmoidal functions, Interval function, $\kappa-$ function, "Prototype filter", Hausdorff distance

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1. INTRODUCTION AND PRELIMINARIES

Sigmoidal functions (also known as "activation functions") find multiple applications to population dynamics, artificial neural networks, antenna–feeder technique, debugging theory and others [1]–[11], [21]–[35].

Definition 1. The κ function is defined by

$$\kappa(t) = \begin{cases} 0, \ 0 \le t < 1 \\ [0,1], \ t = 1 \end{cases} \tag{1}$$

Definition 2. [12] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$.

More precisely,

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \tag{2}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

Definition 3. The new class of sigmoid functions in the interval $[0,1)$ is defined by:

$$M(t) = 1 - \left(\frac{\frac{1+d^{\frac{1}{m}}}{1-d^{\frac{1}{m}}} - t^\beta}{\frac{1+d^{\frac{1}{m}}}{1-d^{\frac{1}{m}}} + t^\beta} \right)^m . \tag{3}$$

We will explicitly point out that the functions of type $\left(\frac{\alpha^\beta + t^\beta}{\alpha^\beta - t^\beta}\right)^k$ have been used substantially by Dombi, Jonas, Toth and Arva in generating the proposed new "omega probability distribution" [10] (see, also [36]).

It is known that for the value of the best Hausdorff approximation of the function

$$\kappa^*(t) = \begin{cases} 0, \ 0 \le t \in [-1,1) \\ 1, \ t = 1 \end{cases}$$

by algebraic polynomial of degree less than n

$$P(t) = \delta T_n\left(\frac{2t+\delta}{2-\delta}\right)$$

($T_n(t)$ is the Chebyshev polynomial, see, Fig. 1) the following is valid [12]:

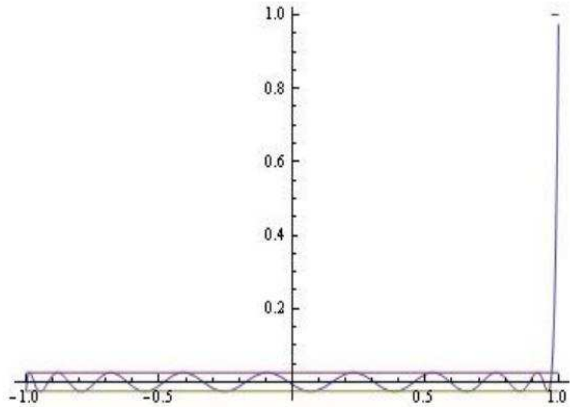

Figure 1: The polynomial of the best Hausdorff distance; $n = 19$; $E_n(\kappa^*) = 0.02571612$ (see, [11].

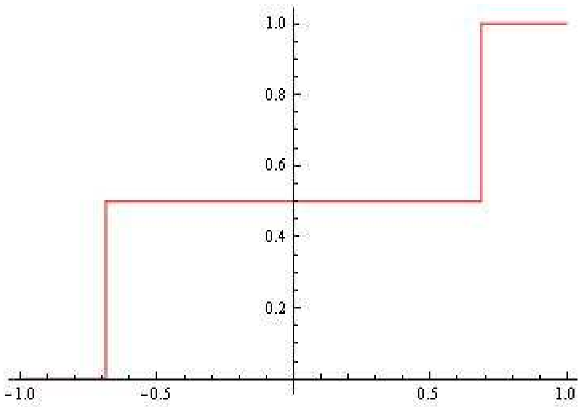

Figure 2: The impulse function $v(t)$ for $\beta = 11$; $m = 32$.

$$\delta = 2\left(\frac{\ln n}{n}\right)^2 + O\left(\frac{\ln n}{n^2}\right).$$

Definition 4. The typical example of impulse function $v(t)$ is the following (see, Fig. 2)

$$v(t) = \begin{cases} 0, \ -1 \leq t \leq -\frac{2\beta}{m} \\ \frac{1}{2}, \ -\frac{2\beta}{m} < t < \frac{2\beta}{m} \\ 1, \ \frac{2\beta}{m} \leq t \leq 1. \end{cases} \tag{4}$$

Approximating of this function is related to the analysis of electric steps and

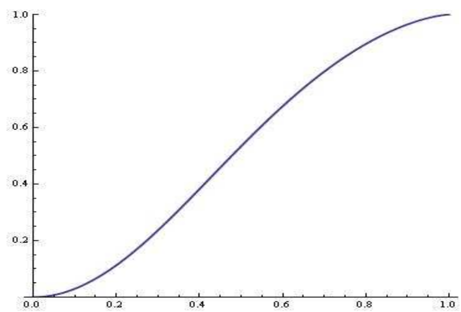

Figure 3: The sigmoid $M(t)$ for $\beta = 2$; $d = 0.0001$; $m = 1.5$.

chains.

Definition 5. We define the following generalized Yun's activation function [7]–[8] by:

$$W(t) = \frac{(1+t^\beta)^m}{(1+t^\beta)^m + (1-t^\beta)^m} \tag{5}$$

where β is odd and $\frac{2\beta}{m} < 1$.

The basic approaches for approximation of functions and point sets of the plane by algebraic and trigonometric polynomials with respect to Hausdorff distance (H–distance) are connected to the work and achievements of Bl. Sendov who established a Bulgarian school in Approximation theory, particularly developing the theory of Hausdorff approximations.

For some basic results about H–continuous functions and their application to problems in abstract areas such as Real Analysis, Approximation Theory and Set–valued Analysis see, [13]–[19]. For other applications of Hausdorff distance, see [52].

