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QUALITATIVE ANALYSIS OF A DIFFUSIVE PREDATOR-PREY MODEL WITH BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE

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ABSTRACT. This paper is concerned with a diffusive predator-prey model with Beddington-DeAngelis functional response under Robin boundary conditions. We establish the existence and nonexistence of coexistence solutions and give some sufficient and necessary conditions. In addition, the stability of coexistence solutions is investigated. Furthermore, the extinction and permanence of time-dependent system are discussed.

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1. INTRODUCTION

In population dynamics, the relationship between predators and their prey plays an important role due to its universal existence. To model various different situations in ecological applications, many significant functional responses and more reasonable interactions have been developed. Early in 1975, Beddington [1] and DeAngelis et al. [9] originally introduced the Beddington-DeAngelis type predator-prey model taking the form

$$\begin{cases} \frac{du}{dt} = r_1 u - g_1 u^2 - \frac{f_1 uv}{\beta+\gamma u+\delta v}, \\ \frac{dv}{dt} = r_2 v - g_2 v^2 + \frac{f_2 uv}{\beta+\gamma u+\delta v}, \end{cases} \tag{1.1}$$

where u and v, respectively, stand for the population densities of the prey and predator. The Beddington-DeAngelis functional response $uv/(\beta + \gamma u + \delta v)$ is similar to the Holling type-II functional response but has an extra term δv in the denominator modelling mutual interference among predators. Hence, this kind of type functional response is affected by both predator and prey. The ordinary differential system (1.1) has been widely studied in recent years, for example, see [5, 12, 13, 27].

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On the other hand, taking into account the inhomogeneous distribution of the predators and prey in different spatial locations at any given time and the natural tendency of each species to diffuse to areas of smaller population concentration, we are lead to consider the following PDE system of reaction-diffusion type under Robin boundary conditions:

$$
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u = u\left(a - u - \frac{cv}{1+u+mv}\right), & x \in \Omega, t \in (0,\infty), \\
\frac{\partial v}{\partial t} - \Delta v = v\left(b - v + \frac{du}{1+u+mv}\right), & x \in \Omega, t \in (0,\infty), \\
\kappa_1 \frac{\partial u}{\partial \nu} + u = \kappa_2 \frac{\partial v}{\partial \nu} + v = 0, & x \in \partial\Omega, t \in (0,\infty), \\
u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in \overline{\Omega}.
\end{cases}
\tag{1.2}
$$

In the above, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$; ν is the outward unit normal vector of the boundary $\partial\Omega$. κ_1, κ_2 are nonnegative constants. From the viewpoint of biology, the Robin boundary condition reflects the situation that the species can escape across the boundary, which appears often in nature. All the parameters of (1.2) are positive due to their biological sense.

In recent years, many authors have focused on diffusive predator-prey systems with various functional responses and different boundary conditions. For example, Chen and Wang [7] studied following diffusive system under homogeneous Neumann boundary conditions:

$$
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u = u\left(1 - u - \frac{bv}{a+u+mv}\right), & x \in \Omega, t \in (0,\infty), \\
\frac{\partial v}{\partial t} - \Delta v = v\left(-d + \frac{eu}{a+u+mv}\right), & x \in \Omega, t \in (0,\infty), \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,\infty), \\
u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in \overline{\Omega}.
\end{cases}
\tag{1.3}
$$

Later, Chen and Wang [6] further studied system (1.3) subject to homogeneous Dirichlet boundary conditions. Ryu and Ahn [23] and Ko and Ryu [14], respectively, studied diffusive ratio-dependent Holling type II predator-prey system and Holling type IV predator-prey system under Robin boundary conditions. The authors of these works mentioned above mainly investigated the existence and nonexistence of positive solutions of the stationary problem, i.e., the corresponding elliptic problems. In [25], Wu studied a diffusive Holling type II predator-prey system and investigated the stability of steady-state solutions, maximal attractor and uniform persistence of the system. For more works on diffusive predator-prey systems, one can see [2, 10, 11, 15, 17, 20, 21, 24, 26] and references therein.

Motivated by the previous works, the main goal in this paper is to study the coexistence states of (1.2), i.e., the positive solutions of the following elliptic system:

$$
\begin{cases}
-\Delta u = u\left(a - u - \frac{cv}{1+u+mv}\right), & x \in \Omega, \\
-\Delta v = v\left(b - v + \frac{du}{1+u+mv}\right), & x \in \Omega, \\
\kappa_1 \frac{\partial u}{\partial \nu} + u = \kappa_2 \frac{\partial v}{\partial \nu} + v = 0, & x \in \partial\Omega.
\end{cases}
\tag{1.4}
$$

After this, the stability of positive solutions under the assumption that the capturing rate c is sufficiently small is investigated. Meantime, some sufficient conditions for the extinction and permanence of parabolic system (1.2) are given.

The rest of this paper is arranged as follows. In section 2, we collect some known results including the eigenvalue problem and the fixed point index on positive cones. In Section 3, some results on the calculation of fixed point indexes are given. In Section 4, we establish the existence and nonexistence of positive solutions of system (1.4) and give some sufficient and necessary conditions. In addition, the stability of positive solutions is investigated. In the last section, the extinction and permanence of time-dependent system (1.2) are discussed by using the method of upper and lower solutions.

2. **PRELIMINARIES**

In this section, we give some preliminaries, which will serve as the basic tools in the sequel. First, we introduce the fixed point index of compact maps on positive cones, see [8, 16, 22].

Let E be a real Banach space and $W \subset E$ a closed convex set. W is called a wedge if $\beta W \subset W$ for all $\beta \geq 0$. A wedge W is said to be a cone if $W \cap (-W) = \{0\}$. For $y \in W$, define $W_y = \{x \in E : y + \gamma x \in W$ for some $\gamma > 0\}$ and $S_y = \{x \in \overline{W}_y : -x \in \overline{W}_y\}$. Then $\overline{W}_y$ is a wedge containing W, y, $-y$, while S_y is a closed subspace of E containing y. We always assume that $E = \overline{W - W}$. Let $\mathcal{T} : \overline{W}_y \to \overline{W}_y$ be a compact linear operator on E. We say that $\mathcal{T}$ has property α on $\overline{W}_y$ if there exist $t \in (0, 1)$ and $w \in \overline{W}_y \backslash S_y$ such that $w - t\mathcal{T}w \in S_y$. Assume that $\mathcal{F} : W \to W$ is a compact operator with a fixed point $y \in W$. If F is Fréchet differential at y, then the derivative $\mathcal{F}'(y)$ has the property that $\mathcal{F}'(y) : \overline{W}_y \to \overline{W}_y$. For an open subset $U \subset W$, define

$$\mathrm{index}_W(\mathcal{F}, U) = \mathrm{index}(\mathcal{F}, U, W) = \deg_W(I - \mathcal{F}, U, 0),$$

where I is the identity map. If y is an isolated fixed point of F, then the fixed point index of $\mathcal{F}$ at y related to W is defined by

$$\mathrm{index}_W(\mathcal{F}, y) = \mathrm{index}(\mathcal{F}, y, W) = \mathrm{index}(\mathcal{F}, U(y), W),$$

where $U(y)$ is a small open neighborhood of y in W.

