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GLOBAL ATTRACTOR FOR NEUTRAL PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH FINITE DELAY

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ABSTRACT. This work is devoted to investigating the existence of global attractors for a class of neutral partial functional integrodifferential equation with delay. Using the classic theory about global attractors in infinite dimensional dynamical systems, we obtain some sufficient conditions for guaranteeing the existence of a global attractor.

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1. PRELIMINARIES

This paper is devoted to investigating the existence of a global attractor for the following neutral partial functional integrodifferential equations with finite delay:

$$(1.1)\qquad \begin{cases} \frac{d}{dt}\mathcal{F}(t,u_t) = A\mathcal{F}(t,u_t) + \int_0^t B(t-s)\mathcal{F}(s,u_s)ds + G(t,u_t),\ t \geq 0, \\ u_0 = \phi \in C, \end{cases}$$

where $(A, D(A))$ is the infinitesimal generator of strongly continuous semigroup on a Banach space $(X, |\cdot|)$; $C := C([-r,0],X)$, $r > 0$, is the space of continuous functions from $[-r,0]$ to the Banach space X, equipped with the uniform norm $\|\phi\| = \sup_{-r\leq\theta\leq 0}|\phi(\theta)|$; $(B(t))_{t\geq 0}$ is closed linear operator from $D(A)$ to X and $B(t) \in L(D(A),X)$, for any $y \in D(A)$, the map $t \to B(t)y$ is bounded, differentiable and the derivative $t \to B'(t)y$ is uniformly bounded continuous on $\mathbb{R}^+$, here $L(D(A),X)$ is a Banach space of bounded linear operators from $D(A)$ to X; the history function $x_t \in C$ is defined by

$$x_t(\theta) = x(t+\theta),\ \text{for } \theta \in [-r,0];$$

$\mathcal{F} : \mathbb{R}^+ \times C \to X$ is defined by

$$\mathcal{F}(t,\phi) = \phi(0) - F(t,\phi),\ (t,\phi) \in \mathbb{R}^+ \times C;$$

F and G are $X-$valued functions on $\mathbb{R}^+ \times C$.

In [1], using the theory of resolvent operator developed in R. Grimmer [2], the authors obtained the existence of strict solutions for (1.1). Since the pioneering work of J. Hale [3], neutral partial functional differential equations have been extensively

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investigated, and this investigation has also inspired rapid development in nonlinear analysis and nonlinear dynamical systems, see [4]–[8] and the references therein. The simplest scalar case is the following neutral partial functional differential equation on the unit circle:

$$\frac{\partial}{\partial t}D(v_t) = K\frac{\partial^2}{\partial x^2}D(v_t) + H(v_t).$$

Abstract neutral partial functional differential equations originate in the theory of viscoelastic materials. In [9], it was illustrated that the equation:

$$\dot{u}(t) = A_T\left[u(t) + \int_{-\infty}^{t} F(t-s)u(s)ds + \int_{-\infty}^{t} K(t-s)u(s)ds\right],\ t \geq 0$$

can be regarded as abstract formulation of the model proposed. After that, Hernandez and Henriquez [10, 11] established some results concerning the existence and uniqueness of solutions of the following partial neutral functional differential equations with infinite delay:

$$\begin{cases} \frac{d}{dt}(u(t) - F(t, u_t)) = Au(t) + G(t, u_t), & t \geq 0, \\ u_0 = \varphi, & \varphi \in \mathcal{B}. \end{cases}$$

Motivated by the above work, in this paper we will establish some sufficient conditions for guaranteeing the existence of a global attractor for (1.1). It is know that the global attractor is a very useful tool, which is valid for more general situations than those for stability to study the asymptotical behavior. Hence, our work enriches the content of partial neutral functional differential equations.

For the sake of convenience, we list the following conditions which will be needed in our study of (1.1).