2. MAIN RESULTS

2.1. SOME PROPERTIES OF THE FUNCTION $M(T)$

As long as the function $M(t)$ can be considered as an "activation function" in the interval $[0,1)$ (see for instance Fig. 3), in this article we will consider the possibility of approximating the function $\kappa(t)$ with the new family in respect of Hausdorff distance.

The Hausdorff distance d between the $\kappa(t)$ and the sigmoidal function $M(t)$ satisfies (see, Fig. 4 – Fig. 6)

$$M(1) = 1 - d \tag{6}$$

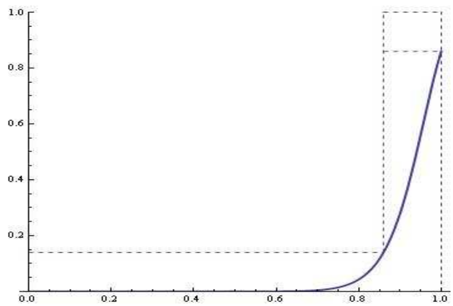

Figure 4: The sigmoid $M(t)$ for $m = 14$; $\beta = 17.0138$; H–distance $d = 0.14$.

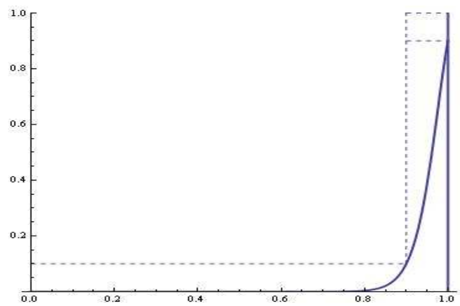

Figure 5: The sigmoid $M(t)$ for $m = 2$; $\beta = 28.3028$; H–distance $d = 0.1$.

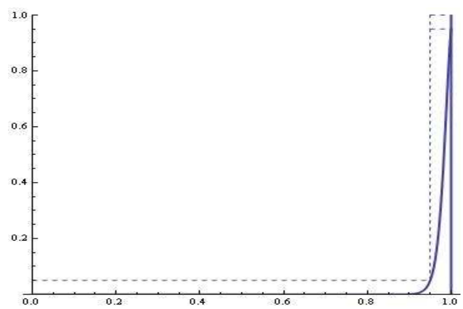

Figure 6: The sigmoid $M(t)$ for $m = 2$; $\beta = 76.0654$; H–distance $d = 0.05$.

$$M(1-d) = 1 - \left(\frac{\frac{1+d^{\frac{1}{m}}}{1-d^{\frac{1}{m}}} - (1-d)^{\beta}}{\frac{1+d^{\frac{1}{m}}}{1-d^{\frac{1}{m}}} + (1-d)^{\beta}} \right)^{m} = d. \tag{7}$$

Obviously, equality (6) is fulfilled. At set values of parameters m and β, the value d searched is calculated from the nonlinear equation (7). With some constraints

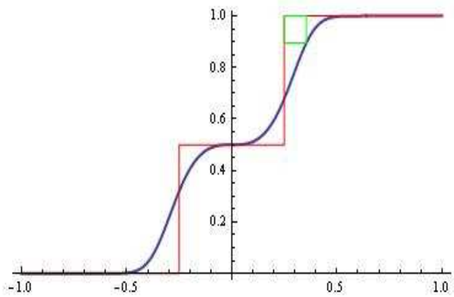

Figure 7: The function $W(t)$ for $m = 24$; $\beta = 3$; H–distance $d = 0.104791$.

imposed on these parameters, it can be shown that the nonlinear equation (7) has a positive root for d. A precise result can be found in Section 4. Approximations of the $\kappa(t)$ by family $M(t)$ for various m and β are visualized on Fig. 4 – Fig. 6.

2.2. SOME PROPERTIES OF THE GENERALIZED YUN'S ACTIVATION FUNCTION $W(T)$

For the one–sided Hausdorff distance between $v(t)$ and $W(t)$ is valid:

$$W\left(\frac{\beta}{m} + d\right) = 1 - d. \tag{8}$$

At set values of parameters m and β, the value d searched is calculated from the nonlinear equation (8).

Approximations of the $v(t)$ by family $W(t)$ for various m and β are visualized on Fig. 7 – Fig. 9.

From Figures 7 – 9, we see that the "saturation" is faster.

3. SOME APPLICATIONS

3.1. APPROXIMATING THE "REAL WEALTH DATA"

For example the appropriate lest–square fitting of the real wealth data by the model

$$M^*(t) = \omega M(t) \tag{9}$$

yields for $m = 68$; $\beta = 16.0979$, $d = 1. \times 10^{-14}$, $\omega = 27638.4$ (see, Fig. 10).

```
β = 7
m = 24
G[t_] := (1 + t^β)^m / ((1 + t^β)^m + (1 - t^β)^m)
g4 = Plot[(1 + t^β)^m / ((1 + t^β)^m + (1 - t^β)^m), {t, -L, L},
    PlotRange → {-0.02, 1.02}, PlotStyle → {Thick}, AxesOrigin → {0, 0},
    AspectRatio → 0.5, PlotStyle → {Thick}];
Clear[d]
FindRoot[(1 + (2 * β / m + d)^β)^m / ((1 + (2 * β / m + d)^β)^m + (1 - (2 * β / m + d)^β)^m) - 1 + d, {d, 0.4}]
d = 0.0734997
Show[g4, ListLinePlot[{{-2 * β / m, 0}, {-2 * β / m, 0.5}}, PlotStyle → {Red}],
 ListLinePlot[{{-2 * β / m, 0.5}, {2 * β / m, 0.5}}, PlotStyle → {Red}],
 ListLinePlot[{{2 * β / m, 0.5}, {2 * β / m, 1}}, PlotStyle → {Red}],
 ListLinePlot[{{2 * β / m, 1}, {1, 1}}, PlotStyle → {Red}], ListLinePlot[{{-1, 0}, {-2 * β / m, 0}},
  PlotStyle → {Red}], Plot[1, {t, 2 * β / m, 2 * β / m + d},
   PlotRange → Full, PlotStyle → {Green}], Plot[1 - d, {t, 2 * β / m, 2 * β / m + d}, PlotRange → Full,
  PlotStyle → {Green}],
 ListLinePlot[{{2 * β / m, 1 - d}, {2 * β / m, 1}}, PlotStyle → {Green}],
 ListLinePlot[{{2 * β / m + d, 1 - d}, {2 * β / m + d, 1}}, PlotStyle → {Green}],
 AspectRatio → 0.5]

7

24

{d → 0.0734997}

0.0734997
```