The following results of fixed point index can be obtained from [8, 16, 22, 23].

Lemma 2.1. *Assume that $I - \mathcal{F}'(y)$ is invertible on $\overline{W}_y$. Then we have*

(i) *if $\mathcal{F}'(y)$ has property α, then* $\mathrm{index}_W(\mathcal{F}, y) = 0$.

(ii) *if $\mathcal{F}'(y)$ does not have property α, then* $\mathrm{index}_W(\mathcal{F}, y) = (-1)^\sigma$, *where σ is the the sum of multiplicities of all eigenvalues of $\mathcal{F}'(y)$ which are greater than one.*

Now, we introduce some some known results about the eigenvalue problem under Robin boundary conditions. For $q(x) \in C^{\alpha}(\overline{\Omega})$ and $\kappa \geq 0$, denote the principal eigenvalue of the following problem:

$$\begin{cases} -\Delta u + q(x)u = \lambda u, & x \in \Omega, \\ \kappa \frac{\partial u}{\partial \nu} + u = 0, & x \in \partial\Omega \end{cases} \tag{2.1}$$

by $\lambda_{1,\kappa}(q(x))$ and simply denote $\lambda_{1,\kappa}(0)$ by $\lambda_{1,\kappa}$. It is known that $\lambda_{1,\kappa}(q(x))$ is strictly increasing, namely, $\lambda_{1,\kappa}(q_1(x)) < \lambda_{1,\kappa}(q_2(x))$ if $q_1(x) \leq q_2(x)$ and $q_1(x) \not\equiv q_2(x)$. Furthermore, the eigenfunction ϕ_1 of (2.1) corresponding to the eigenvalue $\lambda_{1,\kappa}(q(x))$ is unique and positive. In [3, 4], the authors discussed the eigenvalue problem (2.1) in detail and established most of the necessary existence and comparison results for (2.1).

Now, we cite the following lemma which can be found in [23].

Lemma 2.2. *Let $q(x) \in C^{\alpha}(\overline{\Omega})$ and $u \geq 0$, $u \not\equiv 0$ in Ω.*

(a1) *If $0 \not\equiv -\Delta u + q(x)u \leq 0$, then $\lambda_{1,\kappa}(q(x)) < 0$.*

(b1) *If $0 \not\equiv -\Delta u + q(x)u \geq 0$, then $\lambda_{1,\kappa}(q(x)) > 0$.*

(c1) *If $-\Delta u + q(x)u \equiv 0$, then $\lambda_{1,\kappa}(q(x)) = 0$.*

And in addition, if M is a positive constant such that $-q(x) + M > 0$ on $\overline{\Omega}$, then we have the following conclusions:

(a2) $\lambda_{1,\kappa}(q(x)) < 0 \Rightarrow r[(-\Delta + M)^{-1}(-q(x) + M)] > 1$,

(b2) $\lambda_{1,\kappa}(q(x)) > 0 \Rightarrow r[(-\Delta + M)^{-1}(-q(x) + M)] < 1$,

(c2) $\lambda_{1,\kappa}(q(x)) = 0 \Rightarrow r[(-\Delta + M)^{-1}(-q(x) + M)] = 1$,

where $r(\cdot)$ is the spectral radius of an operator.

3. CALCULATIONS OF THE FIXED POINT INDEX

First, we give a priori estimates for positive solutions of (1.4). It is well-known that the following equation

$$\begin{cases} -\Delta u = u(a - u), & x \in \Omega, \\ \kappa_1 \frac{\partial u}{\partial \nu} + u = 0, & x \in \partial\Omega \end{cases}$$

has a unique positive solution $\Theta_{[a]}$ when $a > \lambda_{1,\kappa_1}$, and $\Theta_{[b]}$ is the unique positive solution of the equation

$$\begin{cases} -\Delta v = v(b - v), & x \in \Omega, \\ \kappa_2 \frac{\partial v}{\partial \nu} + v = 0, & x \in \partial\Omega, \end{cases}$$

when $b > \lambda_{1,\kappa_2}$. The nonnegative solutions $(\Theta_{[a]}, 0)$ and $(0, \Theta_{[b]})$ are usually called semi-trivial solutions of system (1.4).

By the method of upper and lower solutions and maximum principle, we have the following results of which the proof is omitted here.

Proposition 3.1. Any nonnegative solution (u, v) of (1.4) has an a priori bounds:

$$u(x) \le a, \quad v(x) \le R := b + \frac{da}{1+a}.$$

Now we are in the position to calculate the fixed point index. We introduce the following notations:

$$E = C_{\kappa_1}(\overline{\Omega}) \times C_{\kappa_2}(\overline{\Omega}), \quad \text{where} \quad C_{\kappa_i}(\overline{\Omega}) = \{w \in C(\overline{\Omega}) : \kappa_i \frac{\partial w}{\partial \nu} + w = 0, x \in \partial\Omega\}.$$

$$W = K_1 \times K_2, \quad \text{where} \quad K_i = \{w \in C_{\kappa_i}(\overline{\Omega}) : 0 \le w(x), x \in \overline{\Omega}\}.$$

$$D = \{(u, v) \in E : u \le a+1, v \le R+1\}, \quad D' = (\text{int} D) \cap W.$$

It is easy to verify that

$$\begin{aligned}
&\overline{W}_{(0,0)} = K_1 \times K_2, \quad S_{(0,0)} = \{(0,0)\};\\
&\overline{W}_{(\Theta_{[a]},0)} = C_{\kappa_1}(\overline{\Omega}) \times K_2, \quad S_{(\Theta_{[a]},0)} = C_{\kappa_1}(\overline{\Omega}) \times \{0\};\\
&\overline{W}_{(0,\Theta_{[b]})} = K_1 \times C_{\kappa_2}(\overline{\Omega}), \quad S_{(0,\Theta_{[b]})} = \{0\} \times C_{\kappa_2}(\overline{\Omega}).
\end{aligned}$$

From Proposition 3.1, we can see that the nonnegative solution of (1.4) must lie in D'.

