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NECESSARY CONDITIONS AND ALGORITHMIC STABILITY TESTS FOR CERTAIN HIGHER ODD ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we obtain necessary conditions and robust algorithmic criteria for asymptotic stability of the zero solution of higher odd order linear neutral delay differential equations of the form

$$y^{(2m+1)}(t) + \alpha y^{(2m+1)}(t-\tau) = \sum_{j=0}^{2m} a_j y^{(j)}(t) + \sum_{j=0}^{2m} b_j y^{(j)}(t-\tau)$$

where a_j, b_j, and $\alpha \neq 0$ are real constants. Here $\tau > 0$ is a constant delay. In proving our results we make use of Pontryagin's theory for quasi-polynomials.

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1. INTRODUCTION

The aim of this paper is to derive robust algorithmic criteria for asymptotic stability of the zero solution of the neutral delay differential equation

$$y^{(2m+1)}(t) + \alpha y^{(2m+1)}(t-\tau) = \sum_{j=0}^{2m} a_j y^{(j)}(t) + \sum_{j=0}^{2m} b_j y^{(j)}(t-\tau) \tag{1.1}$$

where $\tau > 0$, $-1 < \alpha < 1$, $\alpha \neq 0$, and a_j, b_j are constants. In a previous paper [1] we considered the higher even order case which has a different analysis and results than the odd case. In [2], we considered equation (1.1) with $\alpha = 0$, and in [3] we considered equation (1.1) with $\alpha = 0$ and $m = 1$ which arose from a robotic model with damping and delay, and in [4] we considered (1.1) with $\alpha = m = 0$ and a_0, b_0 complex. There are no practical stability criteria of the zero solution of (1.1) for $m > 1$. For study of asymptotic stability of restricted special cases of (1.1) with special values of m see [5,6,7]. For stability and oscillation of certain third order equations see [8,9]. Generally, including delays in a differential equation has a destabilizing effect. Our work on non-neutral delay equations certainly upholds this, but when the order is 2 or higher there are rare cases when the delay has a stabilizing effect. We raise the

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same question as whether inclusion of a "neutral term" can have a stabilizing effect. In Example 4.1, we obtain an affirmative answer. In [10,11,12] study of necessary conditions or sufficient conditions are derived using Lyapunov's direct (or second) method. For studies of systems that may shed light on (1.1) see [11,12]. The study on systems does not, however, yield practical stability criteria of (1.1). For further study on asymptotic stability see [11-16]. It is clear that with $4m+3$ independent parameters in (1.1) one cannot expect to get regions of stability. Our goal is to derive robust algorithmic type stability criteria.

Regarding the stabilizing effect of delay and/or neutrality, our view is that part of the jth derivative term of the equation

$$y^{(2m+1)}(t) = \sum_{j=0}^{2m} p_j y^{(j)}(t) \tag{1.2}$$

is delayed and the remaining part is not. Note that with $\tau = 0$ the zero solution of (1.1) or (1.2) is asymptotically stable if and only if all the characterstic roots of a real polynomial

$$x^{2m+1} - p_{2m}x^{2m} - p_{2m-1}x^{2m-1} - p_{2m-2}x^{2m-2} - \cdots - p_0 = 0 \tag{1.3}$$

are in complex left half plane. Relative to (1.1), we view

$$p_j = \frac{a_j + b_j}{1+\alpha},\ j = 0, 1, \ldots, 2m. \tag{1.4}$$

Here we incorporate both delay and neutrality. By Routh-Hurwitz Criterion [20] all roots of (1.3) have negative real parts if and only if

$$\beta_j > 0, \quad j = 1, 2, \ldots, 2m+1, \tag{1.5}$$

where the β_j are the following determinants:

$$\beta_1 = -p_{2m},$$

$$\beta_2 = \begin{vmatrix} -p_{2m} & -p_{2m-2} \\ 1 & -p_{2m-1} \end{vmatrix},$$

$$\beta_k = \begin{vmatrix} -p_{2m} & -p_{2m-2} & -p_{2m-4} & \cdots & -p_{2(m-k)+2} \\ 1 & -p_{2m-1} & -p_{2m-3} & \cdots & -p_{2(m-k)+3} \\ 0 & -p_{2m} & -p_{2m-2} & \cdots & -p_{2(m-k)+4} \\ 0 & 1 & -p_{2m-1} & \cdots & -p_{2(m-k)+5} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -p_{2m-k+1} \end{vmatrix}, \qquad k = 3, \ldots 2m,$$

with $-p_{2m+1-j} = 0$ for $j > 2m+1$. In previous papers, [3,21] we have found cases (although rare) when the zero solution of (1.2) is unstable while the zero solution of (1.1) is asymptotically stable. We are also interested in whether neutrality alone can stabilize the solutions. In this case, the terms on the left side of (1.1) are merged into $(1+\alpha)y^{(2m+1)}(t)$ while the right side is unchanged. If we maintain delays but compare nonneutral with neutral we would use results in [2] rather than Routh-Hurwitz Criteria. As noted above, we will produce an example where the neutral term has a stabilizing effect with the other delays in place.

This paper is organized as follows. In Section 2, we present the tools used in our asymptotic stability analysis. In Section 3 we give our main results and some special cases. In Section 4 we present some examples.

2. BACKGROUND

In this section, we identify the characteristic function of (1.1) in order to study the asymptotic stability of the zero solution. We also cite the main results of Pontryagin related to asymptotic stability [20] and the applications of Pontryagin's results [21, §13.7–13.9].

The characteristic function of (1.1) is given by

$$\widehat{H}(s) = s^{2m+1} + \alpha e^{-s\tau} s^{2m+1} - \sum_{j=0}^{2m} a_j s^j - \sum_{j=0}^{2m} b_j e^{-s\tau} s^j. \tag{2.1}$$

Multiplying (2.1) by $e^{s\tau}$ yields

$$e^{s\tau}\widehat{H}(s) = e^{s\tau} s^{2m+1} + \alpha s^{2m+1} - \sum_{j=0}^{2m} a_j s^j e^{s\tau} - \sum_{j=0}^{2m} b_j s^j. \tag{2.2}$$

Letting $s = \frac{z}{\tau}$, we examine the zeros of

$$H(z) = \tau^{2m+1} e^z \widehat{H}\left(\frac{z}{\tau}\right) = z^{2m+1} e^z + \alpha z^{2m+1} - \sum_{j=0}^{2m} A_j z^j e^z - \sum_{j=0}^{2m} B_j z^j \tag{2.3}$$

where

$$A_j = a_j \tau^{2m+1-j} \text{ and } B_j = b_j \tau^{2m+1-j}, \quad j = 0, \ldots, 2m. \tag{2.4}$$

The following can be found in [22, Theorem 6.1].

Theorem 2.1 *In order that all solutions of* (1.1) *approach zero as* $t \to \infty$ *it is necessary and sufficient that all zeros of* (2.1), *or equivalently* (2.3), *have negative real parts and are bounded away from the imaginary axis, i.e., there is a positive real number* ν *such that* $Re\ z \leq -\nu$ *for every zero* z *of* $H(z)$.

We first determine the conditions under which all zeros of (2.1), or equivalently (2.2), have negative real parts and then find conditions under which the zeros are bounded uniformly away from the imaginary axis. The function (2.3) is a special function, usually called an exponential polynomial or a quasi-polynomial. The problem of analyzing the distribution of the zeros in the complex plane of such functions has received a great deal of attention.

Definition 2.1 *Let $h(z,w)$ be a polynomial in the two variables z and w (with complex coefficients),*

$$h(z,w)=\sum_{m,n} a_{mn}z^m w^n$$

where m and n are nonnegative integers. We call the term $a_{rs}z^r w^s$ the principal term of $h(z,w)$ if $a_{rs}\neq 0$, and for each term $a_{mn}z^m w^n$ with $a_{mn}\neq 0$, we have $r\geq m$ and $s\geq n$.

Note that $H(z)=h(z,e^z)$ where

$$h(z,w)=z^{2m+1}w+\alpha z^{2m+1}-\sum_{j=0}^{2m} A_j z^j w-\sum_{j=0}^{2m} B_j z^j. \tag{2.5}$$

It is clear from Definition 2.1 that $h(z,w)$ of (2.5) has principal term $z^{2m+1}w$. We now cite two theorems of Pontryagin, see [22,23].