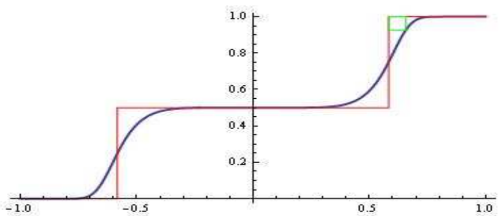

Figure 8: The function $W(t)$ for $m = 24$; $\beta = 7$; H–distance $d = 0.0734997$.

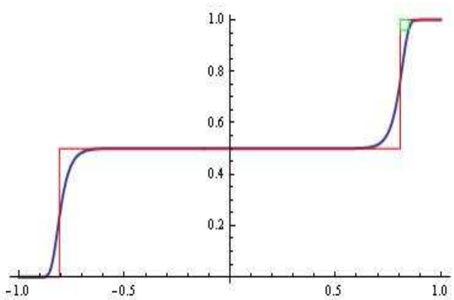

Figure 9: The function $W(t)$ for $m = 52$; $\beta = 21$; H–distance $d = 0.0394412$.

3.2. APPLICATION IN THE THEORY OF IMPULSE TECHNICS

The results have independent significance in the study of issues related to neural networks and impulse technics.

For example, after the substitution $t = kl\cos\theta + a$, where

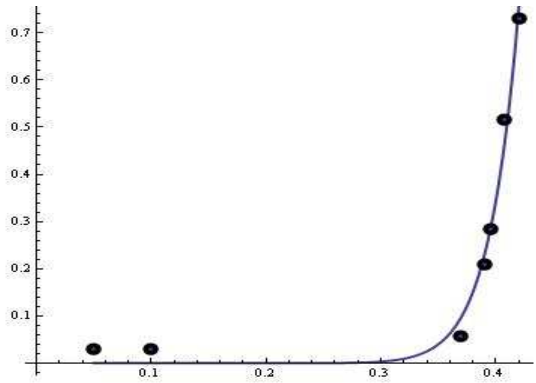

Figure 10: The fitted model $M^*(t)$.

– $k = \frac{2\pi}{\lambda}$, λ is the wave length;
– a is the phase difference;
– θ is the azimuthal angle;
– l is the distance between the emitters ($l = \frac{\lambda}{2}$ is fixed),
the our model gives typical emitting chart of antenna factor (see, Fig. 11 – Fig. 12).

Of course, the question of the practical realization of the activation functions which are generated as emitting charts remains open.

Remark. In many cases it is important for the specialists working in this field to pre-set the magnitude d associated with the "noise in the antenna".

In this setting and fixed parameters: a, m; λ, the unknown quantity β can be calculated as the root of the nonlinear equation (7).

3.3. APPROXIMATING THE "ACTUAL DATA TO ESTIMATE THE NUMBER OF SOFTWARE RESIDUAL FAULTS"

We analyze the following data [37]–[38] (see, Fig. 13)

After that using the model $M^*(t)$ for $\omega = 5186$, $m = 10$, $d = 0.97$ and $\beta = 1.1752$ we obtain the fitted model (see, Fig. 14).

3.4. FILTERS, DESIGNED BY APPROXIMATION OF THE FUNCTION κ

For the first time similar problem was discussed by Sendov, Shinev and Kyurkchiev in [39] as a natural research continuation of the possibility for design of Hausdorff type diagram functions and examination of linear antenna grids.

Details could be found in [40]–[43].

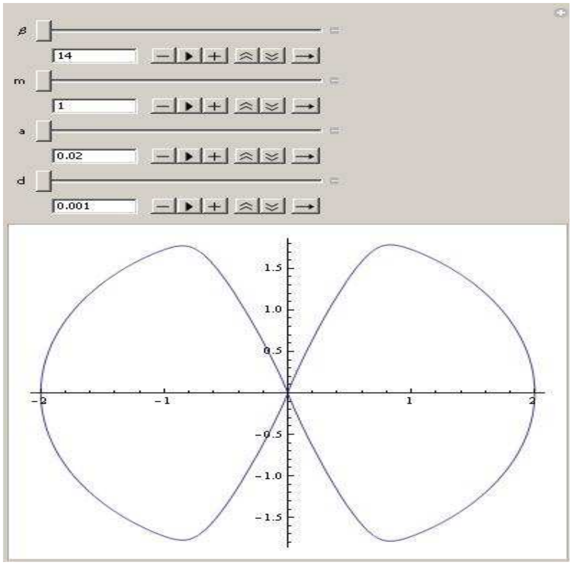

Figure 11: Typical emitting chart for $\beta = 14$; $a = 0.02$; $m = 1$; $d = 0.001$.

