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NONDECREASING SOLUTIONS OF A FRACTIONAL QUADRATIC INTEGRAL EQUATION OF URYSOHN-VOLTERRA TYPE

MOHAMED ABDALLA DARWISH

Department of Mathematics, Sciences Faculty for Girls
King Abdulaziz University, Jeddah, Saudi Arabia
E-mail: darwishma@yahoo.com
and
Department of Mathematics, Faculty of Science, Damanhour University
22511 Damanhour, Egypt

ABSTRACT. In this paper we study a very general quadratic integral equation of fractional order. We show that the quadratic integral equations of fractional orders has at least one monotonic solution in the Banach space of all real functions defined and continuous on a bounded and closed interval. The concept of a measure of noncompactness related to monotonicity, introduced by J. Banaś and L. Olszowy, and a fixed point theorem due to Darbo are the main tools in carrying out our proof. In fact we generalize, improve the results of the paper [M.A. Darwish, On quadratic integral equation of fractional orders, J. Math. Anal. Appl. 311 (2005), 112–119]. Also, we extend and generalize the results of the paper [J. Banaś and B. Rzepka, Monotonic solutions of a quadratic integral equation of fractional order, J. Math. Anal. Appl. 332 (2007), 1370–1378].

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1. INTRODUCTION

In this paper we study the fractional quadratic integral equation of Urysohn-Volterra type

$$y(t) = g(t, y(t)) + \frac{f(t, y(t))}{\Gamma(\beta)} \int_0^t \frac{v(t, s, (Hy)(s))}{(t-s)^{1-\beta}}\, ds,\ t \in J = [0, 1],\ 0 < \beta < 1. \tag{1.1}$$

Here, $f : J \times \mathbb{R} \to \mathbb{R}$, $g : J \times \mathbb{R} \to \mathbb{R}$ and $v : J \times J \times \mathbb{R} \to \mathbb{R}$ are given functions and $H : C(J) \to C(J)$ is an operator which satisfy special assumptions, see Section 3. Let us recall that the functions $f = f(t, y)$ and $g = g(t, y)$ involve in Eq. (1.1) generate the superposition operators F and G, respectively, defined by

$$(Fy)(t) = f(t, y(t)), \text{ and } (Gy)(t) = g(t, y(t)), \tag{1.2}$$

where $y = y(t)$ is an arbitrary function defined on J, see [1].

We remark that:

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- If $g(t,y) = p(t)$ and $Hy = y$ in Eq. (1.1) then we have an equation studied by Banaś and O'Regan in [10].
- If $g(t,y) = a(t)$, $f(t,y) = y$, $v(t,s,z) = u(s,z)$ and $Hy = y$ in Eq. (1.1) then we have an equation studied by Darwish in [22].
- If $g(t,y) = a(t)$, $v(t,s,z) = u(s,z)$ and $Hy = y$ in Eq. (1.1) then we have an equation studied by Banaś and Rzepka in [9].
- If $g(t,y) = a(t)$, $v(t,s,z) = b(t,s)z$ in Eq. (1.1) then we have an equation studied by Banaś and Rzepka in [11].

Consider the case $\beta = 1$. Let $g(t,y) = h(t)$, $f(t,y) = -y$, $Hy = y$ and $v(t,s,u) = k(t,s)u$, Eq. (1.1) takes the form

$$y(t) + y(t)\int_0^t k(t,s)\, y(s)\, ds = h(t),\ t \in J. \tag{1.3}$$

The nonlinear integral equation (1.3) is a generalization of a Volterra counterpart of a famous equation in the transport theory, the so-called Chandrasekhar $H-$equation in which t ranges from 0 to 1, $h(t) = 1$, y must be identified with the $H-$function, and

$$k(t,s) = -\frac{t\phi(s)}{t+s}$$

for a nonnegative characteristic function ϕ, see [15, 20, 36, 38, 46]. Also, Eq. (1.1) includes several integral equations of Volterra and Uryshon-Volterra types studied earlier as special cases, we refer to [14, 21, 40, 42, 47, 48] and references therein.

Quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport, and in the traffic theory. For more details, we refer to [12, 13, 15, 20, 31, 34, 36] and the references therein.

In the last 35 year or so, many authors have studied the existence of solutions for several classes of nonlinear quadratic integral equations with nonsingular kernels. For example, Argyros [2], Banaś *et al.* [4, 6, 8], Banaś and Martinon [7], Caballero *et al.* [16, 17, 18, 19], Darwish [23], Darwish and Ntouyas [29], Hu and Yan [35], Leggett [38], Liu and Kang [39], Stuart [45] and Spiga *et al.* [46].

More recently, following the appearance of the paper [22], there has been significant interest in the study of the existence of solutions for singular quadratic integral equations or fractional quadratic integral equations, see [9, 10, 11, 24, 25, 26, 27, 28, 30].

The aim of this paper is to establish simple criteria for the existence of monotone solutions of Eq. (1.1). The concept of measure of noncompactness related to monotonicity, introduced by J. Banaś and L. Olszowy [5], and a fixed point theorem due to Darbo are the main tools in carrying out our proof.