Theorem 2.2 *Let $H(z)=h(z,e^z)$, where $h(z,w)$ is a polynomial with a principal term. We separate the function $H(iy)$ into real and imaginary parts; that is, we set $H(iy)=F(y)+iG(y)$. If all the zeros of the function $H(z)$ lie in the open left half plane, then the zeros of the functions $F(y)$ and $G(y)$ are real, are interlacing, and*

$$\Delta(y)=G'(y)F(y)-G(y)F'(y)>0 \tag{2.6}$$

for all real y. Moreover, in order that all the zeros of the function $H(z)$ lie in the open left half plane, it is sufficient that any one of the following conditions be satisfied:

- **(a):** *All the zeros of the functions $F(y)$ and $G(y)$ are real and interlace, and the inequality (2.6) is satisfied for at least one value of y.*
- **(b):** *All the zeros of the function $F(y)$ are real and for each of these zeros $y=y_0$ the inequality (2.6) is satisfied; that is, $F'(y_0)G(y_0)<0$.*
- **(c):** *All the zeros of the function $G(y)$ are real and for each of these zeros $y=y_0$ the inequality (2.6) is satisfied; that is, $G'(y_0)F(y_0)>0$.*

In our case,

$$H(iy)=(iy)^{2m+1}e^{iy}+\alpha(iy)^{2m+1}-\sum_{j=0}^{2m} A_j(iy)^j e^{iy}-\sum_{j=0}^{2m} B_j(iy)^j; \tag{2.7}$$

equivalently,

$$\begin{aligned}H(iy)&=(iy)^{2m+1}e^{iy}+\alpha(iy)^{2m+1}-\sum_{j=0}^{m}A_{2j}(iy)^{2j}e^{iy}-\sum_{j=0}^{m-1}A_{2j+1}(iy)^{2j+1}e^{iy}\\&-\sum_{j=0}^{m}B_{2j}(iy)^{2j}-\sum_{j=0}^{m-1}B_{2j+1}(iy)^{2j+1}.\end{aligned} \tag{2.8}$$

(Here and in the rest of this paper if a summation goes from 0 to -1, then the sum is taken to be zero.)

Now

$$
\begin{aligned}
H(iy) = \; & (-1)^m y^{2m+1} i(\cos y + i\sin y) + i\alpha(-1)^m y^{2m+1} \\
& - \sum_{j=0}^{m} A_{2j}(-1)^j y^{2j}(\cos y + i\sin y) \\
& - \sum_{j=0}^{m-1} A_{2j+1}(-1)^j i y^{2j+1}(\cos y + i\sin y) - \sum_{j=0}^{m} B_{2j}(-1)^j y^{2j} \\
& - \sum_{j=0}^{m-1} B_{2j+1}(-1)^j i y^{2j+1} = F(y) + iG(y)
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{aligned}
F(y) = \; & (-1)^{m+1} y^{2m+1}\sin y - \sum_{j=0}^{m} A_{2j}(-1)^j y^{2j}\cos y \\
& - \sum_{j=0}^{m-1} A_{2j+1}(-1)^{j+1} y^{2j+1}\sin y - \sum_{j=0}^{m} B_{2j}(-1)^j y^{2j}
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
G(y) = \; & (-1)^m y^{2m+1}\cos y + \alpha(-1)^m y^{2m+1} - \sum_{j=0}^{m} A_{2j}(-1)^j y^{2j}\sin y \\
& - \sum_{j=0}^{m-1} A_{2j+1}(-1)^j y^{2j+1}\cos y - \sum_{j=0}^{m-1} B_{2j+1}(-1)^j y^{2j+1}.
\end{aligned}
\tag{2.11}
$$

In order to study the location of the zeros of $H(z)$ we study the zeros of F and G. To do so, we need the following result which is useful in determining whether all roots of F and G are real. Let $f(z,u,v)$ be a polynomial in z, u, and v, which we write in the form

$$
f(z,u,v) = \sum_{m,n} z^m \phi_m^{(n)}(u,v)
\tag{2.12}
$$

where $\phi_m^{(n)}(u,v)$ is a polynomial of degree n, homogeneous in u and v, and let $z^r\phi_r^{(s)}(u,v)$ be the principal term of $f(z,u,v)$, and let $\phi^{*(s)}(u,v)$ denote the coefficient of z^r in $f(z,u,v)$, so that

$$
\phi^{*(s)}(u,v) = \sum_{n\le s} \phi_r^{(n)}(u,v).
$$

(The Principal term for the polynomials of the form (2.12) is analogous to that defined in Definition 2.1, see [23, pages 440-443]). Also we let

$$
\Phi^{*(s)}(z) = \phi^{*(s)}(\cos z, \sin z).
$$

Theorem 2.3 *Let $f(z,u,v)$ be a polynomial with principal term $z^r\phi_r^{(s)}(u,v)$ and assume that u^2+v^2 is not a factor of $\phi_r^{(s)}(u,v)$. If ϵ is such that $\Phi^{*(s)}(\epsilon+iy)\neq 0$ for all real y, then in the strip $-2\pi k+\epsilon \le Re\ z \le 2\pi k+\epsilon$, the function $F(z) = f(z,\cos z,\sin z)$ will have, for all sufficiently large values of k, exactly $4sk+r$ zeros.*

Thus, in order for the function $F(z)$ to have only real roots, it is necessary and sufficient that in the real interval $-2\pi k+\epsilon \le x \le 2\pi k+\epsilon$, it has exactly $4sk+r$ real roots for all sufficiently large k.

Note that the functions $F(y)$ and $G(y)$ in (2.10) and (2.11) have principal terms $(-1)^{m+1}y^{2m+1}\sin y$ and $(-1)^m y^{2m+1}\cos y$, respectively. The condition that u^2+v^2 not be a factor of $z^r\phi_r^{(s)}(u,v)$ is frequently overlooked. When $s=1$, it is not an issue. This condition is satisfied by polynomials $f(z,u,v)$ derived from function $h(z,w)$ in (2.12) with a principal term. As well, if $f(z,u,v)$ is derived from function involving $\sin z$ and $\cos z$, the Pythagorean identity could be used to make this condition satisfied. None the less this condition is needed for Theorem 2.3 to be true as stated.

For the case in point, $s=1$ and $r=2m+1$. Therefore $G(y)$ (given in (2.11)) has all real zeros if and only if $G(y)$ has $4k+2m+1$ zeros in $(-2k\pi, 2k\pi)$ for k sufficiently large, and the same holds for F given in (2.10) with $(-2k\pi, 2k\pi)$ replaced by $(-2k\pi+\epsilon, 2k\pi+\epsilon)$ where $0<\epsilon<\pi$.

3. MAIN RESULTS

In this section we present the main results of this paper. We first describe the asymptotic behavior of the zeros of G. Throughout this paper for x real and $a>0$, $[x]_a$ denotes the unique real number in the interval $[0,a)$ for which $x-[x]_a$ is an integer multiple of a. We will use $a=\pi$ and $a=2\pi$.

See Kuang [16, p. 65] for the following result:

Lemma 3.1 *A necessary condition for the zero solution of* (1.1) *to be asymptotically stable is that $|\alpha| \le 1$.*

In this paper we will only consider $|\alpha|<1$. The root analysis of (2.1) or (2.3) is an open problem for the case $|\alpha|=1$. See special examples of $\alpha=-1$ in [23] . We consider the following necessary conditions

Lemma 3.2 *If the zero solution of* (1.1) *is asymptotically stable, then $(A_0+B_0)(A_0+A_1+B_1)>0$.*

Proof. Theorem 2.2 and the fact that $y=0$ is a zero of G yield $\Delta(0) = G'(0)F(0) = (A_0+B_0)(A_0+A_1+B_1)>0$.

Theorem 3.1 *Suppose that $|\alpha|<1$ and that all zeros of $H(z)$ are in the open left half plane (i.e. Re $z<0$ for every zero z of $H(z)$). Then all zeros of $H(z)$ are bounded away from imaginary axis (i.e. there $\eta>0$ for which $\Re z<-\eta$ for every zero z of $H(z)$).*

Proof. Assume otherwise. Then there is a sequence $z_n=\alpha_n+i\beta_n$ of zeros of $H(z)$ where $\alpha_n<0$, and $\alpha_n\to 0$. If $\{\beta_n\}$ were bounded, then $H(z)$ would have a zero on the imaginary axis. Thus we may assume that $\beta_n\to\infty$ and $\beta_n>0$. From $H(z)=0$

and (2.3)

$$|1+\alpha e^{-z_n}| = \left| \frac{\sum_{j=0}^{m} A_j z_n^j + \sum_{j=0}^{m} B_j z_n^j e^{-z_n}}{z_n^{2m+1}} \right| \leq \sum_{j=0}^{m} \frac{|A_j|}{|z_n|^{2m+1-j}} + \sum_{j=0}^{m} \frac{|B_j| e^{-\alpha_n}}{|z_n|^{2m+1-j}} \to 0 \tag{3.1}$$

Since $\alpha_n \to 0$ and $\beta_n \to \infty$, $|z_n| \to \infty$ and the right hand side of (3.7) tends to 0 as $n \to \infty$. But

$$|1+\alpha e^{-z_n}| = (1+\alpha e^{-\alpha_n}\cos\beta_n)^2 + (\alpha e^{-\alpha_n}\sin\beta_n)^2 \geq (1-|\alpha| e^{-\alpha_n})^2 \to (1-|\alpha|)^2. \tag{3.2}$$

Since $|\alpha| < 1$, we have arrived at a contradiction. When $\alpha = -1$, the implication of Theorem 3.1 can fail. See [23] for an example where all roots lie in the open left half plane but are not bounded away from the imaginary axis, and yet there are unbounded solutions. The authors have not seen such phenomena when $\alpha = 1$.