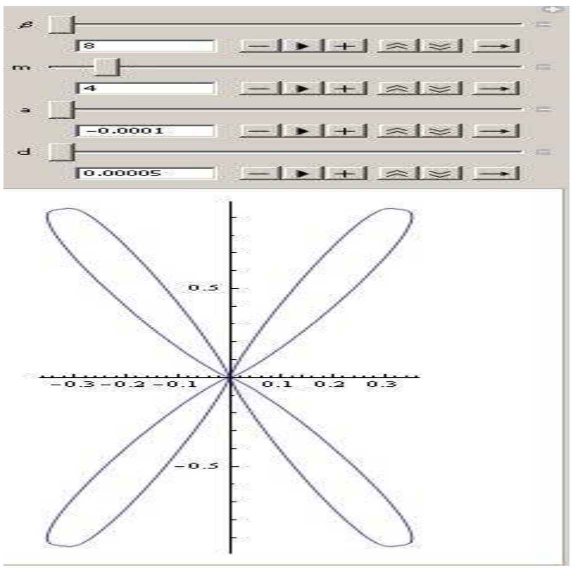

Figure 12: Typical emitting chart for $\beta = 8$; $a = -0.0001$; $m = 4$; $d = 0.00005$.

The polynomial of the best Hausdorff approximation of the function κ, is used in practice for different goals in the field of filter synthesis [44]–[45].

Week	Cumulative number of software faults	Week	Cumulative number of software faults
1	248	31	4351
2	262	32	4401
3	372	33	4439
4	526	34	4488
5	742	35	4548
6	958	36	4596
7	1215	37	4629
8	1471	38	4680
9	1738	39	4713
10	1936	40	4749
11	1971	41	4783
12	2147	42	4817
13	2258	43	4849
14	2418	44	4877
15	2567	45	4901
16	2688	46	4928
17	2809	47	4950
18	2925	48	4970
19	3026	49	4998
20	3205	50	5024
21	3348	51	5060
22	3476	52	5085
23	3573	53	5088
24	3719	54	5090
25	3750	55	5110
26	3952	56	5129
27	4048	57	5139
28	4137	58	5167
29	4251	59	5186
30	4301		

Figure 13: the "actual data to estimate the number of software residual faults" [37]–[38].

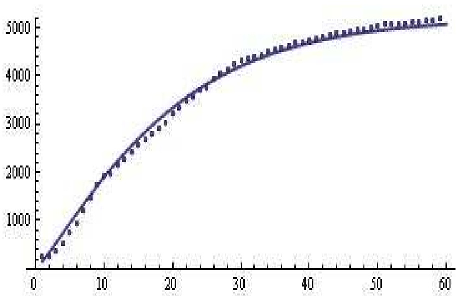

Figure 14: The fitted model $M^*(t)$.

For example a typical filter with a "pass-band" in the interval $[0, 1-d]$ is shown on the Fig. 15.

Consider the function

$$D(t) = 1 - M(t). \tag{10}$$

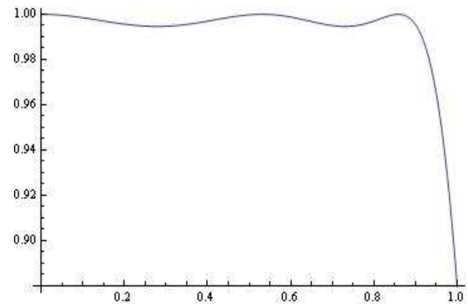

Figure 15: A typical filter with a "pass-band" in the interval $[0, 1-d]$ (see, for example [44]–[45]).

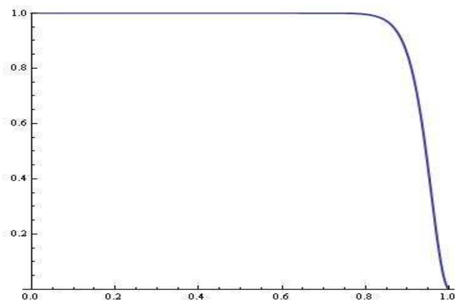

Figure 16: A typical "prototype filter" – $D(t)$ for $\beta = 30$; $m = 2$; $d = 0.001$.

A typical "prototype filter" – $D(t)$ is plotted on Fig. 16.

3.5. APPROXIMATING THE SPECIFIC "CANCER STEM CELL DATA"

We conclude that, the proposed model $M(t)$ has three free parameters leading to greater flexibility in modeling various data types.

We will demonstrate this with another example - approximating the specific "Can-

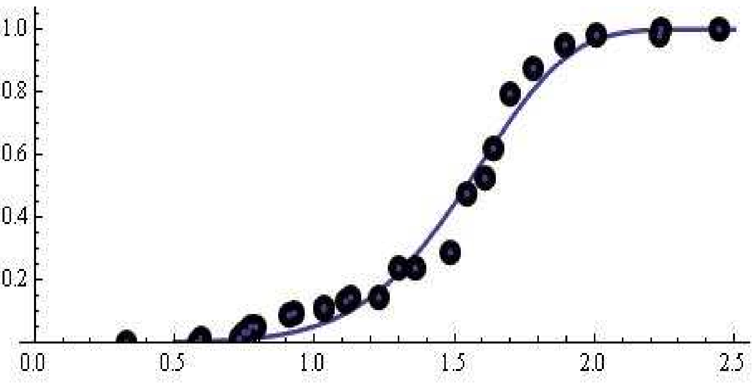

Figure 17: The fitted model $M^*(t)$.