Choosing $M > \max\left\{a + \frac{c}{m}, |-b + 2R + \frac{d}{2}|\right\}$, then the functions

$$u\left(a - u - \frac{cv}{1+u+mv}\right) + Mu \quad \text{and} \quad v\left(b - v + \frac{du}{1+u+mv}\right) + Mv$$

are nonnegative for all $(u, v) \in [0, a] \times [0, R]$. Define an operator $\mathcal{F} : E \to E$ by

$$\mathcal{F}(u, v) \;=\; (-\Delta + M)^{-1} \begin{pmatrix} u\left(a - u - \frac{cv}{1+u+mv}\right) + Mu \\ v\left(b - v + \frac{du}{1+u+mv}\right) + Mv \end{pmatrix}^T.$$

By strong maximum principle, $(-\Delta + M)^{-1}$ is a compact positive linear operator, and $\mathcal{F}$ is a direct sum of compact positive operators. Clearly, system (1.4) is equivalent to $\mathcal{F}(u, v) = (u, v)$ (observe that this is independent of the choice of M as long as M is large enough). Thus, Finding a positive solution of system (1.4) is equivalent to prove that $\mathcal{F}$ has a nontrivial fixed point in D'. Without loss of generality, we may assume that $(0, 0)$, $(\Theta_{[a]}, 0)$ and $(0, \Theta_{[b]})$ are isolated fixed points of $\mathcal{F}$ if exist, and so the corresponding index related to W is well-defined.

For $t \in [0, 1]$, define a homotopy

$$\mathcal{F}_t(u, v) \;=\; (-\Delta + M)^{-1} \begin{pmatrix} tu\left(a - u - \frac{cv}{1+u+mv}\right) + Mu \\ tv\left(b - v + \frac{du}{1+u+mv}\right) + Mv \end{pmatrix}^T.$$

then $\mathcal{F} = \mathcal{F}_1$.

Lemma 3.2. *Assume that $a > \lambda_{1,\kappa_1}$. We have*

(i) *for an open set D' in W,* $\text{index}_W(\mathcal{F}, D') = 1$*;*

(ii) $\text{index}_W(\mathcal{F}, (0, 0)) = 0$*;*

(iii) *if* $b > \lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$, *then* $\text{index}_W\left(\mathcal{F}, (\Theta_{[a]}, 0)\right) = 0$;

(iv) *if* $b < \lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$, *then* $\text{index}_W\left(\mathcal{F}, (\Theta_{[a]}, 0)\right) = 1$.

Proof. (i) First, we can see that $\text{index}_W(\mathcal{F}, D')$ is well-defined since $\mathcal{F}$ has no fixed point on $\partial D'$. For $t \in [0,1]$, a fixed point of $\mathcal{F}_t$ is a solution of the following problem

$$\begin{cases} -\Delta u = tu\left(a - u - \frac{cv}{1+u+mv}\right), & x \in \Omega, \\ -\Delta v = tv\left(b - v + \frac{du}{1+u+mv}\right), & x \in \Omega, \\ \kappa_1 \frac{\partial u}{\partial \nu} + u = \kappa_2 \frac{\partial v}{\partial \nu} + v = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

In view of Proposition 3.1, the fixed points of $\mathcal{F}_t$ satisfies $u(x) \le a$ and $v(x) \le R$ on $\overline{\Omega}$ for all $t \in [0,1]$, and so all fixed points of $\mathcal{F}_t$ must lie in D', and $\text{index}_W(\mathcal{F}, D')$ is independent of t. Hence, by the homotopy invariance,

$$\text{index}_W(\mathcal{F}, D') = \text{index}_W(\mathcal{F}_1, D') = \text{index}_W(\mathcal{F}_0, D').$$

Since problem (3.1) with $t = 0$ has only the trivial solution $(0,0)$, we have

$$\text{index}_W(\mathcal{F}_0, D') = \text{index}_W(\mathcal{F}_0, (0,0)).$$

Denote

$$\mathcal{L} := \mathcal{F}_0'(0,0) = (-\Delta + M)^{-1}\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}.$$

Thus, it follows from Lemma 2.2 that $r(\mathcal{L}) < 1$, which indicates that $I - \mathcal{L}$ is invertible on $\overline{W}_{(0,0)}$ and $\mathcal{L}$ does not have property α on $\overline{W}_{(0,0)}$. So, we may conclude $\text{index}_W(\mathcal{F}_0, (0,0)) = 1$ by Lemma 2.1.

(ii) Note that $\mathcal{F}(0,0) = (0,0)$ and $\mathcal{F}$ is compact. First consider the case of $b \ne \lambda_{1,\kappa_2}$. Then

$$\mathcal{L} := \mathcal{F}'(0,0) = (-\Delta + M)^{-1}\begin{pmatrix} a+M & 0 \\ 0 & b+M \end{pmatrix}.$$

Assume that $\mathcal{L}(\xi,\eta) = (\xi,\eta) \in \overline{W}_{(0,0)}$. Then $-\Delta\xi = a\xi$, $x \in \Omega$ and $\kappa_1\frac{\partial \xi}{\partial \nu} + \xi = 0$, $x \in \partial\Omega$. If $\xi > 0$, then $a = \lambda_{1,\kappa_1}$ by Lemma 2.2, which is a contradiction to the assumption. Thus $\xi \equiv 0$. Similarly, since $b \ne \lambda_{1,\kappa_2}$, we can prove $\eta \equiv 0$. Therefore, $I - \mathcal{L}$ is invertible on $\overline{W}_{(0,0)}$.

Since $a > \lambda_{1,\kappa_1}$, by Lemma 2.2, we have $r_a := r[(-\Delta + M)^{-1}(a+M)] > 1$. From Krein-Rutman theorem, r_a is the principle eigenvalue of the operator $(-\Delta+M)^{-1}(a+M)$ with a corresponding eigenfunction $\phi \in K_1\backslash\{0\}$. Set $t_0 = 1/r_a$. Then we have $0 < t_0 < 1$ and $(I - t_0\mathcal{L})(\phi, 0) = (0,0) \in S_{(0,0)}$. This implies that $\mathcal{L}$ has property α. It follows from Lemma 2.1 that $\text{index}_W(\mathcal{F}, (0,0)) = 0$.

If $b = \lambda_{1,\kappa_2}$, similar to the method of Lemma 3.4 in [23], we can also prove that $\text{index}_W(\mathcal{F}, (0,0)) = 0$.