We now examine asymptotic behavior of the zeros of G. We denote

$$\delta_1 = \cos^{-1}(-\alpha) = \pi - \cos^{-1}(\alpha), \tag{3.3}$$

$$\delta_2 = 2\pi - \cos^{-1}(-\alpha) = \pi + \cos^{-1}(\alpha). \tag{3.4}$$

Lemma 3.3 *Suppose* $-1 < \alpha < 0$ *or* $0 < \alpha < 1$. *For* n *sufficiently large, the interval* $(n\pi, (n+1)\pi)$ *contains exactly one zero* ρ_n *of* G, *and* $\lim_{k\to\infty}[\rho_{2k}]_{2\pi} = \delta_1$ *and* $\lim_{k\to\infty}[\rho_{2k+1}]_{2\pi} = \delta_2$.

Proof. From (2.11), $y = 0$ is zero of G, and

$$\begin{aligned} G(n\pi) = \; & (-1)^{m+n}(n\pi)^{2m+1} + \alpha(-1)^m (n\pi)^{2m+1} - \sum_{j=0}^{m-1} A_{2j+1}(-1)^{j+n}(n\pi)^{2j+1} \\ & - \sum_{j=0}^{m-1} B_{2j+1}(-1)^j (n\pi)^{2j+1}. \end{aligned} \tag{3.5}$$

Since $G(n\pi)$ is a polynomial of degree $2m+1$ in $n\pi$ there can be at most $2m+1$ zeros of G that are multiples of π. All other zeros of G are the roots of the equation

$$w(y) = \eta(y) \tag{3.6}$$

where

$$w(y) = y^2(\cot y + \alpha \csc y) + \sum_{j=0}^{m-1} \frac{A_{2j+1}(-1)^{m+j-1}}{y^{2(m-j)-2}} \cot y + \sum_{j=0}^{m-1} \frac{B_{2j+1}(-1)^{m+j-1}}{y^{2(m-j)-2}} \csc y. \tag{3.7}$$

and

$$\eta(y) = A_{2m} y + \sum_{j=0}^{m-1} \frac{A_{2j}(-1)^{m+j}}{y^{2(m-j)-1}}. \tag{3.8}$$

For n sufficiently large, w resembles the function $\cot y + \alpha \csc y$ on $(n\pi, (n+1)\pi)$ in that $w(n\pi^+) = -w((n+1)\pi^-) = \infty$, and thus $(n\pi, (n+1)\pi)$ contains at least one root of (3.2). Here $w(a^+)$ and $w(a^-)$ denote the right and left hand limits of w at a, respectively. Now (2.11) yields

$$\begin{aligned}\cos y + \alpha = \frac{A_{2m}\sin y}{y} + \sum_{j=0}^{m-1} \frac{(-1)^{m+j} A_{2j}}{y^{2(m-j)+1}} \sin y - \sum_{j=0}^{m-1} \frac{(-1)^{m+j-1} B_{2j+1}}{y^{2(m-j)}} \\ - \sum_{j=0}^{m-1} \frac{(-1)^{m+j-1} A_{2j+1}}{y^{2(m-j)}} \cos y.\end{aligned} \tag{3.9}$$

It follows from (3.9) that

$$\lim_{\substack{G(y)=0 \\ y\to\infty}} \cos y + \alpha = 0. \tag{3.10}$$

If the root ρ_n of G in $(n\pi, (n+1)\pi)$ is unique for n sufficiently large, then (3.10) yields

$$\lim_{k\to\infty} [\rho_{2k}]_{2\pi} = \delta_1$$

and

$$\lim_{k\to\infty} [\rho_{2k+1}]_{2\pi} = \delta_2.$$

Let $\epsilon = min(\pi/2, \delta_1/2)$. For j sufficiently large, it is easily seen that $w'(y) < \zeta'(y)$ for all $y \in (2j\pi + \delta_1 - \epsilon/2, 2j\pi + \delta_1 + \epsilon/2)$, or $y \in (2j\pi + \delta_2 - \epsilon/2, , 2j\pi + \delta_2 + \epsilon/2)$ and uniqueness of the zero ρ_n of G now follows. (A more detailed analysis of this inequality appears in Lemma 3.5 below.)

We now give a far reaching necessary condition for the asymptotic stability of the zero solution of (1.1).

Theorem 3.2 *Assume $-1 < \alpha < 1$. If the zero solution of* (1.1) *is asymptotically stable, then $A_1 + A_0 + B_1 < 0$ and $A_0 + B_0 < 0$.*

Proof. Assume the zero solution of (1.1) is asymptotically stable. From Theorems 2.1 and 2.2 and equations (2.10) and (2.11)

$$\Delta(0) = (A_1 + A_0 + B_1)(A_0 + B_0) > 0. \tag{3.11}$$

It follows from Theorems 2.1-2.3 that G has all real zeros and for k sufficiently large $[-2k\pi, 2k\pi]$ contains precisely $4k + 2m + 1$ zeros of G. Since $y = 0$ is a zero of G and G is odd, $(0, 2k\pi)$ contains precisely $2k + m$ zeros $r_1 < r_2 < \cdots < r_{2k+m}$ of G where k is sufficiently large. By Lemma 3.1, $r_{2k+m} \in ((2k-1)\pi, 2k\pi)$ and $[r_{2k+m}]_{2\pi} \to \delta_2$ as $k \to \infty$. From (2.10) it follows that $F(r_{2k+m})$ has sign $(-1)^m$ for k sufficiently large. By Theorems 2.1 and 2.2, the zeros of F and G interlace and thus the $F(r_j)$

must strictly alternate in sign (where $r_0 = 0$). Thus $(-1)^m F(0)F(r_{2k+m}) > 0$, and since $(-1)^m F(r_{2k+m}) > 0$, $F(0) = -(A_0 + B_0) > 0$. Thus $A_0 + B_0 < 0$, and by (3.8) $A_1 + A_0 + B_1 < 0$. The proof is complete.

Evidently if $A_1 + A_0 + B_1 \geq 0$ or $A_0 + B_0 \geq 0$, then the zero solution of (1.1) is not asymptotically stable. In this paper Z^+ denotes the set of all nonnegative integers.

The following theorem combines Theorems 2.1, 3.1, and 3.2 to obtain a characterization for asymptotic stability of the zero solution of (1.1). It involves rather complex conditions of requiring that G has all real zeros and infinitely many sign conditions on F. Subsequently, we will reduce these to a finite number of conditions.

Theorem 3.3 *The zero solution of* (1.1) *is asymptotically stable if and only if*

1.: $A_0 + B_0 < 0$, $A_1 + A_0 + B_1 < 0$,
2.: *G has all real zeros, and*
3.: $(-1)^n F(r_n) > 0$, $(n = 1, 2, \ldots)$

where $r_1 < r_2 < r_3 < \cdots$ are the positive zeros of G.

Proof. Necessity of 1. and 2. follows from Theorems 3.1, 2.1 and 2.2. Between consecutive zeros of G, G' must properly change sign. Since $G'(0) = -(A_1 + A_0 + B_1) > 0$, $G'(r_n)$ has sign $(-1)^n$ for $n = 1, 2, \ldots$, and now 3. follows from Theorems 2.1 and 2.2. For sufficiency, 1. and 2. yield that $G'(r_n)$ has sign $(-1)^n$ as above for $n = 1, 2, \ldots$. Now 3. yields that $G'(r_n)F(r_n) > 0$ for $n = 1, 2, \ldots$. Sufficiency now follows from the parities of F and G and from Theorems 2.1, 3.1, and 2.2c.

Remark 3.1 We first consider the case of pure delay, i.e. $A_j = 0$, $j = 0, 1, \ldots, 2m$. In this case,

$$G(y) = (-1)^m y^{2m+1} \cos y + \alpha(-1)^m y^{2m+1} - \sum_{j=0}^{m-1} B_{2j+1}(-1)^j y^{2j+1} \tag{3.12}$$

and

$$F(y) = (-1)^{m+1} y^{2m+1} \sin y - \sum_{j=0}^{m} B_{2j}(-1)^j y^{2j}. \tag{3.13}$$

The nonzero zeros of G are the roots of

$$\cos y + \alpha = \zeta(y) \tag{3.14}$$

where

$$\zeta(y) = \sum_{j=0}^{m-1} \frac{B_{2j+1}(-1)^{j+m}}{y^{2(m-j)}}. \tag{3.15}$$

By Theorem 3.2, $B_1 < 0$ is necessary for the zero solution of (1.1) to be asymptotically stable. We assume $B_1 < 0$. Observe that $\lim_{y \to 0^+} \zeta(y) = (-1)^{m+1}\infty$. Let ℓ

be the largest index so that $B_{2\ell+1} \neq 0$, and thus

$$\zeta(y) = \sum_{j=0}^{\ell} \frac{B_{2j+1}(-1)^{j+m}}{y^{2(m-j)}}.$$

If $B_{2\ell+1}(-1)^{\ell+m} > 0$, then ζ is eventually decreasing and convex, and if $B_{2\ell+1}(-1)^{\ell+m} < 0$, then ζ is eventually increasing and concave. In either case, $\lim_{y\to\infty} \zeta(y) = 0$. We obtain a value Y_1 so that ζ is either decreasing and convex or increasing and concave on $[Y_1, \infty)$. It can be seen that if $\zeta'' > 0$ (respectively, $\zeta'' < 0$) in an interval $[Y_1, \infty)$, then $\zeta > 0$ and $\zeta' < 0$ (respectively, $\zeta < 0$ and $\zeta' > 0$) on $[Y_1, \infty)$.