cer Stem Cell data" (see, [51]):

$$Empirical\_cdf :=$$
$$\{\{0.3253, 0\}, \{0.58, 0\}, \{0.5964, 0.013\}, \{0.73, 0.013\}, \{0.747, 0.031\},$$
$$\{0.76, 0.031\}, \{0.7711, 0.048\}, \{0.79, 0.048\}, \{0.91, 0.084\},$$
$$\{0.9277, 0.093\}, \{1.035, 0.1022\}, \{1.036, 0.111\}, \{1.11, 0.1289\},$$
$$\{1.127, 0.1422\}, \{1.23, 0.1422\}, \{1.3012, 0.2356\}, \{1.3614, 0.2356\},$$
$$\{1.4819, 0.2844\}, \{1.5422, 0.4756\}, \{1.6084, 0.5244\}, \{1.6386, 0.6178\},$$
$$\{1.699, 0.7911\}, \{1.7831, 0.8756\}, \{1.8916, 0.9511\}, \{2.006, 0.9822\},$$
$$\{2.2349, 0.9822\}, \{2.241, 1\}, \{2.4458, 1\}\}.$$

After that using the model $M^*(t)$ for $\omega = 1$, $m = 8$, $d = 0.95$ and $\beta = 5.99671$ we obtain the fitted model (see, Fig. 17).

Remark. When β is even, the model $W(t)$ can be used to approximate the impulse function shown in Fig. 18.

For $\beta = 10$ and $m = 30$ for the Hausdorff distance we get $d = 0.0655016$ (see, Fig. 19).

4. APPENDIX.

Recall that the sigmoid functions in the interval $[0, 1)$ are of the form

$$M(t) = 1 - \left(\frac{\frac{1+d^{\frac{1}{m}}}{1-d^{\frac{1}{m}}} - t^{\beta}}{\frac{1+d^{\frac{1}{m}}}{1-d^{\frac{1}{m}}} + t^{\beta}} \right)^{m}. \tag{11}$$

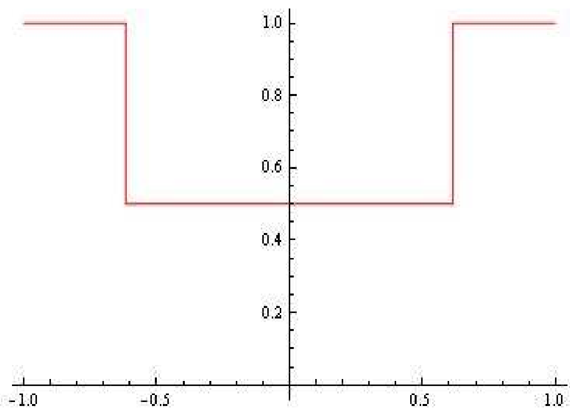

Figure 18: The impulse function $v(t)$ for $\beta = 8$; $m = 26$.

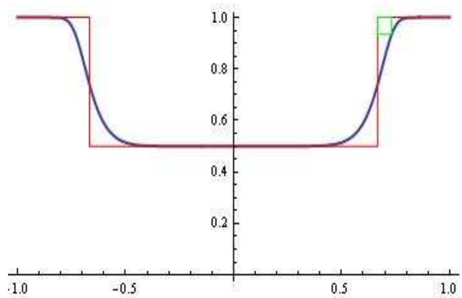

Figure 19: The function $W(t)$ for $m = 30$; $\beta = 10$; H–distance $d = 0.0655016$.

We assume in what follows that parameters d, m and β obey the restrictions

$$\begin{aligned} &d \in (0, 1/2)\,, \\ &m > 0\,, \\ &\beta > 0\,. \end{aligned} \tag{12}$$

It follows from (11) that $M(1) = 1 - d$ for all admissible m and β, and we are interested in the behavior of parameters m and β ensuring the equation (7),

$$M(1 - d) = d\,. \tag{13}$$

We shall prove the following statement.

Theorem 6. *(i) For any fixed* $d \in (0, 1/2)$ *and* $m > 0$ *there exists a unique* $\beta = \beta(d, m) > 0$ *such that equation* (13) *holds true;*

(ii) $\beta(d,m)$ is a monotonically increasing function of m and

$$\beta(d,m) < \frac{\ln\left(\frac{\ln(1-d)}{\ln d}\right)}{\ln(1-d)} =: \widetilde{\beta}\,. \tag{14}$$

For $0 < m \le 1$, a sharper upper bound for $\beta(d,m)$ is given by

$$\beta(d,m) < \frac{\ln\left(\frac{1-(1-d)^{\frac{1}{m}}}{1+(1-d)^{\frac{1}{m}}}\right)}{\ln(1-d)} =: \overline{\beta}(d,m)\,. \tag{15}$$

Before proving Theorem 6, let us mention that (13) is equivalent, under assumptions (12), to the equation

$$(1-d)^{\beta} = \frac{1 - \frac{(1-d)^{\frac{1}{m}} - d^{\frac{1}{m}}}{1-\left[d(1-d)\right]^{\frac{1}{m}}}}{1 + \frac{(1-d)^{\frac{1}{m}} - d^{\frac{1}{m}}}{1-\left[d(1-d)\right]^{\frac{1}{m}}}} =: \frac{1-g(d,m)}{1+g(d,m)}\,. \tag{16}$$

We shall need the following

Lemma 7. *(i) For every fixed $m > 0$,*

$$g(d,m) = \frac{(1-d)^{\frac{1}{m}} - d^{\frac{1}{m}}}{1-\left[d(1-d)\right]^{\frac{1}{m}}}$$

is a monotonically decreasing function of d in the interval $(0,1/2)$. Moreover,

$$0 < g(d,m) < (1-d)^{\frac{1}{m}} < 1\,; \tag{17}$$

(ii) For every fixed $d \in (0,1/2)$, $g(d,m)$ is a monotonically increasing function of m in the interval $(0,\infty)$ and

$$0 = \lim_{m\to 0+} g(d,m) < g(d,m) < \frac{\ln d - \ln(1-d)}{\ln d + \ln(1-d)} < 1\,. \tag{18}$$

Proof. (i) The monotonicity of g with respect to d is obvious, and only the upper bound in (17) needs to be proved; it follows from the inequality

$$\frac{y-x}{1-xy} \le y\,, \qquad 0 < x < y < 1\,,$$

which is easily verified.