(H_1) There exist a positive constant H and function $K(\cdot), M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and for $a > 0$, if $u : (-\infty, \sigma + a] \to X$, $u_\sigma \in \mathcal{B}$ and $u(\cdot)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, \sigma + a]$, the following conditions hold:

(i) $u_t \in \mathcal{B}$,

(ii) $|u(t)| \leq H\|u_t\|$, which is equivalent to $\phi(0) \leq H\|\phi\|$ for each $\phi \in \mathcal{B}$,

(iii) $\|u_t\| \leq K(t-\sigma)\sup_{\sigma \leq s \leq t}|u(s)| + M(t-\sigma)\|u_\sigma\|$,

where $\mathcal{B}$ is a Banach space of of functions mapping $(-\infty, 0]$ into X endowed with the norm $\|\cdot\|$.

(H_2) For the function $u(\cdot)$ in (H_1), $t \to u_t$ is a $\mathcal{B}-$value continuous on $[\sigma, \sigma + a]$.

(H_3) (i) $F : \mathbb{R}^+ \times C \to X$ is globally Lipschitz continuous, i.e., there exists a constant $L_1 > 0$ such that $L_1 K(0) < 1$ and

$$|F(t, \phi_1) - F(t, \phi_2)| \leq L_1\|\phi_1 - \phi_2\| \text{ for any } t \geq 0 \text{ and } \phi_1, \phi_2 \in C.$$

(ii) $G : \mathbb{R}^+ \times C \to X$ is globally Lipschitz continuous, i.e., there exists a constant $L_2 > 0$ such that

$$|G(t,\phi_1) - G(t,\phi_2)| \le L_1\|\phi_1 - \phi_2\| \text{ for any } t \ge 0 \text{ and } \phi_1, \phi_2 \in C.$$

(H_4) (i) $F \in C^1(\mathbb{R}^+ \times C; X)$ and the partial derivatives $D_tF(\cdot,\cdot)$ and $D_\phi F(\cdot,\cdot)$ are locally Lipschitzians with respect to the second argument.
(ii) $G \in C^1(\mathbb{R}^+ \times C; X)$ and the partial derivatives $D_tG(\cdot,\cdot)$ and $D_\phi G(\cdot,\cdot)$ are locally Lipschitzians with respect to the second argument.
(H_5) If $(\varphi_n)_{n\ge 0}$ is a Cauchy sequence in C and if $(\varphi_n)_{n\ge 0}$ converges compactly to φ on $[-r,0]$, then φ is in C and $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$.

Definition 1.1. Let $T > 0$. A function $u : [-r,T] \to X$ is said to be a strict solution of (1.1) if u is continuous on $[0,T]$ and the following conditions hold

(i) $t \to \mathcal{F}(t,u_t) \in C^1([0,T];X) \cap C([0,T];D(A))$,
(ii) u satisfies (1.1) on $[0,T]$,
(iii) $u(t) = \phi(t)$ for $-r \le t \le 0$.

Lemma 1.2 ([1, Theorem 3.7]). *Assume that (H1)-(H5) hold. Let $\phi \in C$ be a continuously differentiable such that*

$$\phi' \in C,\ \mathcal{F}(0,\phi) \in D(A) \text{ and } D_\phi\mathcal{F}(0,\phi)\phi' + D_t\mathcal{F}(0,\phi) = A\mathcal{F}(0,\phi) + G(0,\phi). \tag{1.2}$$

Then (1.1) *possess a unique strict solution, which can be expressed by*

$$u(t) = \begin{cases} R(t)\mathcal{F}(0,\phi) + F(t,u_t) + \int_0^t R(t-s)G(s,u_s)ds, & 0 \le t \le T, \\ \phi(t), & -r \le t \le 0, \end{cases} \tag{1.3}$$

where $R(t) \in L(X,X)$ having the following properties:

(i) *$R(0) = I$ and $\|R(t)\| \le e^{-\alpha t}$ for some constant $\alpha > 0$,*
(ii) *For each $x \in X$, $R(t)x$ is strongly continuous semigroup for $t \ge 0$,*
(iii) *$R(t) \in L(D(A))$ for $t \ge 0$. For $x \in D(A)$, $R(\cdot)x \in C^1([0,+\infty);X) \cap C([0,+\infty); D(A))$ and*

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds. \end{aligned}$$

Denote $\Sigma_0 = \{\phi \in C : \phi \text{ satisfying (1.2)}\}$. Then from Lemma 1.2, for each $\phi \in \Sigma_0$, we define the following operator on Σ_0 by

$$U(t)\phi = u_t(\cdot,\phi),\ t \ge 0, \tag{1.4}$$

where $u_t(\cdot,\phi)$ is unique strict solution of (1.1) in Lemma 1.2. Clearly, $(U(t))_{t\ge 0}$ is a strongly continuous semigroup on Σ_0.

Definition 1.3 ([12])**.** An invariant set $\mathscr{A}$ is said to be a global attractor if $\mathscr{A}$ is a maximal compact invariant set which attracts each bounded set $B \subset X$.

Definition 1.4 ([12])**.** A semigroup $U(t) : X \to X$, $t \geq 0$, is said to be point dissipative if there is a bounded set $B \subseteq X$ that attracts each point of E under $U(t)$.

Lemma 1.5 ([13])**.** *If*

(i) *there is a $t_0 \geq 0$ such that $U(t)$ is compact for $t > t_0$,*
(ii) *$U(t)$ is point dissipative in X, then there exists a nonempty global attractor $\mathscr{A}$ in X.*

2. THE GLOBAL ATTRACTOR FOR (1.1)

In this section, we apply Lemma 1.5 to the strongly continuous semigroup $(U(t))_{t\geq 0}$ to obtain the existence of a global attractor of (1.1). For this purpose, we first give the following generalized Gronwall inequality, which is crucial for the estimate.

Lemma 2.1 ([14])**.** *If*

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)ds,\ t \in [t_0, T),$$

where all the functions involved are continuous on $[t_0, T)$, $T \leq +\infty$, and $k(t) \geq 0$, then $x(t)$ satisfies

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s)e^{\int_s^t k(u)du}ds,\ t \in [t_0, T).$$

Lemma 2.2. *Assume that assumptions* (H_1)–(H_5) *hold. Then, for each $\phi \in \Sigma_0$, if $L_1 < e^{-\gamma r}$, there exists a constant $\gamma > \alpha$ such that the strict solution $u(\cdot, \phi)$ of* (1.1) *satisfies the following inequality:*

$$\begin{aligned} e^{\alpha t}\|u_t\| \leq{}& \frac{1}{e^{-\gamma r} - L_1}\left(\frac{(\alpha c_1 + c_2)L_2(e^{-\gamma r} - L_1)^{-1}}{\alpha^2 - \alpha L_2(e^{-\gamma r} - L_1)^{-1}} + c_1 + \frac{c_2}{\alpha}\right)e^{\alpha t} \\ &+ \frac{L_2}{(e^{-\gamma r} - L_1)^2}\left[\frac{L_2(c_1 + (L_1 + 1)\|\phi\| - \frac{c_2}{\alpha})}{e^{-\gamma r} - L_1}\right. \\ &\left. - \frac{\alpha c_1 + c_2}{\alpha^2 - \alpha L_2(e^{-\gamma r} - L_1)^{-1}}\right]e^{L_2(e^{-\gamma r} - L_1)^{-1}t}, \end{aligned}$$

where $c_1 = F(0, 0)$, $c_2 = G(0, 0)$.