2. AUXILIARY FACTS AND RESULTS

This section collects some definitions and results which will be needed further on.

First, we recall the definition of the Riemann-Liouville fractional integral, see [33, 37, 41, 43, 44].

Definition 2.1. Let $f \in L_1(a,b)$, $0 \leq a < b < \infty$, and let $\beta > 0$ be a real number. The Riemann-Liouville fractional integral of order β of the function $f(t)$ is defined by

$$I^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{f(s)}{(t-s)^{1-\beta}}\, ds,\ a < t < b.$$

Now, let us assume that $(E, \|.\|)$ is a real infinite dimensional Banach space with zero element θ. Let $B(y,r)$ denote the closed ball centered at y and with radius r. The symbol B_r stands for the ball $B(\theta, r)$.

If Y is a subset of E, then $\bar{Y}$ and $ConvY$ denote the closure and convex closure of Y, respectively. Moreover, we denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

Next we give the concept of a measure of noncompactness [3]:

Definition 2.2. A mapping $\mu : \mathcal{M}_E \to [0, +\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1) The family $\ker\mu = \{Y \in \mathcal{M}_E : \mu(Y) = 0\}$ is nonempty and $\ker\mu \subset \mathcal{N}_E$.
2) $Y \subset X \Rightarrow \mu(Y) \leq \mu(X)$.
3) $\mu(\bar{Y}) = \mu(ConvY) = \mu(Y)$.
4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\, \mu(X) + (1-\lambda)\, \mu(Y)$ for $0 \leq \lambda \leq 1$.
5) If $Y_n \in \mathcal{M}_E$, $Y_n = \bar{Y}_n$, $Y_{n+1} \subset Y_n$ for $n = 1,\ 2,\ 3,\ ...$ and $\lim_{n\to\infty} \mu(Y_n) = 0$ then $\cap_{n=1}^{\infty} Y_n \neq \phi$.

We will work in the Banach space $C(J)$ consisting of all real functions defined and continuous on $J = [0,1]$. The space $C(J)$ is equipped with the standard norm

$$\|y\| = \max\{|y(t)| : t \in J\}.$$

Now, we consider the construction of the measure of noncompactness which will be used in the next section, see [3, 7].

Let us fix a nonempty and bounded subset Y of $C(J)$. For $y \in Y$ and $\varepsilon \geq 0$ denoted by $\omega(y, \varepsilon)$, the modulus of continuity of the function y, i.e.,

$$\omega(y,\varepsilon) = \sup\{|y(t) - y(s)| : t,\ s \in J,\ |t-s| \leq \varepsilon\}.$$

Further, let us put

$$\omega(Y,\varepsilon) = \sup\{\omega(y,\varepsilon) : y \in Y\}$$

and

$$\omega_0(Y) = \lim_{\varepsilon \to 0} \omega(Y, \varepsilon).$$

Define

$$d(y) = \sup\{|y(s) - y(t)| - [y(s) - y(t)] : t,\ s \in J,\ t \le s\}$$

and

$$d(Y) = \sup\{d(y) : y \in Y\}.$$

All functions belonging to Y are nondecreasing on J if and only if $d(Y) = 0$.

Now, let us define the function μ on the family $\mathcal{M}_{C(J)}$ by the formula

$$\mu(Y) = \omega_0(Y) + d(Y).$$

The function μ is a measure of noncompactness in the space $C(J)$.

We will make use of the following fixed point theorem due to Darbo [32]. To quote this theorem, we need the following definition.

Definition 2.3. Let M be a nonempty subset of a Banach space E and let $\mathcal{P} : M \to E$ be a continuous operator which transforms bounded sets onto bounded ones. We say that $\mathcal{P}$ satisfies the Darbo condition (with a constant $k \ge 0$) with respect to a measure of noncompactness μ if for any bounded subset Y of M we have

$$\mu(\mathcal{P}Y) \le k\, \mu(Y).$$

If $\mathcal{P}$ satisfies the Darbo condition with $k < 1$ then it is called a contraction operator with respect to μ.

Theorem 2.4. *Let Q be a nonempty, bounded, closed and convex subset of the space E and let*

$$\mathcal{P} : Q \to Q$$

be a contraction with respect to the measure of noncompactness μ.

Then $\mathcal{P}$ has a fixed point in the set Q.

Remark 2.5. Under the assumptions of the above theorem it can be shown that the set Fix$\mathcal{P}$ of fixed points of $\mathcal{P}$ belonging to Q is an element of kerμ.