(iii) By a direct computation, we have

$$\mathcal{L} := \mathcal{F}'(\Theta_{[a]}, 0) = (-\Delta + M)^{-1} \begin{pmatrix} a - 2\Theta_{[a]} + M & -\frac{c\Theta_{[a]}}{1+\Theta_{[a]}} \\ 0 & b + \frac{d\Theta_{[a]}}{1+\Theta_{[a]}} + M \end{pmatrix}.$$

Assume that $\mathcal{L}(\xi, \eta) = (\xi, \eta)$ for some $(\xi, \eta) \in \overline{W}_{(\Theta_{[a]},0)}$. Then

$$\begin{cases} -\Delta\xi + (2\Theta_{[a]} - a)\xi = -\frac{c\Theta_{[a]}}{1+\Theta_{[a]}}\eta, & x \in \Omega, \\ -\Delta\eta - \frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\eta = b\eta, & x \in \Omega, \\ \kappa_1 \frac{\partial \xi}{\partial \nu} + \xi = \kappa_2 \frac{\partial \eta}{\partial \nu} + \eta = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

For $\eta \in K_2$, in the second equation of (3.2), $b = \lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$ if $\eta \not\equiv 0$ by Lemma 2.2. Since $b > \lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$, we have $\eta \equiv 0$. If $\xi \not\equiv 0$, then 0 is an eigenvalue of the problem $-\Delta\xi + \left(2\Theta_{[a]} - a\right)\xi = \lambda\xi$ in Ω and $\kappa_1 \frac{\partial \xi}{\partial \nu} + \xi = 0$ on $\partial\Omega$, and thus $\lambda_{1,\kappa_1}\left(2\Theta_{[a]} - a\right) < 0$. Using the comparison property of eigenvalue, we have $\lambda_{1,\kappa_1}\left(2\Theta_{[a]} - a\right) > \lambda_{1,\kappa_1}\left(\Theta_{[a]} - a\right) = 0$, since $(\Theta_{[a]}, 0)$ is the positive semi-trivial solution of (1.4), which yields a contradiction. Therefore, $(\xi, \eta) = (0, 0)$. This implies that $I - \mathcal{L}$ is invertible on $\overline{W}_{(\Theta_{[a]},0)}$.

Now we shall prove that $\mathcal{L}$ has property α on $\overline{W}_{(\Theta_{[a]},0)}$. In fact, denote

$$\mathcal{A} := (-\Delta + M)^{-1}\left(b + \frac{d\Theta_{[a]}}{1 + \Theta_{[a]}} + M\right).$$

Since $b > \lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$, it follows from Lemma 2.2 that $r_b := r(\mathcal{A}) > 1$. From Krein-Rutman theorem, r_b is the principle eigenvalue of operator $\mathcal{A}$ with a corresponding eigenfunction $\phi \in K_2 \backslash \{0\}$. Set $t_0 = 1/r_b$. Then we have $0 < t_0 < 1$ and $(0, \phi) \in \overline{W}_{(\Theta_{[a]},0)} \backslash S_{(\Theta_{[a]},0)}$. It is easy to verify that $(I - t_0\mathcal{L})(0, \phi) \in S_{(\Theta_{[a]},0)}$. This implies that $\mathcal{L}$ has property α. By Lemma 2.1, $\mathrm{index}_W\left(\mathcal{F}, (\Theta_{[a]}, 0)\right) = 0$.

(iv) Similar to the proof of (iii), $I - \mathcal{L}$ is invertible on $\overline{W}_{(\Theta_{[a]},0)}$. Once we prove that $\mathcal{L}$ does not have property α on $\overline{W}_{(\Theta_{[a]},0)}$ and has no eigenvalues being greater than one, the desired result follows directly from Lemma 2.1. Since $b < \lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$, we have $r(\mathcal{A}) < 1$. Suppose, for contradiction, that $\mathcal{L}$ has property α on $\overline{W}_{(\Theta_{[a]},0)}$. Then there exist $t \in (0, 1)$ and $(\phi_1, \phi_2) \in \overline{W}_{(\Theta_{[a]},0)} \backslash S_{(\Theta_{[a]},0)}$ such that $(I - t\mathcal{L})(\phi_1, \phi_2) \in S_{(\Theta_{[a]},0)}$. A straightforward calculation yields

$$\phi_2 - t(-\Delta + M)^{-1}\left(b + \frac{d\Theta_{[a]}}{1 + \Theta_{[a]}} + M\right)\phi_2 = 0.$$

From $\phi_2 \in K_2 \backslash \{0\}$, $1/t$ is an eigenvalue of the operator $\mathcal{A}$. Since $r(\mathcal{A}) < 1$, we derive a contradiction. Hence $\mathcal{L}$ does not have property α.

Assume that $\lambda > 1$ is an eigenvalue of $\mathcal{L}$ with a corresponding eigenfunction (ξ, η). Then

$$(-\Delta + M)^{-1}\begin{pmatrix} \left(a - 2\Theta_{[a]} + M\right)\xi - \frac{c\Theta_{[a]}}{1+\Theta_{[a]}}\eta \\ \left(b + \frac{d\Theta_{[a]}}{1+\Theta_{[a]}} + M\right)\eta \end{pmatrix} = \lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

that is,

$$\begin{cases} -\Delta\xi + M\xi = \frac{1}{\lambda}\left(\left(a - 2\Theta_{[a]} + M\right)\xi - \frac{c\Theta_{[a]}}{1+\Theta_{[a]}}\eta\right), & x \in \Omega, \\ -\Delta\eta + M\eta = \frac{1}{\lambda}\left(b + \frac{d\Theta_{[a]}}{1+\Theta_{[a]}} + M\right)\eta, & x \in \Omega, \\ \kappa_1\frac{\partial\xi}{\partial\nu} + \xi = \kappa_2\frac{\partial\eta}{\partial\nu} + \eta = 0, & x \in \partial\Omega. \end{cases} \tag{3.3}$$

If $\eta \not\equiv 0$, it follows from Lemma 2.2 and the comparison property of eigenvalue that

$$0 = \lambda_{1,\kappa_2}\left(M(1 - \frac{1}{\lambda}) - \frac{1}{\lambda}\left(b + \frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)\right) \geq -b + \lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right),$$

which yields a contradiction to the assumption. So $\eta \equiv 0$. Thus $\xi \not\equiv 0$. Substituting $\eta \equiv 0$ into the first equation of (3.3) and in view of $\Theta_{[a]} \leq a$, we have

$$0 = \lambda_{1,\kappa_1}\left(M(1 - \frac{1}{\lambda}) - \frac{1}{\lambda}(a - 2\Theta_{[a]})\right) > \lambda_{1,\kappa_1}\left(-(a - \Theta_{[a]})\right) = 0.$$

This is a contradiction again. Consequently,

$$\mathrm{index}_W\left(\mathcal{F}, (\Theta_{[a]}, 0)\right) = (-1)^\sigma = (-1)^0 = 1,$$

where σ is the the sum of multiplicities of all eigenvalues of $\mathcal{L}$ which are greater than one. Hence, the theorem is proven. □

Similarly, we have the following lemma of which the proof is a slight modification of the above.