We have

$$\zeta''(y) = \sum_{j=0}^{\ell} \frac{2(m-j)(2m-2j+1)B_{2j+1}(-1)^{j+m}}{y^{2m-2j+2}}, \tag{3.16}$$

and ζ'' is of constant sign on $[Y_1, \infty)$ if

$$|B_{2\ell+1}| > \frac{1}{(m-\ell)(2m-2\ell+1)} \sum_{j=0}^{\ell-1} \frac{(m-j)(2m-2j+1)|B_{2j+1}|}{Y_1^{2(\ell-j)}}. \tag{3.17}$$

Of course, (3.17) only applies when $ell > 1$. If $\ell \geq 1$. If $\ell = 0$, we only need $|B_1| > 0$ (see Theorem 3.2) and we take $Y_1 = 1$. When $\ell > 1$, (3.17) holds if

$$|B_{2\ell+1}| > \frac{1}{(m-\ell)(2m-2\ell+1)} \sum_{j=0}^{\ell-1} \frac{(m-j)(2m-2j+1)|B_{2j+1}|}{Y_1^{2(\ell-j)}}.$$

Further this holds if

$$Y_1 = \max\left(1, \left(\frac{\sum_{j=0}^{\ell-1}(m-j)(2m-2j+1)|B_{2j+1}|}{(m-\ell)(2m-2\ell+1)|B_{2\ell+1}|}\right)^{\frac{1}{2}}\right). \tag{3.18}$$

Figures 1-4 depict equation (3.12) for all combinations of signs of α and $(-1)^{\ell+m}B_{2\ell+1}$ for $y > Y_1$. Two periods of $\cos y + \alpha$ are shown. Selection of a constant Y below is broken into the four cases delineated by these figures. Specifically, solutions of (3.12) are restricted to certain subintervals revealed in the figures.

If $\alpha > 0$ and $(-1)^{\ell+m}B_{2\ell+1} < 0$ or if $\alpha < 0$ and $(-1)^{\ell+m}B_{2\ell+1} > 0$, we select $Y_2 \geq 1$ sufficiently large so that if $y \geq Y_2$, then

$$|\zeta(y)| = |\sum_{j=0}^{m-1} \frac{B_{2j+1}(-1)^{j+m}}{y^{2(m-j)}}| \leq \sum_{j=0}^{\ell} \frac{|B_{2j+1}|}{y^{2(m-j)}} \leq \frac{1}{y^{2(m-\ell)}} \sum_{j=0}^{\ell} |B_{2j+1}| \leq 1 - |\alpha|.$$

That is, we take

$$Y_2 = \max\left(1, \left(\sum_{j=0}^{\ell} |B_{2j+1}|/(1-|\alpha|)\right)^{\frac{1}{2(m-\ell)}}\right). \tag{3.19}$$

We let

$$Y = \max(Y_1, Y_2) \tag{3.20}$$

If $\alpha > 0$ and $(-1)^{\ell+m}B_{2\ell+1} > 0$ or if $\alpha < 0$ and $(-1)^{\ell+m}B_{2\ell+1} < 0$, we select $Y_2 \geq 1$ sufficiently large so that if $y \geq Y_2$, then

$$|\zeta(y)| \leq |\alpha|.$$

As in (3.18), we take

$$Y_2 = \max\left(1, \left(\sum_{j=0}^{\ell} \frac{|B_{2j+1}|}{|\alpha|}\right)^{\frac{1}{2(m-\ell)}}\right). \tag{3.21}$$

In these latter cases we also choose $Y_3 \geq 1$ sufficiently large so that if $y \geq Y_3$, then

$$|\zeta'(y)| \leq \sum_{j=0}^{\ell} \frac{|B_{2j+1}|2(m-j)}{y^{2(m-j)+1}} \leq \frac{1}{y^{2m-2\ell+1}} \sum_{j=0}^{\ell} 2(m-j)|B_{2j+1}| < \sqrt{1-\alpha^2}. \tag{3.22}$$

That is, we take

$$Y_3 = \max\left(1, \left(\sum_{j=0}^{\ell} 2(m-j)|B_{2j+1}|/\sqrt{1-\alpha^2}\right)^{\frac{1}{2m-2\ell+1}}\right). \tag{3.23}$$

Let

$$Y = \max(Y_1, Y_2, Y_3). \tag{3.24}$$

Now select the integer θ so that

$$Y \in (2(\theta-1)\pi, 2\theta\pi]. \tag{3.25}$$

Lemma 3.4 *Suppose that $A_j = 0$, $j = 0, 1, \ldots, 2m$, and $B_1 < 0$. The function G has all real zeros if and only if G has $2\theta + m + 2$ zeros in the interval* $(0, 2(\theta+1)\pi)$.

Proof. From Remark 3.1 using the intermediate value theorem and opposing convexity (on appropriate subsets of $(2(n-3/2)\pi, 2n\pi)$ of $\cos y + \alpha$ and $\zeta(y)$ for or $(-1)^{\ell+m}B_{2\ell+1}\alpha < 0$, it can be seen that (3.12) has precisely two roots in $(2(n-1)\pi, 2n\pi)$ when $n \geq \theta+1$. For $(-1)^{\ell+m}B_{2\ell+1}\alpha > 0$ it can be seen that (3.12) has precisely two roots in $(2(n-1)\pi, 2n\pi)$ when $n \geq \theta+1$ making use of the fact the derivative of $\cos y + \alpha$ dominates $\zeta'(y)$ in the appropriate subset of $(2(n-1)\pi, 2n\pi)$. Now the proof follows from Theorem 2.2 since G has precisely $4k + 2m + 1$ zeros in $[-2k\pi, 2k\pi]$ for k sufficiently large if and only if G has exactly $2\theta + m + 2$ zeros in $(0, 2(\theta+1)\pi]$.

Remark 3.2 In the pure delay case ($A_j = 0$, $j = 0, 1, \ldots, 2m$), we now obtain a stopping criterion for checking condition 3. in Theorem 3.4. If $\alpha > 0$ and $(-1)^{\ell+m}B_{2\ell+1} > 0$ or if $\alpha < 0$ and $(-1)^{\ell+m}B_{2\ell+1} < 0$, then $|\cos r_n| < |\alpha|$ and so $|\sin r_n| > \sqrt{1-\alpha^2}$ where $r_n > 2\pi\theta$ (See Figures 1 and 2). In this case, we choose the first index $N > \theta + 1$ so that

$$\frac{|B_{2j}|}{r_N^{2(m-j)+1}} < \frac{\sqrt{1-\alpha^2}}{m+1} \quad (j = 0, ..., m) \tag{3.26}$$

If $\alpha > 0$ and $(-1)^{\ell+m}B_{2\ell+1} < 0$ or if $\alpha < 0$ and $(-1)^{\ell+m}B_{2\ell+1} > 0$, then for $r_n > 2\pi\theta$, $|\cos r_n|$ decreases to $|\alpha|$ so that $|\sin r_n|$ increases to $\sqrt{1-\alpha^2}$. For this case we choose the first index $N_1 > \theta$ so that (see Figures 3 and 4)

$$|\sin r_{N_1}| > \frac{\sqrt{1-\alpha^2}}{2}.$$

We choose the first index $N_2 > \theta + 1$ so that

$$\frac{|B_{2j}|}{r_{N_2}^{2(m-j)+1}} < \frac{\sqrt{1-\alpha^2}}{2(m+1)} \quad (j = 0, ..., m). \tag{3.27}$$

Let $N = \max(N_1, N_2)$. In either case, we can see from (3.13) that when $n \geq N$

$$sgnF(r_n) = (-1)^{m+1} sgn(\sin r_n). \tag{3.28}$$

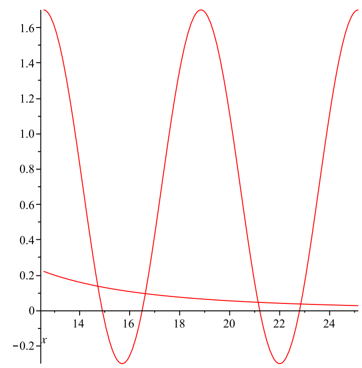


Figure 1: roots of $w = \zeta$, $\alpha > 0$, $(-1)^{\ell+m}B_{2\ell+1} > 0$

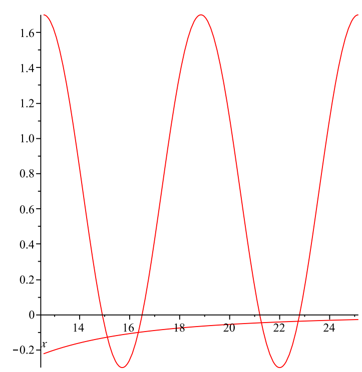


Figure 2: roots of $w = \zeta$, $\alpha < 0$, $(-1)^{\ell+m}B_{2\ell+1} < 0$

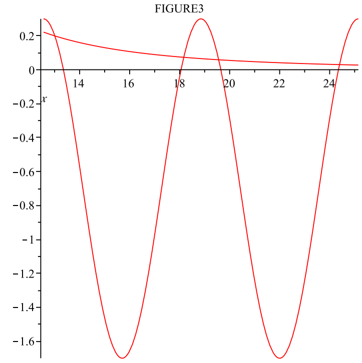


Figure 3: roots of $w = \zeta$, $\alpha < 0$, $(-1)^{\ell+m}B_{2\ell+1} > 0$

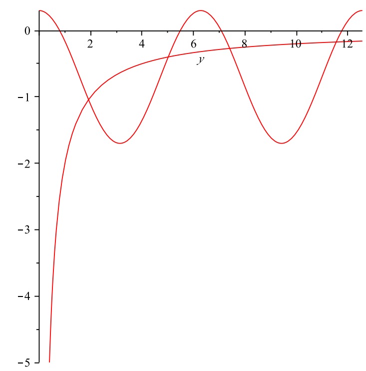


Figure 4: roots of $w = \zeta$, $\alpha > 0$, $(-1)^{\ell+m}B_{2\ell+1} < 0$

If $k > \theta + 1$, then there are $2k + m$ roots in $(0, 2k\pi)$, and the two roots in $(2(k-1)\pi, 2k\pi)$ are r_{2k+m-1} and r_{2k+m}. At these two points the sin function is positive and negative, respectively, and (3.25) yields that $(-1)^n F(r_n) > 0$ for $n = 2k + m - 1$ and $n = 2k + m$.