Proof. By (H_3), for each $\phi \in \Sigma_0$, we have

$$\begin{aligned} |F(t, \phi)| &= |F(t, \phi) - F(0, 0) + F(0, 0)| \\ &\leq |F(0, 0)| + |F(t, \phi) - F(0, 0)| \\ &\leq c_1 + L_1\|\phi\|. \end{aligned}$$

Similar to the above proof, we have

$$\begin{aligned}|\mathcal{F}(0,\phi) &= |\phi(0) - F(0,\phi)| \\ &\le c_1 + (L_1+1)\|\phi\|\end{aligned}$$

and

$$|G(t,\phi)| \le c_2 + L_2\|\phi\|.$$

Instead of considering the norm $\|u_t\|$ directly, we firstly estimate $\|e^{\gamma\cdot}u_t\|$ for some constant $\gamma > \alpha$.

Case 1. For $0 \le t \le r$, by (1.3) we have

$$\begin{aligned}
&\sup_{-r\le\theta\le 0} |e^{\gamma\theta}u_t(\theta)| = \max\{\sup_{-r\le\theta\le -t} |e^{\gamma\theta}\phi(t+\theta)|, \sup_{-t\le\theta\le 0} |e^{\gamma\theta}u_t(\theta)|\} \\
&\le \max\{e^{-\gamma t}\|\phi\|, \sup_{-t\le\theta\le 0} e^{\gamma\theta}e^{-\alpha(t+\theta)}[c_1 + (L_1+1)\|\phi\|] + \sup_{-t\le\theta\le 0}[c_1 + L_1\|u_{t+\theta}\|] \\
&+ \sup_{-t\le\theta\le 0} e^{\gamma\theta}\int_0^{t+\theta} e^{-\alpha(t+\theta-s)}(c_2 + L_2\|u_s\|)ds\} \\
&\le \max\{e^{-\gamma t}\|\phi\|, e^{-\alpha t}[c_1 + (L_1+1)\|\phi\|] + c_1 + L_1\|u_t\| \\
&+ \sup_{-t\le\theta\le 0} c_2 e^{-\alpha(t+\theta)}e^{\gamma\theta}\int_0^{t+\theta} e^{\alpha s}ds + \sup_{-t\le\theta\le 0} L_2 e^{-\alpha(t+\theta)}e^{\gamma\theta}\int_0^{t+\theta} e^{\alpha s}\|u_s\|ds\} \\
&\le e^{-\alpha t}[c_1 + (L_1+1)\|\phi\|] + c_1 + L_1\|u_t\| \\
&+ c_2 e^{-\alpha t}\int_0^t e^{\alpha s}ds + L_2 e^{-\alpha t}\int_0^t e^{\alpha s}\|u_s\|ds \\
&= e^{-\alpha t}[c_1 + (L_1+1)\|\phi\|] + c_1 + L_1\|u_t\| \\
&+ \frac{c_2}{\alpha}(1 - e^{-\alpha t}) + L_2 e^{-\alpha t}\int_0^t e^{\alpha s}\|u_s\|ds.
\end{aligned} \tag{2.1}$$

Case 2. For $t \ge r$, we have

$$\begin{aligned}
&\sup_{-r\le\theta\le 0} |e^{\gamma\theta}u_t(\theta)| = \sup_{0\le t+\theta\le t} |e^{\gamma\theta}u(t+\theta)| \\
&\le \sup_{0\le t+\theta\le t} e^{\gamma\theta}e^{-\alpha(t+\theta)}[c_1 + (L_1+1)\|\phi\|] + c_1 + L_1\|u_t\| \\
&+ \sup_{0\le t+\theta\le t} e^{\gamma\theta}\int_0^{t+\theta} e^{-\alpha(t+\theta-s)}(c_2 + L_2\|u_s\|)ds \\
&\le e^{-\alpha t}[c_1 + (L_1+1)\|\phi\|] + c_1 + L_1\|u_t\| \\
&+ c_2 e^{-\alpha t}\int_0^t e^{\alpha s}ds + L_2 e^{-\alpha t}\int_0^t e^{\alpha s}\|u_s\|ds \\
&= e^{-\alpha t}[c_1 + (L_1+1)\|\phi\|] + c_1 + L_1\|u_t\| \\
&+ \frac{c_2}{\alpha}(1 - e^{-\alpha t}) + L_2 e^{-\alpha t}\int_0^t e^{\alpha s}\|u_s\|ds.
\end{aligned} \tag{2.2}$$