Lemma 3.3. *Assume that* $b > \lambda_{1,\kappa_2}$.

(i) *If* $a > \lambda_{1,\kappa_1}\left(\frac{c\Theta_{[b]}}{1+m\Theta_{[b]}}\right)$, *then* $\mathrm{index}_W\left(\mathcal{F}, (0, \Theta_{[b]})\right) = 0$.

(ii) *If* $a < \lambda_{1,\kappa_1}\left(\frac{c\Theta_{[b]}}{1+m\Theta_{[b]}}\right)$, *then* $\mathrm{index}_W\left(\mathcal{F}, (0, \Theta_{[b]})\right) = 1$.

4. EXISTENCE AND STABILITY OF POSITIVE SOLUTIONS

In this section, we first establish the nonexistence and existence results of positive solutions to system (1.4) and give some sufficient and necessary conditions. After this, we investigate the stability of positive solutions under the assumption that c is sufficiently small.

Theorem 4.1. (i) *If* $a \leq \lambda_{1,\kappa_1}$, *then* (1.4) *has no positive solution and in addition if* $b \leq \lambda_{1,\kappa_2}$, *then* (1.4) *has no nonnegative nonzero solution.*

(ii) *Assume that* $b \leq \lambda_{1,\kappa_2}$. *Then* $a > \lambda_{1,\kappa_1}$ *and* $b > \lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$ *if and only if* (1.4) *has a positive solution.*

(iii) *If* $b > \lambda_{1,\kappa_2}$ *and* $a > \lambda_{1,\kappa_1}\left(\frac{c\Theta_{[b]}}{1+m\Theta_{[b]}}\right)$, *then* (1.4) *has a positive solution.*

(iv) *If* $b > \lambda_{1,\kappa_2}$ *and* (1.4) *has a positive solution, then*

$$\lambda_{1,\kappa_1}\left(-a+\frac{c\Theta_{[b]}}{1+\Theta_{[a]}+m\Theta_{[b]}}\right) < 0.$$

Proof. (i) Suppose on the contrary that $(\bar{u},\bar{v})$ is a positive solution of (1.4), then $(\bar{u},\bar{v})$ satisfies the equation $-\Delta\bar{u} = \bar{u}\left(a-\bar{u}-\frac{c\bar{v}}{1+\bar{u}+m\bar{v}}\right)$ in Ω and $\kappa_1\frac{\partial\bar{u}}{\partial\nu}+\bar{u}=0$ on $\partial\Omega$, and so $\lambda_{1,\kappa_1}\left(-a+\bar{u}+\frac{c\bar{v}}{1+\bar{u}+m\bar{v}}\right)=0$ by Lemma 2.2. Using the comparison property of eigenvalue, we have $a > \lambda_{1,\kappa_1}$, which is a contradiction. Next, assume that $(\bar{u},\bar{v})$ is a nonnegative nonzero solution of (1.4). If $\bar{u}\not\equiv 0$ and $\bar{v}\equiv 0$, then $a>\lambda_{1,\kappa_1}$. Similarly, if $\bar{u}\equiv 0$ and $\bar{v}\not\equiv 0$, then $b>\lambda_{1,\kappa_2}$. A contradiction.

(ii) Assume that $a>\lambda_{1,\kappa_1}$ and $b>\lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$. Since $b\le\lambda_{1,\kappa_2}$, then $\Theta_{[b]}\equiv 0$. By Lemma 3.2, we have

$$\mathrm{index}_W(\mathcal{F},(0,0))+\mathrm{index}_W(\mathcal{F},(\Theta_{[a]},0))=0$$

and $\mathrm{index}_W(\mathcal{F},D')=1$. Therefore, (1.4) has a positive solution in D'.

Conversely, assume that $(\bar{u},\bar{v})$ is a positive solution of (1.4). Then $a>\lambda_{1,\kappa_1}$ and $\bar{u}\le\Theta_{[a]}$. Since $(\bar{u},\bar{v})$ satisfies the equation, $-\Delta\bar{v}=\bar{v}\left(b-\bar{v}+\frac{d\bar{u}}{1+\bar{u}+m\bar{v}}\right)$ in Ω and $\kappa_2\frac{\partial\bar{v}}{\partial\nu}+\bar{v}=0$ on $\partial\Omega$, we have $0=\lambda_{1,\kappa_2}\left(-b+\bar{v}-\frac{d\bar{u}}{1+\bar{u}+m\bar{v}}\right)>\lambda_{1,\kappa_2}\left(-b-\frac{d\bar{u}}{1+\bar{u}}\right)\ge\lambda_{1,\kappa_2}\left(-b-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$ by Lemma 2.2 and using the comparison property of eigenvalue. Hence, $b>\lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$.

(iii) By Lemma 3.2 and 3.3, we have

$$\mathrm{index}_W(\mathcal{F},(0,0))+\mathrm{index}_W\left(\mathcal{F},(\Theta_{[a]},0)\right)+\mathrm{index}_W\left(\mathcal{F},(0,\Theta_{[b]})\right)=0$$

and $\mathrm{index}_W(\mathcal{F},D')=1$. So (1.4) has a positive solution in D'.

(iv) Let $(\bar{u},\bar{v})$ be a positive solution of (1.4). Then $a>\lambda_{1,\kappa_1}$. Thus (1.4) has a semi-trivial solution $(\Theta_{[a]},0)$. From $b>\lambda_{1,\kappa_2}$, (1.4) has a semi-trivial solution $(0,\Theta_{[b]})$. Clearly, by the uniqueness of $\Theta_{[a]}$ and $\Theta_{[b]}$, we have $\bar{u}\le\Theta_{[a]}$ and $\Theta_{[b]}\le\bar{v}$. Thus by using the comparison property of eigenvalue, we have

$$\lambda_{1,\kappa_1}\left(-a+\frac{\Theta_{[b]}}{1+\Theta_{[a]}+m\Theta_{[b]}}\right)<\lambda_{1,\kappa_1}\left(-a+\bar{u}+\frac{c\bar{v}}{1+\bar{u}+m\bar{v}}\right)=0.$$

The proof is completed. ∎

Now, we discuss the stability of the positive solution of (1.4) as $c\to 0$. Let $\hat{v}$ be the unique positive solution of the equation:

$$\begin{cases} -\Delta v = v\left(b-v+\frac{d\Theta_{[a]}}{1+\Theta_{[a]}+mv}\right), & x\in\Omega,\\ \kappa_2\frac{\partial v}{\partial\nu}+v=0, & x\in\partial\Omega, \end{cases} \tag{4.1}$$

when $a>\lambda_{1,\kappa_1}$ and $b>\lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$. If $b\le\lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$, then we define $\hat{v}=0$.