Theorem 3.5 (Algorithmic Stability Test I) *Suppose that $A_j = 0$, $j = 0, 1, \ldots, 2m$. The zero solution of* (1.1) *is asymptotically stable if and only if*

1.: $B_0 < 0$ and $B_1 < 0$,

2.: *G has $2\theta + m + 2$ zeros in $(0, 2(\theta + 1)\pi)$, and*

3.: $(-1)^n F(r_n) > 0$, $n = 0, 1, 2, \ldots, N$ *where θ is given in* (3.22)*, N is given in Remark* 3.2*, and $r_1 < r_2 < \cdots$ are the positive zeros of G.*

Proof. The proof is revealed in Remarks 3.1 and 3.2.

Examples revealing the utility of this algorithm are given in Section 4.

Our next discussion results in a robust algorithmic stability test that applies to the general case of (1.1). It comes at a cost in that it is not as sharp as the development of Algorithmic Stability Tests I. Particularly, the condition for G to have all real zeros is not as straightforward as Lemma 3.4. In addition, the stopping criteria are not as sharp. Never the less, it can be implemented and applied to the general case.

Lemma 3.3 reveals that for n sufficiently large $(n\pi, (n+1)\pi)$ contains exactly one zero of G and that consecutive zeros in $(2(n-1)\pi, 2n\pi)$ converge to δ_1 and δ_2 modulo 2π, respectively, where δ_1 and δ_2 are defined in (3.3) and (3.4).

Lemma 3.5 *Let $k \in Z^+$. If*

$$2k \geq \max(M_1, M_2, M_3)$$

where M_1, M_2, and M_3 are positive integers defined in (3.29, 3.30)*, and* (3.35) *below, then the interval $[2k\pi, (2k+2)\pi)$ contains exactly two roots of G, one in $(2k\pi + \delta_1/2, 2k\pi + \delta_1/2 + \pi/2)$ and one in $(2k\pi + \delta_2/2 + \pi/2, 2k\pi + \delta_2/2 + \pi)$.*

Proof. Let M_1 be a positive integer such that all zeros of G that are multiples of π are in $(0, M_1\pi)$. Using (3.1) and the same approach in deriving Y_1 in Remark 3.1,

$$M_1 = \left[\!\left[1 + \left(\sum_{j=0}^{m-1} \frac{|A_{2j+1}| + |B_{2j+1}|}{(1-|\alpha|)\pi^{2(m-j)}} \right)^{\frac{1}{2}} \right]\!\right] \tag{3.29}$$

suffices. Here $[\![\ ]\!]$ denotes the greatest integer function.

We now use (3.9) to obtain an integer M_2 so that if $2k \geq M_2$, then G has at least one zero in $(2k\pi + \delta_1/2, 2k\pi + \delta_1/2 + \pi/2)$ and at least one zero in $(2k\pi + \delta_2/2 + \pi/2, 2k\pi + \delta_2/2 + \pi)$ and no zero in $[2k\pi, 2(k+1)\pi]$ outside of these two subintervals. The evaluation of the left side of (3.9) at the endpoints of these intervals are of opposite sign. Let $\mu = \min(|\cos(\delta_1/2) + \alpha|, |\cos(\delta_1/2 + \pi/2) + \alpha|) = \min(|\cos(\delta_2/2 + \pi/2) + \alpha|, |\cos(\delta_2 + \pi) + \alpha)|)$. It can been seen that $\mu = \min(|-\sin\delta_1/2 + \alpha|, |\cos\delta_1/2 + \alpha|) = \min(\alpha + \sqrt{\frac{1-\alpha}{2}}, \sqrt{\frac{1+\alpha}{2}} - \alpha) = \sqrt{\frac{1+|\alpha|}{2}} - |\alpha|$. Observe that $|\cos y + \alpha| \geq \mu$ when y is in $[2k\pi, 2(k+1)\pi)$ and outside of these two subintervals. The desire root location is guaranteed if the right side of (3.9) is less than μ on all of $[2k\pi, 2(k+1)\pi)$. This, in turn, holds if

$$\frac{|A_{2m}|}{y} + \sum_{j=0}^{m-1} \frac{|A_{2j}|}{y^{2(m-j)+1}} + \sum_{j=0}^{m-1} \frac{|B_{2j+1}|}{y^{2(m-j)}} + \sum_{j=0}^{m-1} \frac{|A_{2j+1}|}{y^{2(m-j)}} \leq \mu,$$

which is equivalent to

$$y > \frac{1}{\mu} \Bigg(|A_{2m}| + \sum_{j=0}^{m-1} \frac{|A_{2j}|}{y^{2(m-j)}} + \sum_{j=0}^{m-1} \frac{|B_{2j+1}|}{y^{2(m-j)-1}}) + \sum_{j=0}^{m-1} \frac{|A_{2j+1}|}{y^{2(m-j)-1}} \Bigg).$$

The latter inequality holds for $y \geq n\pi$ if

$$n\pi > \frac{1}{\mu} \left(|A_{2m}| + \sum_{j=0}^{m-1} \frac{|A_{2j}|}{(n\pi)^{2(m-j)}} + \sum_{j=0}^{m-1} \frac{|B_{2j+1}|}{(n\pi)^{2(m-j)-1}} + \sum_{j=0}^{m-1} \frac{|A_{2j+1}|}{(n\pi)^{2(m-j)-1}} \right),$$

or, equivalently,

$$n > \frac{1}{\mu} \left(\frac{|A_{2m}|}{\pi} + \sum_{j=0}^{m} \frac{|A_{2j}|}{\pi^{2(m-j)+1}} + \sum_{j=0}^{m-1} \frac{|B_{2j+1}|}{\pi^{2(m-j)}} + \sum_{j=0}^{m-1} \frac{|A_{2j+1}|}{\pi^{2(m-j)}} \right).$$

We take

$$M_2 = \left[\!\left[1 + \frac{1}{\mu} \left(\frac{|A_{2m}|}{\pi} + \sum_{j=0}^{m-1} \frac{|A_{2j}|\pi + |B_{2j+1}| + |A_{2j+1}|}{\pi^{2(m-j)}} \right) \right]\!\right] \tag{3.30}$$

suffices.

Now we determine M_3 so that if $2k \geq M_3$, then $w - \eta$ is strictly decreasing on $(2k\pi + \delta_1/2, 2k\pi + \delta_1/2 + \pi/2)$ and on $(2k\pi + \delta_2/2 + \pi/2, 2k\pi + \delta_2/2 + \pi)$. Recall that the roots of G that are not multiples of π are solutions of (3.6) (that is, $w(y) = \eta(y)$) where w and η are given in (3.7) and (3.8). Thus if $2k \geq \max(M_1, M_2, M_3)$, then

$(2k\pi+\delta_1/2, 2k\pi+\delta_1/2+\pi/2)$ and $(2k\pi+\delta_2/2+\pi/2, 2k\pi+\delta_2/2+\pi)$ each contain exactly one zero of G. Using (3.7) and (3.8),

$$\begin{aligned} w'(y)-\eta'(y) = &-y^2\csc^2 y-\alpha y^2\csc y\cot y+2y\cot y+2\alpha y\csc y \\ &-\sum_{j=0}^{m-1}\frac{A_{2j+1}(-1)^{m+j-1}(2(m-j)-2)}{y^{2(m-j)-1}}\cot y \\ &-\sum_{j=0}^{m-1}\frac{A_{2j+1}(-1)^{m+j-1}}{y^{2(m-j)-2}}\csc^2 y \\ &-\sum_{j=0}^{m-1}\frac{B_{2j+1}(-1)^{m+j-1}((2(m-j)-2)}{y^{2(m-j)-1}}\csc y \\ &-\sum_{j=0}^{m-1}\frac{B_{2j+1}(-1)^{m+j-1}}{y^{2(m-j)-2}}\csc y\cot y \\ &-A_{2m}-\sum_{j=0}^{m-1}\frac{A_{2j}(-1)^{m+j}(2(m-j)-1)}{y^{2(m-j)}} \end{aligned} \tag{3.31}$$