3. MAIN THEOREM

In this section, we will study Eq. (1.1) assuming that the following assumptions are satisfied:

a_1) $g : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists nonnegative constant a such that

$$|g(t, y) - g(t, x)| \le a|y - x|$$

for all $t \in J$ and $x,\ y \in \mathbb{R}$. Moreover $g : J \times \mathbb{R}_+ \to \mathbb{R}_+$.

a_2) The superposition operator G satisfies for any nonnegative function y the condition

$$d(Gy) \leq ad(y),$$

where a is the same constant as in a_1).

a_3) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists nonnegative constant c such that

$$|f(t, y) - f(t, x)| \leq c|y - x|$$

for all $t \in J$ and $x,\ y \in \mathbb{R}$. Moreover $f : J \times \mathbb{R}_+ \to \mathbb{R}_+$.

a_4) The superposition operator F satisfies for any nonnegative function y the condition

$$d(Fy) \leq cd(y),$$

where c is the same constant as in a_3).

a_5) The operator H maps continuously the space $C(J)$ into itself and there exists a nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|Hy\| \leq \psi(\|y\|) \text{ for any } y \in C(J).$$

Moreover, for every function $y \in C(J)$ which is nonnegative on J, the function Hy is nonnegative and nondecreasing on J.

a_6) $v : J \times J \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $v : J \times J \times \mathbb{R}_+ \to \mathbb{R}_+$ and $v(t, s, u)$ is nondecreasing with respect to each variable $t,\ s$ and u, separately. Moreover, there exists a nondecreasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|v(t, s, u)| \leq \Psi(|u|)$$

for all $t,\ s \in J$ and for all $u \in \mathbb{R}$.

a_7) The inequality

$$(ar + g^*)\Gamma(\beta + 1) + (cr + f^*)\Psi(\psi(r)) \leq r\ \Gamma(\beta + 1) \tag{3.1}$$

has a positive solution r_0 such that $a\Gamma(\beta + 1) + c\Psi((\psi(r_0)) < \Gamma(\beta + 1)$, where $f^* = \max_{0 \leq t \leq 1} f(t, 0)$ and $g^* = \max_{0 \leq t \leq 1} g(t, 0)$.

Now, we are in a position to state and prove our main result in the paper.

Theorem 3.1. *Let assumptions a_1)-a_7) be satisfied. Then Eq.* (1.1) *has at least one solution $y \in C(J)$ which is nondecreasing on the interval J.*

Proof. Denote by $\mathcal{F}$ the operator associated with the right-hand side of Eq. (1.1), i.e., equation (1.1) takes the form

$$y = \mathcal{F}y, \tag{3.2}$$

where

$$(\mathcal{F}y)(t) = g(t, y(t)) + f(t, y(t))(\mathcal{V}y)(t),\ t \in J, \tag{3.3}$$

$$(\mathcal{V}y)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{v(t,s,(Hy)(s))}{(t-s)^{1-\beta}}\, ds,\ t \in J,\ 0<\beta<1. \tag{3.4}$$

Solving Eq. (1.1) is equivalent to finding a fixed point of the operator $\mathcal{F}$ defined on the space $C(J)$.

For better readability, we break the proof into a sequence of steps.

Step 1: *$\mathcal{F}$ transforms the space $C(J)$ into itself.*

To do this it suffices to show that if $y \in C(J)$ then $\mathcal{V}y$ is continuous on J, thanks a_1) and a_3). For, fix $\varepsilon > 0$ and take arbitrary numbers $t_1,\ t_2 \in J$ such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality we can assume that $t_2 > t_1$. Then we get

$$\begin{aligned}
&|(\mathcal{V}y)(t_2) - (\mathcal{V}y)(t_1)|\\
&= \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\, ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_1,s,(Hy)(s))}{(t_1-s)^{1-\beta}}\, ds \right|\\
&\le \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\, ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\, ds \right|\\
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\, ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_1,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\, ds \right|\\
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_1,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\, ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_1,s,(Hy)(s))}{(t_1-s)^{1-\beta}}\, ds \right|\\
&\le \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} \frac{|v(t_2,s,(Hy)(s))|}{(t_2-s)^{1-\alpha}}\, ds\\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{|v(t_2,s,(Hy)(s)) - v(t_1,s,(Hy)(s))|}{(t_2-s)^{1-\beta}}\, ds\\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |v(t_1,s,(Hy)(s))|[(t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}]ds.
\end{aligned}$$

Therefore, if

$$\begin{aligned}
\omega_b(v,\varepsilon) = \sup\{&|v(t_2,s,x) - v(t_1,s,x)| :\ s,\ t_1,\ t_2 \in J,\\
&t_1 \ge s,\ t_2 \ge s,\ |t_2 - t_1| \le \varepsilon,\ \text{and } x \in [-b,b]\}
\end{aligned}$$

then we obtain

$$\begin{aligned}
|(\mathcal{V}y)(t_2) - (\mathcal{V}y)(t_1)| \;&\le\; \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{\Psi(\psi(\|y\|))}{(t_2-s)^{1-\beta}}\, ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{\omega_{\psi(\|y\|)}(v,\varepsilon)}{(t_2-s)^{1-\beta}}\, ds\\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \Psi(\psi(\|y\|))\, [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds\\
&\le\; \frac{\Psi(\psi(\|y\|))}{\Gamma(\beta+1)}(t_2-t_1)^{\beta} + \frac{\omega_{\psi(\|y\|)}(v,\varepsilon)}{\Gamma(\beta+1)} t_1^{\beta}\\
&\quad + \frac{\Psi(\psi(\|y\|))}{\Gamma(\beta+1)}[t_1^{\beta} - t_2^{\beta} + (t_2-t_1)^{\beta}].
\end{aligned}$$