Lemma 4.2. *If $a > \lambda_{1,\kappa_1}$, then the positive solution of* (1.4) *(if it exists) converges to* $(\Theta_{[a]}, \hat{v})$ *as* $c \to 0$.

Proof. It is easy to see that the compact operator $\mathcal{F}(u,v)$ converges to the operator

$$\widetilde{\mathcal{F}}(u,v) = (-\Delta + M)^{-1}\left(u(a-u) + Mu, v\left(b - v + \frac{du}{1+u+mv}\right) + Mv\right).$$

as $c \to 0$ in D'. So the fixed points of $\mathcal{F}$ converge to the fixed points of $\widetilde{\mathcal{F}}$ as $c \to 0$ in D'. Since $(\Theta_{[a]}, \hat{v})$ is the only fixed point of $\widetilde{\mathcal{F}}$ in D', then the desired assertion holds evidently. $\square$

Lemma 4.3. *If $a > \lambda_{1,\kappa_1}$ and $b > \lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$, then*

$$\lambda_{1,\kappa_2}\left(-b + 2\hat{v} - \frac{d\Theta_{[a]}(1+\Theta_{[a]})}{(1+\Theta_{[a]}+m\hat{v})^2}\right) > 0.$$

Proof. Since $\hat{v}$ is a positive solution of (4.1), then by Lemma 2.2, we have

$$\lambda_{1,\kappa_2}\left(-b + \hat{v} - \frac{d\Theta_{[a]}}{1+\Theta_{[a]}+m\hat{v}}\right) = 0.$$

Denote $h(x,v) = b - v + \frac{d\Theta_{[a]}}{1+\Theta_{[a]}+mv}$, then $h_v(x,v) < 0$. Hence, by the comparison property of eigenvalue, we have

$$\begin{aligned} 0 &= \lambda_{1,\kappa_2}(-h(x,\hat{v})) \\ &< \lambda_{1,\kappa_2}(-h(x,\hat{v}) - \hat{v}h_v(x,\hat{v})) = \lambda_{1,\kappa_2}\left(-b + 2\hat{v} - \frac{d\Theta_{[a]}(1+\Theta_{[a]})}{(1+\Theta_{[a]}+m\hat{v})^2}\right). \end{aligned}$$

$\square$

Theorem 4.4. *If $a > \lambda_{1,\kappa_1}$, then there exists a positive constant $\overline{C}$ such that* (1.4) *has at most one positive solution when $c \le \overline{C}$. Moreover, the positive solution (if it exists) is nondegenerate and linearly stable.*

Proof. From the Implicit Function Theorem, we can prove the uniqueness result if (1.4) has a positive solution using c as the main parameter. For the purpose to show that such a positive solution is nondegenerate and linearly stable, it suffices to prove that the corresponding linearized eigenvalue problem of (1.4) has no eigenvalue μ with $\mathrm{Re}(\mu) \le 0$. By way of contradiction, assume that (1.4) has a positive solution (u_i, v_i) which is either degenerate or linearly unstable for a sequence $\{c_i\}$ with $c_i \to 0$ where $i \ge 1$. Thus there exist μ_i with $\mathrm{Re}(\mu_i) \le 0$ and $(\xi_i, \eta_i) \ne (0,0)$ satisfying $\|\xi_i\|_2^2 + \|\eta_i\|_2^2 = 1$ such that

$$\begin{cases} -\Delta\xi_i + \left(-a + 2u_i + \frac{c_i v_i(1+mv_i)}{(1+u_i+mv_i)^2}\right)\xi_i + \frac{c_i u_i(1+u_i)}{(1+u_i+mv_i)^2}\eta_i = \mu_i\xi_i, & x \in \Omega, \\ -\Delta\eta_i - \frac{dv_i(1+mv_i)}{(1+u_i+mv_i)^2}\xi_i + \left(-b + 2v_i - \frac{du_i(1+u_i)}{(1+u_i+mv_i)^2}\right)\eta_i = \mu_i\eta_i, & x \in \Omega, \\ \kappa_1\frac{\partial\xi_i}{\partial\nu} + \xi_i = \kappa_2\frac{\partial\eta_i}{\partial\nu} + \eta_i = 0, & x \in \partial\Omega. \end{cases} \tag{4.2}$$

Multiplying the equations of (4.2) by $\overline{\xi}_i$ and $\overline{\eta}_i$, respectively, and integrating them over Ω, then we have

$$\begin{aligned}\mu_i &= \int_\Omega |\nabla\xi_i|^2 + \int_\Omega \left(-a + 2u_i + \frac{c_i v_i(1+mv_i)}{(1+u_i+mv_i)^2}\right)|\xi_i|^2 + \int_\Omega \frac{c_i u_i(1+u_i)}{(1+u_i+mv_i)^2}\eta_i\bar{\xi}_i \\ &\quad + \int_\Omega |\nabla\eta_i|^2 - \int_\Omega \frac{dv_i(1+mv_i)}{(1+u_i+mv_i)^2}\xi_i\bar{\eta}_i + \int_\Omega \left(-b + 2v_i - \frac{du_i(1+u_i)}{(1+u_i+mv_i)^2}\right)|\eta_i|^2 \\ &\quad + \tau_1 \int_{\partial\Omega} |\xi_i|^2 + \tau_2 \int_{\partial\Omega} |\eta_i|^2,\end{aligned}$$

where $\bar{\xi}_i$ and $\bar{\eta}_i$ are the respective complex conjugates of ξ_i and η_i. In addition, $\tau_i = 1/\kappa_i$ for $\kappa_i \neq 0$ and $\tau_i = 0$ for $\kappa_i = 0$, $i = 1, 2$. In the above equality, we can see that $\{\mathrm{Re}(\mu_i)\}$ and $\{\mathrm{Im}(\mu_i)\}$ are bounded since u_i and v_i are bounded, $\mathrm{Re}(\mu_i) \leq 0$ and $\|\xi_i\|_2^2 + \|\eta_i\|_2^2 = 1$. Therefore $\{\mu_i\}$ is bounded. Without loss of generality, assume that $\mu_i \to \mu$. Thus $\mathrm{Re}(\mu) \leq 0$. We can also assume that $\xi_i \to \xi$, $\eta_i \to \eta$ since $\{\xi_i\}$ and $\{\eta_i\}$ are bounded. Note that $u_i \to \Theta_{[a]}$ and $v_i \to \hat{v}$ as $c_i \to 0$. by Lemma 4.2. Taking the limit in (4.2), we obtain

$$\begin{cases} -\Delta\xi + (-a + 2\Theta_{[a]})\xi = \mu\xi, & x \in \Omega, \\ -\Delta\eta - \frac{d\hat{v}(1+m\hat{v})}{(1+\Theta_{[a]}+m\hat{v})^2}\xi + \left(-b + 2\hat{v} - \frac{d\Theta_{[a]}(1+\Theta_{[a]})}{(1+\Theta_{[a]}+m\hat{v})^2}\right)\eta = \mu\eta, & x \in \Omega, \\ \kappa_1 \frac{\partial\xi}{\partial\nu} + \xi = \kappa_2 \frac{\partial\eta}{\partial\nu} + \eta = 0, & x \in \partial\Omega. \end{cases} \tag{4.3}$$