On $(2k\pi+\delta_1/2, 2k\pi+\delta_1/2+\pi/2)\cup(2k\pi+\delta_2/2+\pi/2, 2k\pi+\delta_2/2+\pi)$

$$|\sin y|>\sqrt{\frac{1-|\alpha|}{2}} \quad \text{and} \quad |\cos y|<\sqrt{\frac{1+|\alpha|}{2}} \tag{3.32}$$

so that

$$1\le|\csc y|<\sqrt{\frac{2}{1-|\alpha|}} \quad \text{and} \quad |\cot y|<\sqrt{\frac{1+|\alpha|}{1-|\alpha|}}. \tag{3.33}$$

Let

$$\begin{aligned} \nu &= \frac{2\left(1-|\alpha|\sqrt{\frac{1+|\alpha|}{2}}\right)}{1-|\alpha|}=\frac{2-|\alpha|\sqrt{2}\sqrt{1+|\alpha|}}{1-|\alpha|}, \\ \nu_1 &= \sqrt{\frac{1+|\alpha|}{1-|\alpha|}} \quad \text{and} \quad \nu_2=\sqrt{\frac{2}{1-|\alpha|}} \end{aligned} \tag{3.34}$$

It is evident that $\nu>0$. For y in this union of intervals,

$$\begin{aligned} w'(y)-\eta'(y)\le &-y^2\nu+2y(\nu_1+|\alpha|\nu_2)+\sum_{j=0}^{m-1}\frac{|A_{2j+1}|(2(m-j)-2)}{y^{2(m-j)-1}}\nu_1 \\ &+\sum_{j=0}^{m-1}\frac{|A_{2j+1}|}{y^{2(m-j)-2}}\nu_2^2+\sum_{j=0}^{m-1}\frac{|B_{2j+1}|(2(m-j)-2)}{y^{2(m-j)-1}}\nu_1\nu_2+ \\ &+\sum_{j=0}^{m-1}\frac{|B_{2j+1}|}{y^{2(m-j)-2}}\nu_2+|A_{2m}|+\sum_{j=0}^{m-1}\frac{|A_{2j}|(2(m-j)-1)}{y^{2(m-j)}}. \end{aligned} \tag{3.35}$$

From (3.35), $w'(y) - \eta'(y) < 0$ if

$$y > \frac{1}{\nu}\Bigg(\frac{2(\nu_1 + |\alpha|\nu_2)}{y} + \sum_{j=0}^{m-1} \frac{|A_{2j+1}|(2(m-j)-2)\nu_1}{y^{2(m-j)}} + \sum_{j=0}^{m-1} \frac{|A_{2j+1}|\nu_2^2}{y^{2m-j)+1}}$$
$$+ \sum_{j=0}^{m-1} \frac{|B_{2j+1}(2(m-j)-2)\nu_1\nu_2}{y^{2(m-j)}} + \sum_{j=0}^{m-1} \frac{|B_{2j+1}|\nu_2}{y^{2(m-j)-1}}$$
$$+ \frac{|A_{2m}|}{y} + \sum_{j=0}^{m} \frac{|A_{2j}|(2(m-j)-1)}{y^{2(m-j)+1}}\Bigg). \tag{3.36}$$

The same inequality holds for $y \in (2k\pi + \delta_2/2 + \pi/2, 2k\pi + \delta_2/2 + \pi]$. As above the latter inequality holds for $y \geq n\pi$ if

$$n\pi > \frac{1}{\nu}\Bigg(\frac{2(\nu_1 + |\alpha|\nu_2)}{\pi} + \sum_{j=0}^{m-1}|A_{2j+1}|\Bigg(\frac{2(m-j-1)\nu_1}{\pi^{2(m-j)}} + \frac{\nu_2^2}{\pi^{2(m-j)-1}}\Bigg)$$
$$+ \sum_{j=0}^{m-1}|B_{2j+1}|\Bigg(\frac{2(m-j-1)\nu_1\nu_2}{\pi^{2(m-j)}} + \frac{\nu_2}{\pi^{2(m-j)-1}}\Bigg)$$
$$\frac{|A_{2m}|}{\pi} + \sum_{j=0}^{m-1}\frac{|A_{2j}|(2(m-j)+1)}{\pi^{2(m-j)+1}}\Bigg). \tag{3.37}$$

As above,

$$M_3 = \Bigg[\!\!\Bigg[1 + \frac{1}{\nu}\Bigg(\frac{2(\nu_1 + |\alpha|\nu_2)}{\pi^2} + \sum_{j=0}^{m-1}|A_{2j+1}|\Bigg(\frac{2(m-j-1)\nu_1}{\pi^{2(m-j)+1}} + \frac{\nu_2^2}{\pi^{2(m-j)}}\Bigg)$$
$$+ \sum_{j=0}^{m-1}|B_{2j+1}|\Bigg(\frac{2(m-j-1)\nu_1\nu_2}{\pi^{2(m-j)+1}} + \frac{\nu_2}{\pi^{2(m-j)}}\Bigg)$$
$$\frac{|A_{2m}|}{\pi^2} + \sum_{j=0}^{m-1}\frac{|A_{2j}|(2(m-j)+1)}{\pi^{2(m-j+1)}}\Bigg)\Bigg]\!\!\Bigg] \tag{3.38}$$

suffices.

The proof is now complete.

Remark 3.3 Recall that G has all real zeros if and only if G has $4k+2m+1$ zeros in $(-2k\pi, 2k\pi)$ (or, equivalently, $2k+m$ zeros in $(0, 2k\pi)$) for all sufficiently large k. From Lemma 3.3, it follows that if G has all real zeros and m is even, $[\rho_{2j}]_{2\pi} \to \delta_1$ and $[\rho_{2j+1}]_{2\pi} \to \delta_2$ when G has all real zeros. Also if m is odd $[\rho_{2j+1}]_{2\pi}] \to \delta_1$ and $[\rho_{2j}]_{2\pi} \to \delta_2$ Here $\rho_1 < \rho_2 < \rho_3 < \cdots$ are the positive zeros of G, and $[a]_{2\pi}$ is the unique real number in $[0, 2\pi]$ for which $a - [a]_{2\pi}$ is an integer.

If m is odd, (2.10) yields

$$F(\rho_{2j+1}) = \qquad \rho_{2j+1}^{2m+1}\Bigg(\sin\rho_{2j+1} - \sum_{\ell=0}^{m}\frac{A_{2\ell}(-1)^{\ell}\cos\rho_{2j+1}}{\rho_{2j+1}^{2(m-\ell)+1}}$$

$$-\sum_{\ell=0}^{m-1}\frac{A_{2\ell+1}(-1)^{\ell+1}\sin\rho_{2j+1}}{\rho_{2j+1}^{2(m-\ell)}}-\sum_{\ell=0}^{m}\frac{B_{2\ell}(-1)^{\ell}}{\rho_{2j+1}^{2(m-\ell)+1}}\Bigg) \tag{3.39}$$

$$\begin{aligned} F(\rho_{2j}) = \;& \rho_{2j}^{2m+1}\Bigg(\sin\rho_{2j}-\sum_{\ell=0}^{m}\frac{A_{2\ell}(-1)^{\ell}\cos\rho_{2j}}{r_{2j}^{2(m-\ell)+1}} \\ & -\sum_{j=0}^{m-1}\frac{A_{2\ell+1}(-1)^{\ell+1}\sin\rho_{2j}}{\rho_{2j}^{2(m-\ell)}}-\sum_{j=0}^{m}\frac{B_{2\ell}(-1)^{\ell}}{\rho_{2j}^{2(m-\ell)+1}}\Bigg), \end{aligned} \tag{3.40}$$

and if m is even

$$\begin{aligned} F(\rho_{2j+1}) = \;& \rho_{2j+1}^{2m+1}\Bigg(-\sin\rho_{2j+1}-\sum_{\ell=0}^{m}\frac{A_{2\ell}(-1)^{\ell}\cos\rho_{2j+1}}{r_{2j+1}^{2(m-\ell)+1}} \\ & -\sum_{\ell=0}^{m-1}\frac{A_{2\ell+1}(-1)^{\ell+1}\sin\rho_{2j+1}}{r_{2j+1}^{2(m-\ell)}}-\sum_{\ell=0}^{m}\frac{B_{2\ell}(-1)^{\ell}}{\rho_{2j+1}^{2(m-\ell)+1}}\Bigg), \end{aligned} \tag{3.41}$$

$$\begin{aligned} F(\rho_{2j}) = \;& \rho_{2j}^{2m+1}\Bigg(-\sin\rho_{2j}-\sum_{\ell=0}^{m}\frac{A_{2\ell}(-1)^{\ell}\cos\rho_{2j}}{r_{2j}^{2(m-\ell)+1}} \\ & -\sum_{\ell=0}^{m-1}\frac{A_{2\ell+1}(-1)^{\ell+1}\sin\rho_{2j}}{\rho_{2j}^{2(m-\ell)}}-\sum_{\ell=0}^{m}\frac{B_{2\ell}(-1)^{\ell}}{\rho_{2j}^{2(m-\ell)+1}}\Bigg). \end{aligned} \tag{3.42}$$

As in the proof of Lemma 3.5, for y in either of the intervals $(2k\pi+\delta_1/2, 2k\pi+\delta_1/2+\pi/2)$ and $(2k\pi+\delta_2/2+\pi/2, 2k\pi+\delta_2/2+\pi)$,

$$|\sin y| > \sqrt{\frac{1-|\alpha|}{2}}. \tag{3.43}$$

We further choose the first positive integers N_1, N_2, and N_3 such that

$$\sum_{\ell=0}^{m}\frac{|A_{2\ell}|}{(N_1\pi)^{2(m-\ell)+1}} < \frac{1}{3}\sqrt{\frac{1-|\alpha|}{2}}, \tag{3.44}$$

$$\sum_{\ell=0}^{m-1}\frac{|A_{2\ell+1}|}{(N_2\pi)^{2(m-\ell)}} < \frac{1}{3}\sqrt{\frac{1-|\alpha|}{2}}, \tag{3.45}$$

$$\sum_{\ell=0}^{m}\frac{|B_{2\ell}|}{(N_3\pi)^{2(m-\ell)+1}} < \frac{1}{3}\sqrt{\frac{1-|\alpha|}{2}}. \tag{3.46}$$

(We could provide explicit expression for N_1, N_2, and N_3, but they would be overestimates.)

