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COMPONENT-WISE CONDITIONS FOR THE ASYMPTOTIC EQUIVALENCE FOR NONLINEAR DIFFERENTIAL SYSTEMS WITH MAXIMA

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ABSTRACT. We obtain new sufficient component-wise conditions for the asymptotic equivalence between bounded solutions of linear and nonlinear systems of differential equations with maxima. A Lipschitz component-wise and a spectral condition allow us to obtain vectorial asymptotic formulae. Under a spectral dichotomy condition the equivalences take the form of a homeomorphism which is also extended to unbounded solutions. We also obtain a vectorial Levinson's theorem with maximum about asymptotic integration.

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1. INTRODUCTION

The theory of differential equations with maxima arise naturally when solving practical problems in the study of systems with automatic regulation which are actived when the state attains the maximum in a previous interval $I_t = [t-h, t]$

$$y'(t) = \Lambda y(t) + C \max_{s \in I_t} y(s) + f(t).$$

For example, in the case of an electrogenerator, the mechanism becomes actived when the maximum voltage variation that is permited is reached in an interval of time I_t, with h a positive constant. The equation which describes the actioning of this regulator has the form

$$V'(t) = -\delta V(t) + p \max_{s \in I_t} V(s) + F(t),$$

where δ and p are constants that are determined by the characteristic of the system, $V(t)$ is the voltage and $F(t)$ is the effect of the perturbation that appears associated to the change of voltage [7, 12]. Also, there are many processes in Physics, Engineering or Biology [6, 13, 16, 19] which are modeled by ordinary differential equations of the type

$$y'(t) = f\left(t, y(t), \max_{s \in I_t} y(s)\right), \qquad t \in [0, T].$$

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The most of this study has been done in the scalar case. In this paper we consider the vectorial case. Recently, there is increasing interest towards equations which contain maxima. Much work on these equations has been carried out in the last three decades. We mention the work in [6–8, 12, 15–16, 18–19].

The inequality $x \leq y$ between two real vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ means that every coordinate of y is not less than the corresponding coodinate of x, i.e., for each $i \in \{1, 2, \ldots, n\}$ the inequality $x_i \leq y_i$ is verified. For all $x = (x_1, x_2, \ldots, x_n)$, we define the vectors $|x| = (|x_1|, |x_2|, \ldots, |x_n|)$, $|x|_\infty = (\sup_{t\in I} |x_i(t)|)_{i=1}^n$. For any $n \times n$ matrix $A = (a_{ij})$, we define the matrix $|A| = (|a_{ij}|)$ and $\|x\| = \max_{1\leq i\leq n} |x_i|$ (see [1, 2]). If $h > 0$, for all $t \in R$, denote $J = [-h, \infty)$, $I_t = [t - h, t]$ and we define

$$\max_{s\in I_t} x(s) = \left(\max_{s\in I_t} x_1(s), \max_{s\in I_t} x_2(s), \ldots, \max_{s\in I_t} x_n(s)\right), \quad \tilde{y}(t) = \max_{s\in I_t} y(s).$$

For scalar real function β defined on $J = [-h, \infty)$ and $e(t) = \exp\left(\int_0^t \lambda(\sigma)d\sigma\right)$ we denote $|\beta|_e = \sup_{t\in J} |\beta(t)e(t)^{-1}|$. Also, for $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ we define $\|\varphi\|_e = \max(|\varphi_1|_e, |\varphi_2|_e, \ldots, |\varphi_n|_e)$. Moreover, we consider the Banach space

$$B_e = \{\varphi : J = [-h, \infty) \to \mathbb{R}^n \ / \ \|\varphi\|_e < \infty\}$$

of the e-bounded real vector functions defined on J, provided of the norm $\| \quad \|_e$.

The spectral radious of a matrix Ω (see [1, 2, 5]) is given by $\rho(\Omega) = \lim_{n\to\infty} \|\Omega^n\|^{\frac{1}{n}}$, where $\| \quad \|$ is any norm on the vector space of the $n \times n$ real matrices; this spectral radious is independent of the norm employed to calculate it.