Thus

$$\omega(\mathcal{V}y,\varepsilon)\leq \frac{1}{\Gamma(\beta+1)}[2\varepsilon^{\beta}\Psi(\psi(\|y\|))+\omega_{\psi(\|y\|)}(v,\varepsilon)]. \tag{3.5}$$

In view of the uniform continuity of the function v on $J\times J\times[-\psi(\|y\|),\psi(\|y\|)]$ we have that $\omega_{\psi(\|y\|)}(v,\varepsilon)\to 0$ as $\varepsilon\to 0$. From inequality (3.5) we infer that the function $\mathcal{V}y$ is continuous on the interval J and consequently, the function $\mathcal{F}y$ is continuous on J.

Step 2: *$\mathcal{F}$ transforms the ball B_{r_0} into itself.*

For each $t\in J$ we have

$$\begin{aligned}
|(\mathcal{F}y)(t)| &\leq \left|g(t,y(t))+\frac{f(t,y(t))}{\Gamma(\beta)}\int_0^t \frac{v(t,s,(Hy)(s))}{(t-s)^{1-\beta}}\,ds\right|\\
&\leq |g(t,y(t))-g(t,0)|+|g(t,0)|\\
&\quad +\frac{|f(t,x(t))-f(t,0)|+|f(t,0)|}{\Gamma(\beta)}\int_0^t \frac{|v(t,s,(Hy)(s))|}{(t-s)^{1-\beta}}\,ds\\
&\leq a\|y\|+g^*+\frac{c\|y\|+f^*}{\Gamma(\beta)}\Psi(\psi(\|y\|))\int_0^t \frac{ds}{(t-s)^{1-\beta}}\\
&= a\|y\|+g^*+\frac{c\|y\|+f^*}{\Gamma(\beta+1)}\Psi(\psi(\|y\|)).
\end{aligned}$$

Hence

$$\|\mathcal{F}y\|\leq a\|y\|+g^*+\frac{c\|y\|+f^*}{\Gamma(\beta+1)}\Psi(\psi(\|y\|)). \tag{3.6}$$

Thus, if $\|y\|\leq r_0$ we obtain from assumption a_6) the estimate

$$\|\mathcal{F}y\|\leq ar_0+g^*+\frac{cr_0+f^*}{\Gamma(\beta+1)}\Psi(\psi(r_0)).$$

Consequently, the operator $\mathcal{F}$ transforms the ball B_{r_0} into itself.

Step 3: *$\mathcal{F}$ transforms continuously the ball $B^+_{r_0}$ into itself.*

Consider the operator $\mathcal{F}$ on the subset $B^+_{r_0}$ of the ball B_{r_0} defined by

$$B^+_{r_0}=\{y\in B_{r_0}: y(t)\geq 0,\ \text{for } t\in J\}.$$

Obviously, the set $B^+_{r_0}$ is nonempty, bounded, closed and convex. In view of these facts and assumptions a_1), a_3) and a_5), we deduce that $\mathcal{F}$ transforms the set $B^+_{r_0}$ into itself.

Step 4: *The operator $\mathcal{F}$ is continuous on $B^+_{r_0}$.*

Let us fix $\varepsilon>0$ and arbitrary $x,\ y\in B^+_{r_0}$ such that $\|y-x\|\leq\varepsilon$. Then, for $t\in J$, we have

$$\begin{aligned}
&|(\mathcal{F}y)(t)-(\mathcal{F}x)(t)|\\
&\leq |g(t,y(t))-g(t,x(t))|+\left|\frac{f(t,y(t))}{\Gamma(\beta)}\int_0^t \frac{v(t,s,(Hy)(s))}{(t-s)^{1-\beta}}\,ds\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{f(t,x(t))}{\Gamma(\beta)}\int_0^t \frac{v(t,s,(Hx)(s))}{(t-s)^{1-\beta}}\,ds\Bigg| \\
&\le a|y(t)-x(t)| + \left|\frac{f(t,y(t))}{\Gamma(\beta)}\int_0^t \frac{v(t,s,(Hy)(s))}{(t-s)^{1-\beta}}\,ds\right. \\
&\qquad\left. -\frac{f(t,x(t))}{\Gamma(\beta)}\int_0^t \frac{v(t,s,(Hy)(s))}{(t-s)^{1-\beta}}\,ds\right| \\
&\quad + \left|\frac{f(t,x(t))}{\Gamma(\beta)}\int_0^t \frac{v(t,s,(Hy)(s))}{(t-s)^{1-\beta}}\,ds\right. \\
&\qquad\left. -\frac{f(t,x(t))}{\Gamma(\beta)}\int_0^t \frac{v(t,s,(Hx)(s))}{(t-s)^{1-\beta}}\,ds\right| \\
&\le a|y(t)-x(t)| + \frac{|f(t,y(t))-f(t,x(t))|}{\Gamma(\beta)}\int_0^t \frac{|v(t,s,(Hy)(s))|}{(t-s)^{1-\beta}}\,ds \\
&\quad + \frac{|f(t,x(t))|}{\Gamma(\beta)}\int_0^t \frac{|v(t,s,(Hy)(s))-v(t,s,(Hx)(s))|}{(t-s)^{1-\beta}}\,ds \\
&\le a|y(t)-x(t)| + \frac{c|y(t)-x(t)|}{\Gamma(\beta)}\int_0^t \frac{\Psi(\psi(\|y\|))}{(t-s)^{1-\beta}}\,ds \\
&\quad + \frac{|f(t,x(t))+f(t,0)|+|f(t,0)|}{\Gamma(\beta)}\int_0^t \frac{\gamma_v(\varepsilon)}{(t-s)^{1-\beta}}\,ds.
\end{aligned}
$$