For m even we have a similar result

Theorem 3.6 General Algorithmic Stability Test *Let $2N$ be the smallest even integer greater than or equal to* $\max\{N_1, N_2, N_3, M_1, M_2, M_3\}$ *The zero solution of (1.1) is asymptotically stable if and only if*

: 1. $A_0 + B_0 < 0$, $A_1 + A_0 + B_1 < 0$,

: 2. *G has $2N+m$ distinct zeros $r_1 < r_2 < \cdots < r_{2N+m}$* in $(0, 2N\pi)$, and
: 3. $(-1)^n F(\rho_n) > 0$ $(n = 1, \ldots, 2N+m)$.

Proof. Necessity follows immediately from Theorem 3.3 and Theorem 2.3 and the fact that each interval $(n\pi, (n+1)\pi)$ contains exactly one zero of G for $n \geq 2N$. For sufficiency, Lemma 3.4 yields that G has all real and distinct zeros. Now let $r_{2N+m+1} < r_{2N+m+2} < \cdots$ be the remaining positive zeros of G. Lemma 3.4 and Remark 3.3 now imply that $(-1)^n F(r_n) > 0$ for all $n > 2N+m$. Sufficiency now follows from Theorem 3.6.

4. EXAMPLES

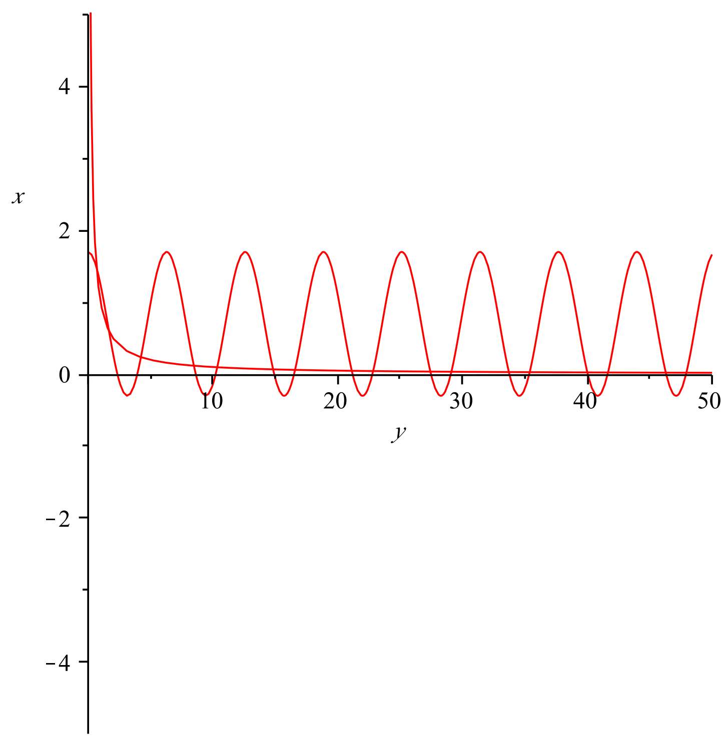
Example 4.1 Consider equation (1.1) with $m = 1$, $a_0 = a_1 = a_2 = 0$, i.e.

$$y^{(3)}(t) + \alpha y^{(3)}(t-\tau) = b_0 y(t-\tau) + b_1 y\prime(t-\tau) + b_2 y''(t-\tau), \tag{4.1}$$

where

$$B_0 = b_0\tau^3 = -0.6,\ B_1 = b_1\tau^2 = -1.0,\ B_2 = b_\tau = -1.8, \quad \text{and } \alpha = 0.7. \tag{4.2}$$

Since $B_0 = -0.6 < 0$ and $B_1 = -1.0 < 0$, the necessary conditions of Theorem 3.2 are satisfied.



The roots of G $(w(y) = \zeta(y)\ )$

In this example, we apply Algorithmic Stability Test I. From (3.17), (3.20), and (3.23), $Y_1 = 1$, $Y_2 \doteq \sqrt{1/0.7} = 1.19522$, and $Y_3 \doteq 1.409$. From (3.24), and (3.25), $Y \doteq 1.409$ and $\theta = 1$. Condition 1. of Theorem 3.5 is satisfied since $B_0 = -0.6 < 0$, $B_1 = -1.0 < 0$. Also, G has five zeros in $(0, 4\pi)$, which are $r_1 \doteq .6754939316$,

$r_2 \doteq 1.6754939316$, $r_3 \doteq 4.230322227$, $r_4 \doteq 8.475177921$, and $r_5 \doteq 10.34786440$. Figure 1 below shows the roots of $w = \zeta$ in $(0, 50)$. Simple calculations yield that $N = 3$. For Condition 3. of Theorem 3.5 we found $F(0) = -B_0 = 0.6 > 0$, and $F(r_1) \doteq -0.0285994403$, $F(r_2) \doteq 0.224720799$, $F(r_3) \doteq -98.68903150$. We also found $F(r_4) \doteq 366.3420167$, and $F(r_5) \doteq -1075.760380$. By Algorithmic Stability Test I (Theorem 3.5) the zero solution of (4.1) is asymptotically stable.

In this example we also examine the effect of the neutrality. We take the left side of equation (4.1) to be $(1+\alpha)y^{(3)}(t)$ while the right side of the equation stays the same, and equation (4.1) turn into a non-neutral equation. We examine the zeros of G for the non-neutral equation. We found that $r_1 = 0.359544451$, $F(r_1) = 0.2324182884 > 0$, and $F(0) = 0.3529411765 > 0$. Here r_1 is the first positive zero of G. Since $F(r_1) > 0$ and $F(0) > 0$ by Theorem 2.2 the zero solution of the non-neutral equation is not asymptotically stable. In this case neutrality can stabilize the solution. Had we chosen to drop the delay term in (4.1) for comparison sake, the zero solution in the non-neutral equation is also not asymptotically stable. For examples where delay stabilized solutions see [1,2]. It would be interesting to find sufficient conditions when delay and/or neutrality has a stabilizing effect.

Example 4.2 Consider (1.1) with $m = 3$ and $a_0 = a_1 = a_2 = a_3 = a_4 = 0$, i.e.

$$\begin{aligned} y^{(7)}(t) + \alpha y^{(7)}(t-\tau) = \; & b_0 y(t-\tau) + b_1 y(t-\tau) + b_1 y'(t-\tau) + b_2 y''(t-\tau) + b_3 y'''(t-\tau) \\ & + b_4 y^{(4)}(t-\tau) + b_5 y^{(5)}(t-\tau) + b_6 y^{(6)}(t-\tau) \end{aligned} \tag{4.3}$$

where $B_0 = -1$, $B_1 = -0.1$, $B_2 = -5$, $B_3 = -0.5$, $B_4 = -0.5$, $B_5 = -0.75$, $B_6 = -0.1$, and $\alpha = 0.5$.

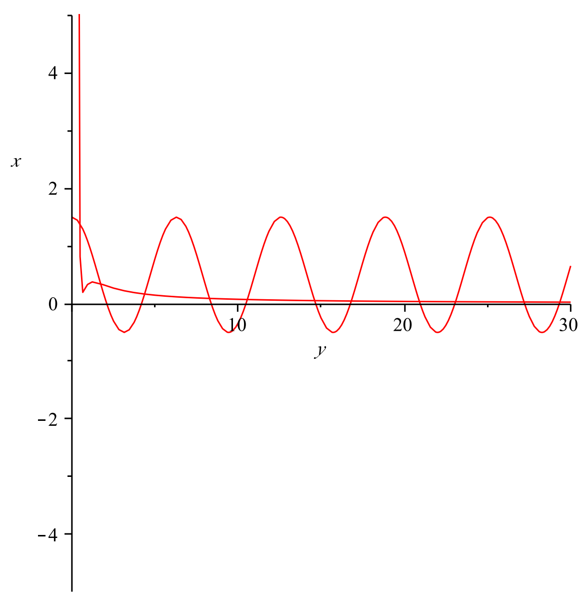


Figure 2

In this example we apply Algorithmic Stability Test I. Here $Y_1 \doteq 1.776388$, $Y_2 \doteq 1.643316$ and $Y_3 = 2$, and thus $\theta = 1$. Here $B_0 = -1 < 0$, $B_1 = -0.1 < 0$, and Condition 1. of Theorem 3.5 is satsified. For the necessary conditions of Lemma 3.2 to be satisfied, G must have seven zeros in $(0, 4\pi]$. However, G has only 5 real zeros in $(0, 4\pi]$ (see Figure 2 above), and therefore the zero solution of (4.2) is not asymptotically stable.