In this work, we study nonlinear differential systems with maxima of the type

$$\begin{cases} y'(t) = A(t)y(t) + F(t, y(t), \tilde{y}(t)), & \text{with } t \in I \text{ and } y(t) \in R^n \\ y(t) = \chi(t), & \text{with } t \in [-h, 0] \end{cases} \tag{1.1}$$

where the $n \times n$ matrix $A(t)$ is continuous in the interval $I = [0, \infty)$ and $h > 0$.

In this paper we consider system (1.1) and we assume that the vectorial Lipschitz condition

$$|F(t, x_1, y_1) - F(t, x_2, y_2)| \leq Q(t)\, |x_1 - x_2| + R(t)\, |y_1 - y_2|, \quad F(t, 0, 0) \equiv 0, \tag{1.2}$$

holds for all $(t, x_i, y_i) \in I \times R^n \times R^n$, $i = 1, 2$, where $Q(t)$ and $R(t)$ are locally integrable matrices with nonnegative entries, and we prove for nonlinear system with maxima (1.1) the asymptotic equivalence among bounded solutions with non perturbed system

$$x'(t) = A(t)x(t). \tag{1.3}$$

Moreover, with this componentwise Lipschitz condition and a spectral condition, we obtain new results about asymptotic equivalence. The vectorial asymptotic formulae

$$y = x + O(\Theta(t)\,|x|_\infty), \tag{1.4}$$

$$y = x + O\left(\Omega\,|x|_\infty\right), \tag{1.5}$$

where Θ and Ω are majorants matrices with $|\Theta(t)| \le \Omega$ and $\rho(\Omega) < 1$, and

$$|y(t)| \le (I - \Omega)^{-1}\,|x|_\infty \tag{1.6}$$

are obtained. Inequality (1.6) gives a vectorial bound of the solution y in terms of the solution x, representing in vectorial way the stability of solution y respect to solution x. Furthermore, using componentwise Lipschitz condition (1.2), we prove some results of the Levinson's theorem type (see ([9]–[11]) for differential systems with maxima. Obviously, these results can be applied to "linear" systems with maxima as $y'(t) = A(t)y(t) + B(t)\tilde{y}(t)$. The asymptotic formulae for the vectorial Levinson's theorem takes the form

$$y = \exp\left(\int_0^t \lambda(s)ds\right)(I + O(\Omega))v.$$

In this paper, we obtain new sufficient conditions for the asymptotic equivalence of linear and nonlinear systems of differential equations with maxima. Equivalences among unbounded solutions are also proved. The equivalences have the form of homeomorphisms, which is the most important version of the correspondence between the solutions. As it is well known, to arrange an equivalence with another set which is then carefully studied is one of the most effective methods to clarify the structure of a complex set, see [3, 4, 9].

The main idea of this method is to apply the properties of the Green matrix of the system (1.1), the techniques developed in the study of differential equations with dichotomies, and to use the spectral radious of a majorant matrix. We give certain examples in which we can appreciate the advantage of the asymptotic representations obtained with this method.

2. **ASYMPTOTIC EQUIVALENCE**

Given a fundamental matrix $\Phi(t)$ of (1.3), we denote by $G(t,s)$ the Green matrix associated to this linear system and the projection P_1, defined by

$$G(t,s) = \begin{cases} \Phi(t)P_1\Phi^{-1}(s), & t \ge s \ge 0 \\ -\Phi(t)P_2\Phi^{-1}(s), & s \ge t \ge 0, \end{cases}$$

where P_1 and P_2 are supplementary projections (see [9, 11, 14, 17]). For fixing idea we will chose P_1 such that $\Phi(t)P_1 \to 0$ as $t \to \infty$, see [9, 14, 15, 17]. Let be the $n \times n$ nonnegative matrix $\Theta(t) = (\theta_{ij})$ defined by