Thus

$$
\|\mathcal{F}y-\mathcal{F}x\| \le \left(a+\frac{c\Psi(\psi(r_0))}{\Gamma(\beta+1)}\right)\|y-x\| + \frac{cr_0+f^*}{\Gamma(\beta+1)}\gamma_v(\varepsilon), \tag{3.7}
$$

where we denoted

$$
\gamma_v(\varepsilon)=\sup\{|v(t,s,x_2)-v(t,s,x_1)| : t,\ s\in J,\ x_1,\ x_2\in[0,\psi(r_0)],\ \|x_2-x_1\|\le\varepsilon\}.
$$

By, the uniform continuity of the function v on the set $J\times J\times[0,\psi(r_0)]$, it is easy to see that $\gamma_v(\varepsilon)\to 0$ as $\varepsilon\to 0$. Thus, from inequality (3.7), we conclude that $\mathcal{F}$ is continuous on the set $B^+_{r_0}$.

Step 5: *An estimate of $\mathcal{F}$ with respect to the term related to continuity.*

Let us take a nonempty set $Y\subset B^+_{r_0}$. Fix an arbitrarily number $\varepsilon>0$ and choose $y\in Y$ and $t_1,\ t_2\in J$ such that $|t_2-t_1|\le\varepsilon$. Without loss of generality we may assume that $t_2\ge t_1$. Then, in view of our assumptions, we obtain