Therefore for $t \geq 0$, from (2.1) and (2.2), we get

$$\sup_{-r\le\theta\le0} |e^{\gamma\theta} u_t(\theta)| \le e^{-\alpha t}[c_1 + (L_1+1)\|\phi\|] + c_1 + L_1\|u_t\| + \frac{c_2}{\alpha}(1-e^{-\alpha t}) + L_2 e^{-\alpha t}\int_0^t e^{\alpha s}\|u_s\|ds. \tag{2.3}$$

On the other hand, we have

$$\sup_{-r\le\theta\le0} |e^{\gamma\theta} u_t(\theta)| = \sup_{-r\le\theta\le0} e^{\gamma\theta}|u_t(\theta)| \ge \sup_{-r\le\theta\le0} e^{-\gamma r}|u_t(\theta)| = e^{-\gamma r}\|u_t\|, \tag{2.4}$$

which combines with (2.3) yields that

$$\|u_t\| \le e^{-\alpha t}[c_1 + (L_1+1)\|\phi\|](e^{-\gamma r}-L_1)^{-1} + c_1(e^{-\gamma r}-L_1)^{-1} + \frac{c_2}{\alpha}(1-e^{-\alpha t})(e^{-\gamma r}-L_1)^{-1} + L_2 e^{-\alpha t}(e^{-\gamma r}-L_1)^{-1}\int_0^t e^{\alpha s}\|u_s\|ds$$

and

$$e^{\alpha t}\|u_t\| \le [c_1 + (L_1+1)\|\phi\|](e^{-\gamma r}-L_1)^{-1} + c_1(e^{-\gamma r}-L_1)^{-1}e^{\alpha t} + \frac{c_2}{\alpha}(e^{\alpha t}-1)(e^{-\gamma r}-L_1)^{-1} + L_2(e^{-\gamma r}-L_1)^{-1}\int_0^t e^{\alpha s}\|u_s\|ds.$$

Using the generalized Gronwall inequality in Lemma 2.1, we have

$$\begin{aligned}
e^{\alpha t}\|u_t\| \le{}& [c_1 + (L_1+1)\|\phi\|](e^{-\gamma r}-L_1)^{-1} + c_1(e^{-\gamma r}-L_1)^{-1}e^{\alpha t} \\
&+ \frac{c_2}{\alpha}(e^{\alpha t}-1)(e^{-\gamma r}-L_1)^{-1} \\
&+ L_2(e^{-\gamma r}-L_1)^{-2}\int_0^t \Big[c_1 + (L_1+1)\|\phi\| + c_1 e^{\alpha s} \\
&+ \frac{c_2}{\alpha}(e^{\alpha s}-1)\Big] e^{\int_s^t L_2(e^{-\gamma r}-L_1)^{-1}d\tau}ds \\
={}& \frac{1}{e^{-\gamma r}-L_1}\left(\frac{(\alpha c_1+c_2)L_2(e^{-\gamma r}-L_1)^{-1}}{\alpha^2-\alpha L_2(e^{-\gamma r}-L_1)^{-1}} + c_1 + \frac{c_2}{\alpha}\right)e^{\alpha t} \\
&+ \frac{L_2}{(e^{-\gamma r}-L_1)^2}\left[\frac{L_2(c_1+(L_1+1)\|\phi\| - \frac{c_2}{\alpha})}{e^{-\gamma r}-L_1} \right. \\
&\left. - \frac{\alpha c_1+c_2}{\alpha^2-\alpha L_2(e^{-\gamma r}-L_1)^{-1}}\right]e^{L_2(e^{-\gamma r}-L_1)^{-1}t}.
\end{aligned}$$