Thus μ must be a real number satisfying $\mu \leq 0$. If $\xi \not\equiv 0$, then μ is an eigenvalue of the problem $-\Delta\phi + (-a + 2\Theta_{[a]})\phi = \mu\phi$ in Ω and $\kappa_1 \frac{\partial\phi}{\partial\nu} + \phi = 0$ on $\partial\Omega$, and so $0 \geq \lambda_{1,\kappa_1}(-a + 2\Theta_{[a]})$. Since $\Theta_{[a]}$ is a semi-trivial solution of the first equation of (1.4), we have $\lambda_{1,\kappa_1}(-a + \Theta_{[a]}) = 0$. By the comparison property of eigenvalue, $\lambda_{1,\kappa_1}(-a+2\Theta_{[a]}) > \lambda_{1,\kappa_1}(-a+\Theta_{[a]})$, from which we get a contradiction. Hence, $\xi \equiv 0$, and thus $\eta \neq 0$. Form (4.3) with $\xi \equiv 0$, we get

$$\begin{cases} -\Delta\eta + \left(-b + 2\hat{v} - \frac{d\Theta_{[a]}(1+\Theta_{[a]})}{(1+\Theta_{[a]}+m\hat{v})^2}\right)\eta = \mu\eta, & x \in \Omega, \\ \kappa_2 \frac{\partial\eta}{\partial\nu} + \eta = 0, & x \in \partial\Omega. \end{cases}$$

Since $\eta \neq 0$, we have $0 \geq \mu \geq \lambda_{1,\kappa_2}\left(-b + 2\hat{v} - \frac{d\Theta_{[a]}(1+\Theta_{[a]})}{(1+\Theta_{[a]}+m\hat{v})^2}\right)$ when $b > \lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$. By Lemma 4.3, we derive a contradiction. Thus $\eta \equiv 0$. For the case of $b \leq \lambda_{1,\kappa_2}\left(-\frac{d\Theta_{[a]}}{1+\Theta_{[a]}}\right)$ (thus $\hat{v} \equiv 0$), we can also have $\eta \equiv 0$. This is a contradiction to $\|\xi\|_2^2 + \|\eta\|_2^2 = 1$. The proof is completed. $\square$

5. LARGE TIME BEHAVIOR OF TIME-DEPENDENT SYSTEM

In this section, we consider the long time behavior of the time-dependent solutions of system (1.2), namely, we give some sufficient conditions for the extinction and permanence of parabolic system (1.2). It's a treatment similar to [23].

First, it can be easily checked that $u\left(a - u - \frac{cv}{1+u+mv}\right)$ and $v\left(b - v + \frac{du}{1+u+mv}\right)$ satisfy the Lipschitz conditions in a bounded set $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$.

Let U and V_U be the respective solutions of the following equations:

$$\begin{cases} U_t - \Delta U = U(a-U), & x \in \Omega, t \in (0,\infty), \\ \kappa_1 \frac{\partial U}{\partial \nu} + U = 0, & x \in \partial\Omega, t \in (0,\infty), \\ U(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{5.1}$$

and

$$\begin{cases} (V_U)_t - \Delta V_U = V_U \left(b - V_U + \frac{dU}{1+U+mV_U} \right), & x \in \Omega, t \in (0,\infty), \\ \kappa_2 \frac{\partial V_U}{\partial \nu} + V_U = 0, & x \in \partial\Omega, t \in (0,\infty), \\ V_U(x,0) = v_0(x) \geq 0, & x \in \Omega. \end{cases}$$

For (5.1), it is easy to see that the trivial solution is global stable if $a \leq \lambda_{1,\kappa_1}$, while when $a > \lambda_{1,\kappa_1}$, the solution $U(x,t)$ of (5.1) converges to $\Theta_{[a]}(x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$.

By applying Theorem 12.5.1 in [18], we can obtain the following proposition.

Proposition 5.1. Let $u_0(x) \geq 0$ and $v_0(x) \geq 0$. Then (1.2) has a unique bounded solution $(u(x,t), v(x,t))$ satisfying the relation

$$(0,0) \leq (u(x,t), v(x,t)) \leq (U, V_U), \quad (x,t) \in \Omega \times (0,\infty).$$

Furthermore, $(u(x,t), v(x,t))$ is positive in $\Omega \times (0,\infty)$ if $u_0(x) \not\equiv 0$ and $v_0(x) \not\equiv 0$.

The following results imply the extinction of the prey and predator.

Theorem 5.2. *Let $(u(x,t), v(x,t))$ be the positive solution of* (1.2)*.*

(i) *If $a \leq \lambda_{1,\kappa_1}$ and $b+d \leq \lambda_{1,\kappa_2}$, then $(u(x,t), v(x,t)) \to (0,0)$ uniformly on $\overline{\Omega}$ as $t \to \infty$.*

(ii) *If $a > \lambda_{1,\kappa_1}$ and $b+d \leq \lambda_{1,\kappa_2}$, then $(u(x,t), v(x,t)) \to (\Theta_{[a]}, 0)$ uniformly on $\overline{\Omega}$ as $t \to \infty$.*

(iii) *If $a \leq \lambda_{1,\kappa_1}$ and $b > \lambda_{1,\kappa_2}$, then $(u(x,t), v(x,t)) \to (0, \Theta_{[b]})$ uniformly on $\overline{\Omega}$ as $t \to \infty$.*

Proof. (i) Notice that $u_t - \Delta u = u\left(a - u - \frac{cv}{1+u+mv}\right) \leq u(a-u)$ in $\Omega \times (0,\infty)$, we have $0 \leq u(x,t) \leq U(x,t)$ by the comparison principle. Since $a \leq \lambda_{1,\kappa_1}$, $U(x,t) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Hence, $u(x,t) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Choose small $\epsilon > 0$ such that

$$v_t - \Delta v = v\left(b - v + \frac{du}{1+u+mv}\right) \leq v\left(b - v + \frac{d\epsilon}{1+\epsilon+mv}\right)$$

for all $x \in \overline{\Omega}$ and sufficiently large t. Similar arguments shows that $b + d \leq \lambda_{1,\kappa_2}$ implies $v(x,t) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$.