$$
\begin{aligned}
&|(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)| \\
&= \left|g(t_2,y(t_2)) + \frac{f(t_2,y(t_2))}{\Gamma(\beta)}\int_0^{t_2}\frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds\right. \\
&\quad \left. -g(t_1,y(t_1)) - \frac{f(t_1,y(t_1))}{\Gamma(\beta)}\int_0^{t_1}\frac{v(t_1,s,(Hy)(s))}{(t_1-s)^{1-\beta}}\,ds\right| \\
&\le |g(t_2,y(t_2))-g(t_1,y(t_1))| \\
&\quad + \left|\frac{f(t_2,y(t_2)))}{\Gamma(\beta)}\int_0^{t_2}\frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds - \frac{f(t_2,y(t_2))}{\Gamma(\beta)}\int_0^{t_2}\frac{v(t_1,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{f(t_2,y(t_2))}{\Gamma(\beta)}\int_0^{t_2}\frac{v(t_1,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds-\frac{f(t_2,y(t_2))}{\Gamma(\beta)}\int_0^{t_1}\frac{v(t_1,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds\right|\\
&+\left|\frac{f(t_2,y(t_2))}{\Gamma(\beta)}\int_0^{t_1}\frac{v(t_1,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds-\frac{f(t_2,y(t_2))}{\Gamma(\beta)}\int_0^{t_1}\frac{v(t_1,s,(Hy)(s))}{(t_1-s)^{1-\beta}}\,ds\right|\\
&+\left|\frac{f(t_2,y(t_2))}{\Gamma(\beta)}\int_0^{t_1}\frac{v(t_1,s,(Hy)(s))}{(t_1-s)^{1-\beta}}\,ds-\frac{f(t_1,y(t_1))}{\Gamma(\beta)}\int_0^{t_1}\frac{v(t_1,s,(Hy)(s))}{(t_1-s)^{1-\beta}}\,ds\right|\\
&\le |g(t_2,y(t_2))-g(t_1,y(t_2))|+|g(t_1,y(t_2))-g(t_1,y(t_1))|\\
&\quad+\frac{|f(t_2,y(t_2))|}{\Gamma(\beta)}\int_0^{t_2}\frac{|v(t_2,s,(Hy)(s))-v(t_1,s,(Hy)(s))|}{(t_2-s)^{1-\beta}}\,ds\\
&\quad+\frac{|f(t_2,y(t_2))|}{\Gamma(\beta)}\int_{t_1}^{t_2}\frac{|v(t_1,s,(Hy)(s))|}{(t_2-s)^{1-\beta}}\,ds\\
&\quad+\frac{|f(t_2,y(t_2))|}{\Gamma(\beta)}\int_0^{t_1}|v(t_1,s,(Hy)(s))|\left[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right]\,ds\\
&\quad+\frac{|f(t_2,y(t_2))-f(t_1,y(t_2))|}{\Gamma(\beta)}\int_0^{t_1}\frac{|v(t_1,s,(Hy)(s))|}{(t_1-s)^{1-\beta}}\,ds\\
&\quad+\frac{|f(t_1,y(t_2))-f(t_1,y(t_1))|}{\Gamma(\beta)}\int_0^{t_1}\frac{|v(t_1,s,(Hy)(s))|}{(t_1-s)^{1-\beta}}\,ds\\
&\le \delta_g(\varepsilon)+a\,\omega(y,\varepsilon)+\frac{c\|y\|+f^*}{\Gamma(\beta)}\omega_{\psi(\|y\|)}(v,\varepsilon)\int_0^{t_2}\frac{ds}{(t_2-s)^{1-\beta}}\\
&\quad+\frac{c\|y\|+f^*}{\Gamma(\beta)}\Psi(\psi(\|y\|))\left\{\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds+\int_0^{t_1}\left[(t_1-s)^{\beta-1}-(t_2-s)^{\alpha-1}\right]\,ds\right\}\\
&\quad+\frac{\delta_f(\varepsilon)+c\,\omega(y,\varepsilon)}{\Gamma(\beta)}\Psi(\psi(\|y\|))\int_0^{t_1}\frac{ds}{(t_1-s)^{1-\beta}}\\
&\le \delta_g(\varepsilon)+a\,\omega(y,\varepsilon)+\frac{cr_0+f^*}{\Gamma(\beta+1)}\omega_{\psi(r_0)}(v,\varepsilon)\;t_2^\beta\\
&\quad+\frac{cr_0+f^*}{\Gamma(\beta+1)}\Psi(\psi(r_0))\left[t_1^\beta-t_2^\beta+2(t_2-t_1)^\beta\right]+\frac{\delta_f(\varepsilon)+c\,\omega(y,\varepsilon)}{\Gamma(\beta+1)}\Psi(\psi(r_0))\;t_1^\beta\\
&\le \delta_g(\varepsilon)+a\,\omega(y,\varepsilon)+\frac{cr_0+f^*}{\Gamma(\beta+1)}[\omega_{\psi(r_0)}(v,\varepsilon)+2\varepsilon^\beta\Psi(\psi(r_0))]+\frac{\delta_f(\varepsilon)+c\,\omega(y,\varepsilon)}{\Gamma(\beta+1)}\Psi(\psi(r_0)),
\end{aligned}
$$

where we denoted

$$\delta_h(\varepsilon)=\sup\left\{|h(t_2,x)-h(t_1,x)|:t_1,\ t_2\in J,\ x\in[0,r_0],\ |t_2-t_1|\le\varepsilon\right\}.$$

Hence, from the last inequality, we obtain

$$
\begin{aligned}
\omega(\mathcal{F}y,\varepsilon)\quad\le\quad&\delta_g(\varepsilon)+\frac{\Psi(\psi(r_0))}{\Gamma(\beta+1)}\delta_f(\varepsilon)+\left(a+\frac{c\Psi(\psi(r_0))}{\Gamma(\beta+1)}\right)\omega(y,\varepsilon)\\
&+\frac{cr_0+f^*}{\Gamma(\beta+1)}[\omega_{\psi(r_0)}(v,\varepsilon)+2\varepsilon^\beta\Psi(\psi(r_0))].
\end{aligned}
$$

Consequently,

$$\omega(\mathcal{F}Y,\varepsilon)\quad\le\quad\delta_g(\varepsilon)+\frac{\Psi(\psi(r_0))}{\Gamma(\beta+1)}\delta_f(\varepsilon)+\left(a+\frac{c\Psi(\psi(r_0))}{\Gamma(\beta+1)}\right)\omega(Y,\varepsilon)$$

$$+\frac{cr_0+f^*}{\Gamma(\beta+1)}[\omega_{\psi(r_0)}(v,\varepsilon)+2\varepsilon^{\beta}\Psi(\psi(r_0))].$$

The last inequality implies

$$\omega_0(\mathcal{F}Y)\le\left(a+\frac{c\Psi(\psi(r_0))}{\Gamma(\beta+1)}\right)\omega_0(Y). \tag{3.8}$$

Step 6: *An estimate of $\mathcal{F}$ with respect to the term related to monotonicity.*