Lemma 2.3. *Assume that the conditions of Lemma 2.2 are satisfied, further more, $\alpha > \frac{L_2}{e^{-\gamma r}-L_1}$, where γ is the constant defined by Lemma 2.2. Then $(U(t))_{t\ge0}$ is point dissipative.*

Proof. From Lemma 2.2, we find that, for each $\phi \in \Sigma_0$, since $\alpha > \frac{L_2}{e^{-\gamma r}-L_1}$, there exits a $t_0 := t_0(\phi) > 0$ such that for $t > t_0$,

$$\|u_t\| \le \frac{1}{e^{-\gamma r}-L_1}\left(\frac{(\alpha c_1+c_2)L_2(e^{-\gamma r}-L_1)^{-1}}{\alpha^2-\alpha L_2(e^{-\gamma r}-L_1)^{-1}} + c_1 + \frac{c_2}{\alpha}\right) + 1. \text{ (independent of } \phi)$$

Therefore,

$$B_{X_0}\left(0,\frac{1}{e^{-\gamma r}-L_1}\Big(\frac{(\alpha c_1+c_2)L_2(e^{-\gamma r}-L_1)^{-1}}{\alpha^2-\alpha L_2(e^{-\gamma r}-L_1)^{-1}}+c_1+\frac{c_2}{\alpha}\Big)+1\right)\cap X_0$$

attracts each point of X_0, where $B_{X_0}\left(0,\frac{1}{e^{-\gamma r}-L_1}\Big(\frac{(\alpha c_1+c_2)L_2(e^{-\gamma r}-L_1)^{-1}}{\alpha^2-\alpha L_2(e^{-\gamma r}-L_1)^{-1}}+c_1+\frac{c_2}{\alpha}\Big)+1\right)$ denotes the open ball in Σ_0 with center 0 and radius $\frac{1}{e^{-\gamma r}-L_1}\left(\frac{(\alpha c_1+c_2)L_2(e^{-\gamma r}-L_1)^{-1}}{\alpha^2-\alpha L_2(e^{-\gamma r}-L_1)^{-1}}+c_1+\frac{c_2}{\alpha}\right)+1$. □

Now, we show the compactness of the operator $U(t)$. The following lemma is similar with Theorem 2.7 in [15]. But, for the reader convenience, we give the details of its proof.

Lemma 2.4. *Assume that assumptions* (H$_1$)–(H$_5$) *hold. Then,* $U(t)$ *is compact for* $t>r$.

Proof. Let $t>r$ and $\{\phi_n\}$ be any bounded sequence of Σ_0. We will use Ascoli-Arzelà theorem to show that $\{U(t)\phi_n : n\in\mathbb{N}\}$ is pre-compact in Σ_0 by two steps.

Step 1. Show for any $\theta\in[-r,0]$, the set

$$Z(\theta)=\{((U(t)\phi_n)(\theta) : n\in\mathbb{N}\}$$

is pre-compact. For $t>r$ and $\theta\in[-r,0]$, by (1.3), we have

$$((U(t)\phi_n)(\theta)=R(t+\theta)\mathcal{F}(0,\phi_n)+F(t+\theta,u^n_{t+\theta})+\int_0^{t+\theta}R(t+\theta-s)G(s,u^n_s)ds, \tag{2.5}$$

where $u^n(\cdot)$ is the strict solution of (1.1) with initial function ϕ_n. Since $\{R(t)\}_{t\geq 0}$ is compact, the boundedness of $\mathcal{F}(0,\phi_n)$ and assumption (H$_3$), we know that

$$R(t+\theta)\mathcal{F}(0,\phi_n) \text{ and } F(t+\theta,u^n_{t+\theta})$$

are pre-compact. Now, considering the third term in (2.5), for sufficiently small $\varepsilon>0$, we have

$$\begin{aligned}\int_0^{t+\theta}R(t+\theta-s)G(s,u^n_s)ds&=R(\varepsilon)\int_0^{t+\theta-\varepsilon}R(t+\theta-s-\varepsilon)G(s,u^n_s)ds\\&\quad+\int_{t+\theta-\varepsilon}^{t+\theta}R(t+\theta-s)G(s,u^n_s)ds.\end{aligned}$$