In the table below we list the zeros of G, r_j, and the values of $F(r_j)$ as well as $j_{2\ell+1} = \delta_1 + 2\ell\pi$ and $j_{2\ell} = \delta_2 + 2\ell\pi$.

Table I

$r_1 = .4579295023$	$F(r_1) = 0.025565527$	
$r_2 = 1.7278059166$	$F(r_2) = 33.27245293$	
$r_3 = 4.371467592$	$F(r_3) = -29360.46730$	
$r_4 = 8.276724857$	$F(r_4) = 2.396404151 \times 10^6$	$j_3 = 8.377580410$
$r_5 = 10.55180474$	$F(r_5) = -1.328590735 \times 10^7$	$j_4 = 10.47197551$
$r_6 = 14.60258862$	$F(r_6) = 1.255746270 \times 10^8$	$j_5 = 14.66076572$
$r_7 = 17.28120581$	$F(r_7) = -3.392953508 \times 10^8$	$j_6 = 16.75516083$
$r_8 = 20.427766131$	$F(r_8) = 1.536210457 \times 10^9$	$j_7 = 20.94395102$
$r_9 = 23.56439208$	$F(r_9) = -3.094519561 \times 10^9$	$j_8 = 23.03834613$
$r_{10} = 26.70869065$	$F(r_{10}) = 9.656954062 \times 10^9$	$j_9 = 27.22713633$
$r_{11} = 29.84736057$	$F(r_{11}) = -1.658144432 \times 10^{10}$	$j_{10} = 29.32153144$
$r_{12} = 32.99067642$	$F(r_{12}) = 4.132616518 \times 10^{10}$	$j_{11} = 33.51032164$
$r_{13} = 36.13031875$	$F(r_{13}) = -6.418103666 \times 10^{10}$	$j_{12} = 35.60471675$
$r_{14} = 39.27310485$	$F(r_{14}) = 1.375970956 \times 10^{11}$	$J_{13} = 39.79350695$
$r_{15} = 42.41330367$	$F(r_{15}) = -1.994559795 \times 10^{11}$	$j_{14} = 41.88790206$

Notice that in this example interlacing holds, $m = 3$ is odd and $[r_{2j}]_{2\pi} \to \delta_1$ and $[r_{2j+1}]_{2\pi} \to \delta_2$ as indicated in Remark 3.2. In the next examples we consider the general case with $m = 1$ and $m = 2$.

Example 4.3 Consider equation (1.1) with $m = 1$,

$$y^{(3)}(t) + \alpha y^{(3)}(t-\tau) = a_0 y(t) + a_1 y'(t) + a_2 y'' + b_0 y(t-\tau) + b_1 y'(t-\tau) + b_2 y''(t-\tau), \tag{4.4}$$

where

$$B_0 = b_0\tau^3 = -0.6,\ B_1 = b_1\tau^2 = -1.0,\ B_2 = b_\tau = -1.8, \quad \text{and } \alpha = 0.5. \tag{4.5}$$

$$A_0 = a_0\tau^3 = -0.1,\ A_1 = a_1\tau^2 = -0.5,\ A_2 = a_\tau = -2.0. \tag{4.6}$$

Since $A_0 + B_0 = -0.7 < 0$ and $A_1 + A_0 + B_1 = -1.6 < 0$, the necessary conditions of Theorem 3.2 are satisfied. In this example, we apply the General Algorithmic Stability Test (Theorem 3.6). From (3.44-3.45) simple calculations yield that $N_1 = 6$, $N_2 = 1$ and $N_3 = 1$. From (3.29), (3.30), and (3.38), $M_1 = 2$, $M_2 = 4$, and $M_3 = 2$. Thus $2N = 6$. Condition 2. of Theorem 3.6 is satisfied since G has seven zeros in $(0, 6\pi)$. The zeros in $(0, 6\pi)$ are $r_1 \doteq .9097378425$, $r_2 \doteq 2.027759247$, $r_3 \doteq 4.234896050$, $r_4 \doteq 8.383600807$, $r_5 \doteq 10.48518367$, $r_6 \doteq 14.66567268$, and $r_7 \doteq 16.76256628$. For Condition 3. of Theorem 3.6 we found $F(0) = -B_0 =$

$0.6 > 0$, and $F(r_1) \doteq -1.609304304$, $F(r_2) \doteq 3.355507965$, $F(r_3) \doteq -80.81862227$, $F(r_4) \doteq 449.9490905$, $F(r_5) \doteq -1091.170401$, $F(r_6) \doteq 2547.927736$ and $F(r_7) \doteq -4316.872095$. By the General Algorithmic Stability Test (Theorem 3.6) the zero solution of (4.3) is asymptotically stable.

Example 4.4 Consider equation (1.1) with $m = 2$, and $\alpha = -0.5$,

$$(4.7)\ y^{(5)}(t) + \alpha y^{(5)}(t-\tau) = a_0 y(t) + a_1 y'(t) + a_2 y''(t) + a_3 y^{(3)}(t) + a_4 y^{(4)}(t)$$
$$+ b_0 y(t-\tau) + b_1 y'(t-\tau) + b_2 y''(t-\tau) + b_3 y^{(3)}(t-\tau) + b_4 y^{(4)}(t-\tau),$$

where

(4.8)
$$B_0 = b_0\tau^5 = -2,\ B_1 = b_1\tau^4 = -1.0,\ B_2 = b_2\tau^3 = 1,\ B_3 = b_3\tau^2 = -3.0\ B_4 = b_4 = 3.0\tau$$

(4.9)
$$A_0 = a_0\tau^5 = -1.0,\ A_1 = a_1\tau^4 = -3.0,\ A_2 = a_\tau^3 = 0,\ A_3 = a_3\tau^2 = 2.0,\ A_4 = a_4\tau = 5.0$$

Since $A_0 + B_0 = -3 < 0$ and $A_1 + A_0 + B_1 = -5 < 0$, the necessary conditions of Theorem 3.2 are satisfied. In this example, we apply the General Algorithmic Stability Test (Theorem 3.6) . From (3.44-3.45) simple calculations yield that $N_1 = 10$, $N_2 = 2$ and $N_3 = 6$. From (3.29), (3.30), and (3.38), $M_1 = 2$, $M_2 = 7$, and $M_3 = 9$. Thus $2N = 10$. Condition 2. of Theorem 3.6 requires that G has 12 distinct zeros in $(0, 10\pi)$. However G has only 10 zeros in $(0, 10\pi)$, and therefore Condition 2. fails, and the zero solution of (4.4) is not asymptotically stable.

In Table II below several of the zeros of G and the values of F at those zeros are listed. This gives a glimpse of the behavior of the zeros of G

Table II

$r_1 = .7547435366$	$F(r_1) = -1.165341368$	
$r_2 = 4.331028041$	$F(r_2) = 1196.878949$	
$r_3 = 6.782099205$	$F(r_3) = -22761.16415$	
$r_4 = 11.06830728$	$F(r_4) = 1.180597 \times 10^5$	$j_3 = 8.377580410$
$r_5 = 13.28129758$	$F(r_5) = -4.846455503 \times 10^5$	$j_4 = 10.47197551$
$r_6 = 17.51126300$	$F(r_6) = 1.22679629 \times 10^6$	$j_5 = 14.66076572$
$r_7 = 19.66080574$	$F(r_7) = -3.103639402 \times 10^6$	$j_6 = 16.75516083$
$r_8 = 23.87156246$	$F(r_8) = 5.940874433 \times 10^6$	$j_7 = 20.94395102$
$r_9 = 25.99745791$	$F(r_9) = -1.191474658 \times 10^7$	$j_8 = 23.03834613$
$r_{10} = 30.19978300$	$F(r_{10}) = 1.966969751 \times 10^7$	$j_9 = 27.22713633$
$r_{11} = 32.31448044$	$F(r_{11}) = -3.428782565 \times 10^{17}$	$j_{10} = 29.32153144$
$r_{12} = 36.51241587$	$F(r_{12}) = 5.15909466 \times 10^7$	$j_{11} = 33.510332164$
$r_{13} = 38.62096342$	$F(r_{13}) = -8.194528127 \times 10^7$	$j_{12} = 35.60471675$
$r_{14} = 42.81633650$	$F(r_{14}) = 1.157095837 \times 10^8$	$j_{13} = 39.79350695$
$r_{15} = 44.92115307$	$F(r_{15}) = --1.719797454 \times 10^8$	$j_{14} = 41.88790206$

While trying to build examples we noticed that it is extremely difficult to come up with the zero solution being asymptotically stable when the order gets higher. Perhaps, there is a physical interpretation to this phenomenon which we do not understand. We leave it open to the reader to come up with more examples with odd higher order with many parameters and some physical interpretation of the difficulties of coming up with zero asymptotic solution of higher order delay differential equations.

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