Fix an arbitrary $y\in Y$ and $t_1,\ t_2\in J$ with $t_2>t_1$. Then, taking into account our assumptions, we have

$$\begin{aligned}
&|(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)|-[(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)]\\
&\le|g(t_2,y(t_2))-g(t_1,y(t_1))|-[g(t_2,y(t_2))-g(t_1,y(t_1))]\\
&\quad+|f(t_2,y(t_2))(\mathcal{V}y)(t_2)-f(t_1,y(t_1))(\mathcal{V}y)(t_1)|\\
&\quad-[f(t_2,y(t_2))(\mathcal{V}y)(t_2)-f(t_1,y(t_1))(\mathcal{V}y)(t_1)]\\
&\le|g(t_2,y(t_2))-g(t_1,y(t_1))|-[g(t_2,y(t_2))-g(t_1,y(t_1))]\\
&\quad+|f(t_2,y(t_2))(\mathcal{V}y)(t_2)-f(t_1,y(t_1))(\mathcal{V}y)(t_2)|\\
&\quad+|f(t_1,y(t_1))(\mathcal{V}y)(t_2)-f(t_1,y(t_1))(\mathcal{V}y)(t_1)|\\
&\quad-[f(t_2,y(t_2))(\mathcal{V}y)(t_2)-f(t_1,y(t_1))(\mathcal{V}y)(t_2)]\\
&\quad-[f(t_1,y(t_1))(\mathcal{V}y)(t_2)-f(t_1,y(t_1))(\mathcal{V}y)(t_1)]\\
&\le|g(t_2,y(t_2))-g(t_1,y(t_1))|-[g(t_2,y(t_2))-g(t_1,y(t_1))]\\
&\quad+\{|f(t_2,y(t_2))-f(t_1,y(t_1))|-[f(t_2,y(t_2))-f(t_1,y(t_1))]\}\\
&\times\frac{1}{\Gamma(\beta)}\int_0^{t_2}\frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds\\
&+\frac{f(t_1,y(t_1))}{\Gamma(\alpha)}\left\{\left|\int_0^{t_2}\frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds-\int_0^{t_1}\frac{v(t_1,s,(Hy)(s))}{(t_1-s)^{1-\beta}}\,ds\right|\right.\\
&\left.-\left[\int_0^{t_2}\frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds-\int_0^{t_1}\frac{v(t_1,s,(Hy)(s))}{(t_1-s)^{1-\beta}}\,ds\right]\right\}
\end{aligned} \tag{3.9}$$

Now, we will prove that

$$\int_0^{t_2}\frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds-\int_0^{t_1}\frac{v(t_1,s,(Hy)(s))}{(t_1-s)^{1-\beta}}\,ds\ge 0.$$

In fact, we have

$$\begin{aligned}
&\int_0^{t_2}\frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds-\int_0^{t_1}\frac{v(t_1,s,(Hy)(s))}{(t_1-s)^{1-\beta}}\,ds\\
&\quad=\int_0^{t_1}\frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds+\int_{t_1}^{t_2}\frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds-\int_0^{t_1}\frac{v(t_1,s,(Hy)(s))}{(t_1-s)^{1-\beta}}\,ds\\
&\quad\ge\int_0^{t_1}v(t_2,s,(Hy)(s))[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]\,ds+\int_{t_1}^{t_2}\frac{v(t_2,s,(Hy)(s))}{(t_2-s)^{1-\beta}}\,ds.
\end{aligned}$$

Since the term

$$(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}$$

is negative for $0 \le s < t_1$, thus taking into account assumption a_6) from the above inequality we get

$$\begin{aligned}
&\int_0^{t_2} \frac{v(t_2, s, (Hy)(s))}{(t_2 - s)^{1-\beta}}\, ds - \int_0^{t_1} \frac{v(t_1, s, (Hy)(s))}{(t_1 - s)^{1-\beta}}\, ds \\
&\ge \int_0^{t_1} v(t_2, t_1, (Hy)(t_1))[(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}]\, ds + \int_{t_1}^{t_2} \frac{v(t_2, t_1, (Hy)(t_1))}{(t_2 - s)^{1-\beta}}\, ds \\
&= v(t_2, t_1, (Hy)(t_1)) \left(\int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}]\, ds + \int_{t_1}^{t_2} \frac{ds}{(t_2 - s)^{1-\beta}} \right) \\
&= v(t_2, t_1, (Hy)(t_1)) \frac{t_2^\beta - t_1^\beta}{\beta} \\
&\ge 0.
\end{aligned}$$

This together with (3.9) yields

$$\begin{aligned}
&|(\mathcal{F}y)(t_2) - (\mathcal{F}y)(t_1)| - [(\mathcal{F}y)(t_2) - (\mathcal{F}y)(t_1)] \\
&\le |g(t_2, y(t_2)) - g(t_1, y(t_1))| - [g(t_2, y(t_2)) - g(t_1, y(t_1))] \\
&\quad + \frac{|f(t_2, y(t_2)) - f(t_1, y(t_1))| - [f(t_2, y(t_2)) - f(t_1, y(t_1))]}{\Gamma(\beta)} \int_0^{t_2} \frac{v(t_2, s, (Hy)(s))}{(t_2 - s)^{1-\beta}}\, ds.
\end{aligned}$$