Noting that Lemma 2.2, we have

$$\sup_{n\in\mathbb{N}}\|u^n_s\|<\infty,\ \ s\in[0,t]. \tag{2.6}$$

By (H$_3$), we get

$$|G(s,u^n_s)|\leq c_2+L_2\|u^n_s\|.$$

Therefore, from (2.5) and (2.6), there exist some constants $M_1, M_2 > 0$ such that

$$\left| \int_0^{t+\theta-\varepsilon} R(t+\theta-s-\varepsilon)G(s,u_s^n)ds \right| \le M_1$$

and

$$\left| \int_{t+\theta-\varepsilon}^{t+\theta} R(t+\theta-s)G(s,u_s^n)ds \right| \le M_2$$

which yields

$$R(\varepsilon)\left\{ \int_0^{t+\theta-\varepsilon} R(t+\theta-s-\varepsilon)G(s,u_s^n)ds : n \in \mathbb{N} \right\} \subset \Gamma_\varepsilon,$$

where Γ_ε is a compact set. Thus, $Z(\theta)$ is pre-compact.

Step 2. Show the equicontinuity of $\{U(t)\phi_n : n \in \mathbb{N}\}$. Let $-r \le \theta_1 < \theta_2 \le 0$, we have

$$\begin{aligned}
(U(t)\phi_n)(\theta_2) - (U(t)\phi_n)(\theta_1) &= [R(t+\theta_2) - R(t+\theta_1)]\mathcal{F}(0,\phi_n) \\
&+ F(t+\theta_2, u_{t+\theta_2}^n) - F(t+\theta_1, u_{t+\theta_1}^n) \\
&+ \int_0^{t+\theta_2} R(t+\theta_2-s)G(s,u_s^n)ds \\
&- \int_0^{t+\theta_1} R(t+\theta_1-s)G(s,u_s^n)ds \\
&= R(t+\theta_1)[R(t+\theta_2) - I]\mathcal{F}(0,\phi_n) \\
&+ F(t+\theta_2, u_{t+\theta_2}^n) - F(t+\theta_1, u_{t+\theta_1}^n) \\
&+ \int_{t+\theta_1}^{t+\theta_2} R(t+\theta_2-s)G(s,u_s^n)ds \\
&+ \int_0^{t+\theta_1} (R(t+\theta_2-s) - R(t+\theta_1-s))G(s,u_s^n)ds
\end{aligned}$$

which leads to

$$\begin{aligned}
|(U(t)\phi_n)(\theta_2) - (U(t)\phi_n)(\theta_1)| &\le \|R(t+\theta_1)[R(\theta_2-\theta_1) - I]\| \times |\mathcal{F}(0,\phi_n)| \\
&+ |F(t+\theta_2, u_{t+\theta_2}^n) - F(t+\theta_1, u_{t+\theta_1}^n)| \\
&+ \int_{t+\theta_1}^{t+\theta_2} |R(t+\theta_2-s)G(s,u_s^n)|ds \\
&+ \|R(\theta_2-\theta_1) - I\| \int_0^{t+\theta_1} |R(t+\theta_1-s)G(s,u_s^n)|ds.
\end{aligned}$$

Since the mapping $t \to R(t)$ is norm-continuous for $t > 0$, for some $\delta \in (0, t-r)$, put

$$R(t+\theta_1)[R(\theta_2-\theta_1) - I] = R(t+\theta_1-\delta)[R(\theta_2-\theta_1+\delta) - R(\delta)].$$