(ii) Let us set $y = 1-d$, $x = d$, $\alpha = 1/m$, then we need to show that

$$\varphi(\alpha) = \frac{y^{\alpha} - x^{\alpha}}{1 - x^{\alpha}y^{\alpha}}, \qquad 0 < x < y < 1,$$

is a monotonically decreasing function in $(0,\infty)$. After differentiation and some simplification we get

$$\varphi'(\alpha)=\frac{(1-x^{2\alpha})(1-y^{2\alpha})}{(1-x^{\alpha}y^{\alpha})^2}\left[\frac{y^{\alpha}}{1-y^{2\alpha}}\,\ln y-\frac{x^{\alpha}}{1-x^{2\alpha}}\,\ln x\right],$$

hence it suffices to show that

$$\psi_1(x)=\frac{x^{\alpha}}{1-x^{2\alpha}}\,\ln x$$

is a decreasing function in $(0,1)$. We have

$$\psi_1'(x)=\frac{x^{\alpha-1}(1+x^{2\alpha})}{1-x^{2\alpha})^2}\left[\frac{1-x^{2\alpha}}{1+x^{2\alpha}}+\alpha\,\ln x\right]=:\frac{x^{\alpha-1}(1+x^{2\alpha})}{1-x^{2\alpha})^2}\,\psi_2(x)\,.$$

Since $\psi_2(1)=0$ and

$$\psi_2'(x)=\frac{\alpha(1-x^{2\alpha})^2}{x(1+x^{2\alpha})^2}>0\,,\qquad 0<x<1,\ \alpha>0\,,$$

it follows that $\psi_2(x)<0$ for $x\in(0,1)$, and therefore $\psi_1'(x)<0$ for $x\in(0,1)$. Consequently, $\varphi(\alpha)$ is decreasing in $(0,\infty)$, whence $g(d,m)$ is an increasing function of m in $(0,\infty)$. Now the upper bound in (18) follows from $g(d,;m)<\lim_{m\to+\infty}g(d,m)$ and application of the L´Hospital rule to the right-hand side. □

Corollary 8. *For every fixed* $d\in(0,1/2)$*, the function*

$$h(d,m)=\frac{1-g(d,m)}{1+g(d,m)}$$

is a monotonically decreasing function of m *in the interval* $(0,\infty)$*. Moreover,*

$$1>h(d,m)>\frac{\ln(1-d)}{\ln d}$$

and

$$h(d,m)>\frac{1-(1-d)^{\frac{1}{m}}}{1+(1-d)^{\frac{1}{m}}}\,.$$

Proof of Theorem 6. Assume that $d\in(0,1/2)$ and $m>0$ are fixed. Since

$$f(\beta)=(1-d)^{\beta}$$

is a continuous and strictly monotonically decreasing function in $(0,\infty)$, with $f(0)=1$ and $\lim_{\beta\to+\infty}f(\beta)=0$, it follows from Corollary 8 and the Weierstrass theorem that there exists a unique $\beta=\beta(d,m)>0$ such that $f(\beta)=h(d,m)$. Thus, equation (16), and thereby (13), has a unique solution $\beta=\beta(d,m)>0$.

Using again Corollary 8, we conclude that if $0 < m_1 < m_2$, then $h(d, m_1) > h(d, m_2)$. Since $f(\beta)$ is monotonically decreasing, it follows that $\beta(d, m_1) < \beta(d, m_2)$, thus $\beta(d, m)$ is a monotonically increasing function of m.

To derive the upper bounds for $\beta(d, m)$ in (14) and (15), we consider the equations

$$f(\beta) = \frac{\ln(1-d)}{\ln d} \text{ and } f(\beta) = \frac{1-(1-d)^{\frac{1}{m}}}{1+(1-d)^{\frac{1}{m}}},$$

whose solutions are respectively

$$\widetilde{\beta}(d) = \frac{\ln\left(\frac{\ln(1-d)}{\ln d}\right)}{\ln(1-d)} \text{ and } \overline{\beta}(d, m) = \frac{\ln\left(\frac{1-(1-d)^{\frac{1}{m}}}{1+(1-d)^{\frac{1}{m}}}\right)}{\ln(1-d)}.$$

According to Corollary 8, the right-hand sides of the above equations are lower bounds for $h(d, m)$. Since f is a decreasing function of β, it follows that $\beta(d, m) < \widetilde{\beta}(d)$ and $\beta(d, m) < \overline{\beta}(d, m)$. $\square$

Remark 9. It can be seen that $\widetilde{\beta}(d)$ provides a good approximation to $\beta(d, m)$ when m is large, while when m is small (i.e., close to 0) a very good approximation to $\beta(d, m)$ is furnished by $\overline{\beta}(d, m)$.

5. CONCLUSION

In this paper we study the possibility of approximating the functions $\kappa(t)$ and $v(t)$ with the new families with respect to Hausdorff distance.

Some applications are also given.