The above estimate implies

$$d(\mathcal{F}y) \le d(Gy) + \frac{\Psi(\psi(r_0))}{\Gamma(\beta + 1)} d(Fy).$$

Therefore

$$d(\mathcal{F}y) \le \left(a + \frac{c\ \Psi(\psi(r_0))}{\Gamma(\beta + 1)} \right) d(y).$$

and consequently

$$d(\mathcal{F}Y) \le \left(a + \frac{c\ \Psi(\psi(r_0))}{\Gamma(\beta + 1)} \right) d(Y). \tag{3.10}$$

Step 7: *$\mathcal{F}$ is contraction with respect to the measure of noncompactness μ.*

From (3.4) and (3.10) and the definition of the measure of noncompactness μ, we obtain

$$\mu(\mathcal{F}Y) \le \left(a + \frac{c\ \Psi(\psi(r_0))}{\Gamma(\beta + 1)} \right) \mu(Y).$$

Now, the above obtained inequality together with the fact that $a\Gamma(\beta+1)+c\Psi(\psi(r_0)) < \Gamma(\beta + 1)$ enable us to apply Theorem 2.4, then Eq. (1.1) has at least one solution $x \in C(J)$. This completes the proof. □

4. EXAMPLE

Consider the perturbed fractional quadratic integral equation

$$y(t) = \frac{ty(t)}{4+t^2} + \frac{y(t)}{25\Gamma(\frac{1}{5})}\int_0^t \frac{\arctan\left(\frac{t\int_0^s \tau y^2(\tau)\,d\tau}{1+s^2}\right)}{(t-s)^{\frac{4}{5}}}\,ds. \tag{4.1}$$

In this example we have $g(t,y) = \frac{t\,y}{1+t^2}$ and this function satisfies assumption a_1) and

$$|g(t,y) - g(t,x)| \leq \frac{1}{5}|y-x|$$

for all $x,\ y \in \mathbb{R}$ and $t \in J$. Moreover, the function g satisfies assumption a_2). Indeed, taking an arbitrary nonnegative function $y \in C(J)$ and $t_1,\ t_2 \in J$ such that $t_2 \geq t_1$, we obtain

$$\begin{aligned}
& |(Gy)(t_2) - (Gy)(t_1)| - [(Gy)(t_2) - (Gy)(t_1)] \\
= \ & |g(t_2, y(t_2)) - g(t_1, y(t_1))| - [g(t_2, y(t_2)) - g(t_1, y(t_1))] \\
= \ & \left|\frac{t_2}{(4+t_2^2)}y(t_2) - \frac{t_1}{(4+t_1^2)}y(t_1)\right| - \left[\frac{t_2}{(4+t_2^2)}y(t_2) - \frac{t_1}{(4+t_1^2)}y(t_1)\right] \\
\leq \ & \frac{t_2}{(4+t_2^2)}\,|y(t_2) - y(t_1)| + \left|\frac{t_2}{(4+t_2^2)} - \frac{t_1}{(4+t_1^2)}\right|\ y(t_1) \\
& \quad - \frac{t_2}{(4+t_2^2)}\,[y(t_2) - y(t_1)] - \left[\frac{t_2}{(4+t_2^2)} - \frac{t_1}{(4+t_1^2)}\right]\ y(t_1) \\
\leq \ & \frac{t_2}{(4+t_2^2)}\,\{|y(t_2) - y(t_1)| - [y(t_2) - y(t_1)]\} \\
\leq \ & \frac{t_2}{(4+t_2^2)}\ d(y) \\
\leq \ & \frac{1}{5}\ d(y).
\end{aligned}$$

The function $f(t,y) = \frac{y}{25}$ satisfies assumptions a_3) and a_4) with $c = \frac{1}{25}$.

Also, $v(t,s,y) = \arctan\frac{ty}{1+s^2}$ and this function satisfies assumption a_5). Indeed, we have

$$|v(t,s,y)| \leq |y|.$$

Therefore, $\Psi(r) = r$. Moreover,

$$(Hy)(t) = \int_0^t s\ y^2(s)\ ds$$

and this operator satisfies assumption a_7) with $\psi(r) = r^2$.

Therefore, the inequality (3.1) takes the form

$$\frac{r}{5}\Gamma(\frac{6}{5}) + \frac{1}{25}r^3 \leq r\Gamma(\frac{6}{5})$$

or

$$r\Gamma(\frac{1}{5}) + r^3 \leq 5r\Gamma(\frac{1}{5})$$

and this admits $r_0 = 0.2$ as a positive solution. Moreover,

$$\begin{aligned} a\,\Gamma(\frac{6}{5}) + c\,\Psi(\psi(r_0)) &= \frac{1}{5}\Gamma(\frac{6}{5}) + \frac{1}{625} \\ &< \Gamma(\frac{6}{5}). \end{aligned}$$

Theorem 3.1 guarantees that equation (4.1) has a nondecreasing solution.

Remark 4.1. In [9] Banaś and Rzepka announced that the proof in [22] is not correct. In this paper we generalized, improved and corrected the results in [22]. Also the condition in [9] that f is nondecreasing with respect to each of the variables separately has been relaxed. It is worthwhile mentioning that we use the same measure of noncompactness used in [22].

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