(ii) From the proof of (i), we have $0 \leq u(x,t) \leq U(x,t) \to \Theta_{[a]}(x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$, since $a > \lambda_{1,\kappa_1}$. This implies that $\limsup_{t\to\infty} u(x,t) \leq \Theta_{[a]}(x)$ uniformly on $\overline{\Omega}$. As a result, for a given $\epsilon > 0$, there exists a $T_\epsilon \geq 0$ such that $u(x,t) \leq \Theta_{[a]}(x)+\epsilon$ for all $t \geq T_\epsilon$. Thus,

$$v_t - \Delta v = v\left(b - v - \frac{d(\Theta_{[a]}(x)+\epsilon)}{1+\Theta_{[a]}(x)+\epsilon+mv}\right) \leq v\left(b - v - \frac{d\epsilon}{1+\epsilon+mv}\right)$$

for all $t \geq T_\epsilon$ on $\overline{\Omega}$. Since $b+d \leq \lambda_{1,\kappa_2}$, we can get that $v(x,t) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Therefore, using the first equation of (1.2) again, we have that there exists $T'_\epsilon \geq 0$ such that $u_t - \Delta u = u(a-\epsilon-u)$ in $\Omega \times (T'_\epsilon, \infty)$. Let $\epsilon > 0$ be sufficiently small satisfying $a - \epsilon > \lambda_{1,\kappa_1}$. Consequently, it follows from the comparison principle that $u(x,t) \geq U_\epsilon(x,t)$ for all $t \geq T'_\epsilon$, where U_ϵ is the solution of the following equation

$$\begin{cases} (U_\epsilon)_t - \Delta U_\epsilon = U_\epsilon(a-\epsilon-U_\epsilon), & x \in \Omega, t \in (T'_\epsilon, \infty), \\ \kappa_1 \frac{\partial U_\epsilon}{\partial \nu} + U_\epsilon = 0, & x \in \partial\Omega, t \in (T'_\epsilon, \infty), \\ U_\epsilon(x, T'_\epsilon) = u(x, T'_\epsilon), & x \in \Omega. \end{cases}$$

Since $U_\epsilon(x,t) \to U(x,t)$ as $\epsilon \to 0^+$ and $U(x,t) \to \Theta_{[a]}(x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$, we have $\liminf_{t\to\infty} u(x,t) \geq \Theta_{[a]}(x)$. Synthetically, we have the conclusion (ii).

(iii) It can be verified similarly as that of (ii). The proof is completed. □

Now, we give some sufficient conditions for the permanence of system (1.2) by using the method of upper and lower solutions. Note that system (1.2) is mixed quasi-monotone, we introduce the definition of upper and lower solutions, see [18, 19].

Definition 5.3. A pair of functions $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ in $C(\overline{\Omega}) \cap C^2(\Omega)$ are called upper and lower solutions of system (1.4) provided that they satisfy the relation $\overline{u} \geq \underline{u}$, $\overline{v} \geq \underline{v}$ and the following inequalities:

$$\begin{cases} -\Delta\overline{u} \geq \overline{u}\left(a - \overline{u} - \frac{c\underline{v}}{1+\overline{u}+m\underline{v}}\right), & x \in \Omega, \\ -\Delta\underline{u} \leq \underline{u}\left(a - \underline{u} - \frac{c\overline{v}}{1+\underline{u}+m\overline{v}}\right), & x \in \Omega, \\ -\Delta\overline{v} \geq \overline{v}\left(b - \overline{v} + \frac{d\overline{u}}{1+\overline{u}+m\overline{v}}\right), & x \in \Omega, \\ -\Delta\underline{v} \leq \underline{v}\left(b - \underline{v} + \frac{d\underline{u}}{1+\underline{u}+m\underline{v}}\right), & x \in \Omega, \\ \kappa_1\frac{\partial\overline{u}}{\partial\nu} + \overline{u} \geq 0 \geq \kappa_1\frac{\partial\underline{u}}{\partial\nu} + \underline{u}, & x \in \partial\Omega, \\ \kappa_2\frac{\partial\overline{v}}{\partial\nu} + \overline{v} \geq 0 \geq \kappa_2\frac{\partial\underline{v}}{\partial\nu} + \underline{v}, & x \in \partial\Omega. \end{cases}$$

Similar to the proof of Theorem 5.4 in [23], we have the following theorem, which shows the permanence of parabolic system (1.2).

Theorem 5.4. *If $a - \frac{c}{m} > \lambda_{1,\kappa_1}$, $b + d > \lambda_{1,\kappa_2}$, then there exists a pair of quasi-solutions $(\widetilde{u}, \widetilde{v})$ and $(\hat{u}, \hat{v})$ of system (1.2) with $\widetilde{u} \geq \hat{u}$ and $\widetilde{v} \geq \hat{v}$, in other words,*

$(\widetilde{u},\widetilde{v})$ and $(\hat{u},\hat{v})$ satisfy the equations:

$$\begin{cases} -\Delta\widetilde{u} = \widetilde{u}\left(a - \widetilde{u} - \frac{c\hat{v}}{1+\widetilde{u}+m\hat{v}}\right), & x \in \Omega, \\ -\Delta\hat{u} = \hat{u}\left(a - \hat{u} - \frac{c\widetilde{v}}{1+\hat{u}+m\widetilde{v}}\right), & x \in \Omega, \\ -\Delta\widetilde{v} = \widetilde{v}\left(b - \widetilde{v} + \frac{d\widetilde{u}}{1+\widetilde{u}+m\widetilde{v}}\right), & x \in \Omega, \\ -\Delta\hat{v} = \hat{v}\left(b - \hat{v} + \frac{d\hat{u}}{1+\hat{u}+m\hat{v}}\right), & x \in \Omega, \\ \kappa_1\frac{\partial\widetilde{u}}{\partial\nu} + \widetilde{u} = 0 = \kappa_1\frac{\partial\hat{u}}{\partial\nu} + \hat{u}, & x \in \partial\Omega, \\ \kappa_2\frac{\partial\widetilde{v}}{\partial\nu} + \widetilde{v} = 0 = \kappa_2\frac{\partial\hat{v}}{\partial\nu} + \hat{v}, & x \in \partial\Omega. \end{cases}$$

Moreover, $[\hat{u},\widetilde{u}] \times [\hat{v},\widetilde{v}]$ is a positive global attractor of system (1.2).

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