We further plan to extend the Distributed Platform for e-Learning (DisPeL) [46]–[50] with specialized modules for simulation of Hausdorff type diagram functions and prototype filters.

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REFERENCES

[1] G. Cybenko, Approximation by superpositions of a sigmoidal function, *Math. Control Signal.*, **2** (1989), 303–314.

[2] Z. Chen, F. Cao, The approximation operators with sigmoidal functions, *Comput. Math. Appl.*, **58** (2009), 758–765.

[3] N. Guliyev, V. Ismailov, A single hidden layer feedforward network with only one neuron in the hidden layer san approximate any univariate function, *Neural Computation*, **28** (2016), 1289–1304.

[4] D. Costarelli, R. Spigler, Approximation results for neural network operators activated by sigmoidal functions, *Neural Networks*, **44** (2013), 101–106.

[5] D. Costarelli, G. Vinti, Convergence for a family of neural network operators in Orlicz spaces, *Mathematische Nachrichten*, **290**, No. 2-3 (2017), 226–235.

[6] D. Costarelli, R. Spigler, Constructive Approximation by Superposition of Sigmoidal Functions, *Anal. Theory Appl.*, **29** (2013), 169–196.

[7] B. I. Yun, A Neural Network Approximation Based on a Parametric Sigmoidal Function, *Mathematics*, **7** (2019), 262.

[8] N. Kyurkchiev, Comments on the Yun's algebraic activation function. Some extensions in the trigonometric case, *Dynamic Systems and Applications*, **28**, No. 3 (2019), 533–543.

[9] J. Dombi, Z. Gera, The approximation of piecewise linear membership functions and Likasiewicz operators, *Fuzzy Sets and Systems*, **154** (2005), 275–286.

[10] J. Dombi, T. Jonas, Z. Toth, G. Arva, The omega probability distribution and its applications in reliability theory, *Quality and Reliability Engineering International*, (2018), 1–27.

[11] N. Kyurkchiev, A. Andreev, *Approximation and Antenna and Filters synthesis. Some Moduli in Programming Environment MATHEMATICA*, LAP LAMBERT Academic Publishing, Saarbrucken, 150 pp., (2014), ISBN 978-3-659-53322-8.

[12] B. Sendov, *Hausdorff Approximations*, Kluwer, Boston (1990).

[13] B. Sendov, V. Popov, The exact asymptotic behavior of the best approximation by algebraic and trigonometric polynomials in the Hausdorff metric, *Math. USSR-Sb,*, Moscow, **89**, No. 131 (1972), 138–147 (in Russian).

[14] P. Petrushev, V. Popov, *Rational Approximation of Real Functions*, Cambridge University Press, (1987) (371 pages).

[15] K. Ivanov, V. Totik, Fast Decreasing Polynomials, Constructive Approx., **6** (1990), 1–20.

[16] S. Tashev, Approximation of bounded sets on the plane in Hausdorff metric, *C.R. Acad. Bulgare Sci.*, **29**, No 4 (1976), 465–468. (in Russian)

[17] R. Anguelov, S. Markov, B. Sendov, On the Normed Linear Space of Hausdorff Continuous Functions. In: Lirkov, I., et al (eds): Lecture Notes in Computer Science, **3743**, Springer, (2006), 281–288.

[18] R. Anguelov, S. Markov, B. Sendov, Algebraic Operations on the Space of Hausdorff Continuous Functions. In: Bojanov, B. (ed.): Constructive Theory of Functions, Prof. M. Drinov Academic Publ. House, Sofia (2006), 35–44.

[19] R. Anguelov, S. Markov, B. Sendov, The Set of Hausdorff Continuous Functions - the Largest Linear Space of Interval Functions, *Reliable Computing* **12** (2006), 337–363.

[20] N. Kyurkchiev, S. Markov, On the Hausdorff distance between the Heaviside step function and Verhulst logistic function, *J. Math. Chem.*, **54**, No. 1 (2016), 109–119.

[21] N. Kyurkchiev, A Note on the Volmer's Activation (VA) Function, *C. R. Acad. Bulg. Sci.*, **70**, No. 6 (2017), 769–776.

[22] N. Kyurkchiev, S. Markov, *Sigmoid functions: Some Approximation and Modelling Aspects*, LAP LAMBERT Academic Publishing, Saarbrucken (2015), ISBN: 978-3-659-76045-7.

[23] N. Kyurkchiev, A. Iliev, S. Markov, *Some techniques for recurrence generating of activation functions*, LAP LAMBERT Academic Publishing, (2017), ISBN: 978-3-330-33143-3.

[24] N. Kyurkchiev, A. Iliev, A. Rahnev, *Some Families of Sigmoid Functions: Applications to Growth Theory*, LAP LAMBERT Academic Publishing, (2019), ISBN: 978-613-9-45608-6.

[25] N. Kyurkchiev, A. Iliev, A. Rahnev, A new class of activation functions based on the correcting amendments of Gompertz-Makeham type, *Dynamic Systems and Applications*, **28**, No. 2 (2019), 243–257.

[26] N. Pavlov, A. Iliev, A. Rahnev, N. Kyurkchiev, *Some software reliability models: Approximation and modeling aspects*, LAP LAMBERT Academic Publishing, (2018), ISBN: 978-613-9-82805-0.

[27] N. Pavlov, A. Iliev, A. Rahnev, N. Kyurkchiev, *Nontrivial Models in Debugging Theory (Part 2)*, LAP LAMBERT Academic Publishing, (2018), ISBN: 978-613-9-87794-2.