Then

$$\|R(\theta_2-\theta_1+\delta) - R(\delta)\| \to 0 \text{ as } \theta_2 \to \theta_1.$$

Thus

$$\|R(t+\theta_1)[T_0(\theta_2-\theta_1) - I]\| \times |\mathcal{F}(0,\phi_n)| \to 0 \text{ as } \theta_2 \to \theta_1.$$

By the property of F,

$$|F(t+\theta_2, u^n_{t+\theta_2}) - F(t+\theta_1, u^n_{t+\theta_1})| \to 0 \text{ as } \theta_2 \to \theta_1.$$

By the boundedness of $|R(t+\theta_2 - s)G(s, u^n_s)|$, then

$$\int_{t+\theta_1}^{t+\theta_2} |R(t+\theta_2 - s)G(s, u^n_s)| ds \to 0 \text{ as } \theta_2 \to \theta_1.$$

Obviously, $\int_0^{t+\theta_1} |R(t+\theta_1 - s)G(s, u^n_s)| ds$ belongs to a compact subset of X, and

$$\|R(\theta_2 - \theta_1) - I\| \int_0^{t+\theta_1} |R(t+\theta_1 - s)G(s, u^n_s)| ds \to 0 \text{ as } \theta_2 \to \theta_1.$$

Hence $\{U(t)\phi_n : n \in \mathbb{N}\}$ is equicontinuity. $\square$

Here, we state our main theorem of this paper, which is an immediate consequence of Lemma 1.5, 2.3 and 2.4.

Theorem 2.5. *Assume that assumptions* (H_1)–(H_5) *hold. If* $\alpha > \frac{L_2}{e^{-\gamma r} - L_1}$, *then* (1.1) *has a nonempty global attractor* $\mathscr{A}$.

3. AN EXAMPLE

Ezzinbi et al [1] considered the following Lotka-Volterra model with diffusion:

(3.1)

$$\begin{cases} \dfrac{\partial}{\partial t}\left[n(t,\xi) - \displaystyle\int_{-\infty}^0 f(\theta, n(t+\theta,\xi))d\theta\right] = \dfrac{\partial^2}{\partial \xi^2}\left[n(t,\xi) - \displaystyle\int_{-\infty}^0 f(\theta, n(t+\theta,\xi))d\theta\right] \\ \qquad + \displaystyle\int_0^t b(t-s)\dfrac{\partial^2}{\partial \xi^2}\left[n(s,\xi) - \displaystyle\int_{-\infty}^0 f(\theta, n(s+\theta,\xi))d\theta\right] ds \\ \qquad + \displaystyle\int_{-\infty}^0 g(\theta, n(t+\theta,\xi))d\theta \text{ for } t \geq 0 \\ n(t,0) - \int_{-\infty}^0 f(\theta, n(t+\theta,0))d\theta = 0 \text{ for } t \geq 0 \\ n(t,\pi) - \int_{-\infty}^0 f(\theta, n(t+\theta,\pi))d\theta = 0 \text{ for } t \geq 0 \\ n(\theta,\xi) = n_0(\theta,\xi) \text{ for } -\infty < \theta \leq 0,\ 0 \leq \xi \leq \pi, \end{cases}$$

where $f, g : \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}$, $n_0 : \mathbb{R}^- \times [0,\pi] \to \mathbb{R}$ and $b : \mathbb{R}^+ \times \to \mathbb{R}$ are continuous functions and obtained the following results:

Theorem 3.1. *Assume that assumptions* (E_1)–(E_5) *hold. Then* (3.1) *has a unique strict solution, where* (E_1)–(E_5) *can be found in* [1].

Based on the above result, we can choose the proper $\alpha, \gamma, r, L_1, L_2$ such that $\alpha > \frac{L_2}{e^{-\gamma r} - L_1}$ and (3.1) has a nonempty global attractor $\mathscr{A}$.

$\square$

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