[28] A. Iliev, N. Kyurkchiev, A. Rahnev, T. Terzieva, *Some models in the theory of computer viruses propagation*, LAP LAMBERT Academic Publishing, (2019), ISBN: 978-620-0-00826-8.

[29] S. Markov, A. Iliev, A. Rahnev, N. Kyurkchiev, A note on the Log-logistic and transmuted Log-logistic models. Some applications, *Dynamic Systems and Applications*, **27**, No. 3 (2018), 593–607.

[30] S. Markov, N. Kyurkchiev, A. Iliev, A. Rahnev, On the approximation of the generalized cut functions of degree p+1 by smooth hyper-log-logistic function, *Dynamic Systems and Applications*, **27**, No. 4 (2018), 715–728.

[31] N. Pavlov, A. Iliev, A. Rahnev, N. Kyurkchiev, N., On Some Nonstandard Software Reliability Models, *Dynamic Systems and Applications*, **27**, No. 4 (2018), 757–771.

[32] S. Markov, A. Iliev, A. Rahnev, N. Kyurkchiev, A Note On the Three-stage Growth Model, *Dynamic Systems and Applications*, **28**, No. 1 (2019), 63–72.

[33] A. Golev, T. Djamiykov, N. Kyurkchiev, Sigmoidal Functions in Antenna-feeder Technique, *International Journal of Pure and Applied Mathematics*, **116**, No. 4 (2017), 1081–1092.

[34] N. Kyurkchiev, A. Andreev, Hausdorff approximation of functions different from zero at one point -implementation in programming environment MATHEMATICA, *Serdica Journal of Computing*, **7**, No. 2 (2013), 135–142.

[35] N. Kyurkchiev, A. Andreev, Synthesis of slot aerial grids with Hausdorff-type directive patterns - implementation in programming environment MATHEMATICA, *Compt. rend. Acad. bulg. Sci.*, **66**, No. 11 (2013), 1521–1528.

[36] I. Okorie, S. Nadarajah, On the omega probability distribution, *Quality and Reliability Engineering International*, (2019), 1–6.

[37] T. Mitsuhashi, *A method of software quality evaluation*, JUSE Press, Tokyo (1981). (in Japanese)

[38] D. Satoh, A discrete Gompertz equation and a software reliability growth model, *IEICE Trans. Inf. and Syst.*, **E83–D**, No. 7 (2000), 1508–1513.

[39] Bl. Sendov, H. Schinev, N. Kjurkchiev, Hausdorff-synthesis of aerial grids in scanning the directive diagram, *Electropromishlenost i Priboroostroene*, ISSN 0013-5763; **16**, No 5 (1981), 203–205 (in Bulgarian).

[40] N. Kyurkchiev, Synthesis of slot aerial grids with Hausdorff type directive patterns, PhD Thesis, Department of Radio-Electronics, VMEI, Sofia, (1979) (in Bulgarian).

[41] N. Kyurkchiev, Bl. Sendov, Approximation of a class of functions by algebraic polynomials with respect to Hausdorff distance, *Ann. Univ. Sofia, Fac. Math.*, **67** (1975), 573–579 (in Bulgarian).

[42] N. Kyurkchiev, S. Markov, On the numerical approximation of the "cross" set, *Ann. Univ. Sofia, Fac. Math.*, **66** (1974), 19–25 (in Bulgarian).

[43] M. Gachev, Synthesis of antenna grids with optimal directivity chart, PhD Thesis, Sofia, VMEI, (1981).

[44] P. Appostolov, Hausdorff-filters, PhD Thesis, Department of Radio-Electronic, TU, Sofia, (2005) (in Bulgarian).

[45] P. Appostolov, General theory, approximation method and design of electrical filters based on Hausdorff polynomials, *Mechanics, Transport, Communications*, **2** (2007), 1–8. (in Bulgarian)

[46] A. Rahnev, N. Pavlov, V. Kyurkchiev, Distributed Platform for e-Learning DisPeL, *European International Journal of Science and Technology*, **3**, No. 1 (2014), 95-109.

[47] V. Kyurkchiev, N. Pavlov, A. Rahnev, Cloud-based architecture of DisPeL, *International Journal of Pure and Applied Mathematics*, **120**, No. 4 (2018), 573–581.

[48] N. Pavlov, A. Rahnev, V. Kyurkchiev, A. Malinova, E. Angelova, Geographic map visualization in DisPeL, In: Proceedings of the Scientific Conference Innovative ICT in Business and Education: Future Trends, Applications and Implementation, Pamporovo, 24-25 November 2016, 13-20, ISBN 978-954-8852-72-2.

[49] T. Terzieva, O. Rahneva, V. Arnaudova, A. Karabov, Application of DisPeL for adaptivity and individualization in the training, In: Proceedings of the Scientific Conference Innovative Software Tools and Technologies with Applications in Research in Mathematics, Informatics and Pedagogy of Education, 23-24 November, (2017), Pamporovo, Bulgaria, 175-182, ISBN: 978-619-202-343-0.

[50] A. Rahnev, N. Pavlov, A. Golev, M. Stieger, T. Gardjeva, New electronic education services using the distributed e-learning platform (DisPel), International Electronic Journal of Pure and Applied Mathematics, **7**, No. 2 (2014), 63-72.

[51] O. Maxwell, A. Chukwu, O. Oyamakin, M. Khalel, The Marshal–Olkin inverse Lomax distribution (MO–ILD) with application on cancer stem cell, *Journal of Advances in Mathematics and Computer Science*, **33**, No. 4 (2019), 1–12.

[52] J. Peters, *Foundations of Computer Vision: Computational Geometry, Visual Image Structures and Object Shape Detection*, Springer, Cham (2017).