$$\Theta(t) = \int_0^\infty |G(t,s)|\,(Q(s) + R(s))ds. \tag{2.1}$$

Let be the constant $n \times n$ matrix $\Omega = (\omega_{ij})$ with nonnegative real entries ω_{ij}, $i, j \in \{1, 2, \ldots, n\}$ defined by

$$\Omega = \sup_{t \in I} \Theta(t), \tag{2.2}$$

The main hypothesis will be the existence and boundedness of $\Theta(t)$, for $t \geq 0$ (and hence the integral in (2.1) is finite). Other useful condition concerns with the spectral radious of the matrix Ω, namely $\rho(\Omega) < 1$. In this case, we will say that system (1.3) has a dichotomy such that $\rho(\Omega) < 1$.

Let be the linear nonhomogeneous system

$$x'(t) = A(t)x(t) + g(t) \tag{2.3}$$

and

$$y'(t) = A(t)y(t) + F(t, y(t), \tilde{y}(t)) + g(t). \tag{2.4}$$

Theorem 2.1. *Assume that there exist locally integrable matrices $Q(t)$ and $R(t)$ with nonnegative entries, such that the vectorial Lipschitz condition* (1.2) *holds for all $(t, x_i, y_i) \in I \times R^n \times R^n$, $i = 1, 2$. If we assume that system* (1.3) *has a dichotomy with $\rho(\Omega) < 1$, then:*

a) *For each bounded solution x of* (2.3) *there exists a unique solution y of* (2.4) *such that* (1.4), (1.5) *and* (1.6) *hold.*
b) *For each bounded solution y of* (2.4) *there exists a solution x of* (2.3) *such that* (1.4) *and* (1.5) *are verified.*
c) *The correspondence $x \longrightarrow y$ is bicontinuous and it represents an asymptotic equivalence between the bounded solutions of* (1.1) *and* (2.3), *when $\Theta(t) \to 0$ as $t \to \infty$.*

Proof. Let x be a bounded solution of (2.3). Consider the integral equation

$$y(t) = x(t) + \int_0^\infty G(t,s)F(s, y(s), \tilde{y}(s))ds, \tag{2.5}$$

where $y(t) = \chi(t)$, for $t \in [-h, 0]$, and $G(t, s)$ is the Green matrix of (1.3) Any solution of (2.5) is a solution of (1.1). We denote by

$$B(J, R^n) = \{\varphi : J = [-h, \infty) \to \mathbb{R}^n \ / \ |\varphi|_\infty < \infty\}$$

and we define the operator $\mathcal{A} : B(J, R^n) \to B(J, R^n)$ by

$$(\mathcal{A}y)(t) = x(t) + \int_0^\infty G(t,s)F(s, y(s), \tilde{y}(s))ds, \quad \text{for } t \in I, \tag{2.6}$$

where $y(t) = \chi(t)$, for all $t \in [-h, 0]$. Then by the componentwise Lipschitz condition (1.2), we obtain the vectorial inequality:

$$|\mathcal{A}\varphi_1(t) - \mathcal{A}\varphi_2(t)| \leq \left(\int_0^\infty |G(t,s)| \, [Q(s) + R(s)] \, ds \right) |\varphi_1 - \varphi_2|_\infty \,, \tag{2.7}$$

where the inequality above means the comparison coordinate to coordinate. By (2.7) and condition $\rho(\Omega) < 1$ we can apply the Banach fixed point theorem to obtain the existence of a unique fixed point $y(t)$ of the operator $\mathcal{A}$. Then $y(t)$ verifies the integral equation (2.5) and condition (1.5). Moreover, $y(t) - x(t) \leq \Omega \, |y|_\infty$, then $|y|_\infty - |x|_\infty \leq \Omega \, |y|_\infty$, and we have $|y|_\infty \leq (I - \Omega)^{-1} \, |x|_\infty$. Since $y(t)$ is a bounded solution and

$$|y(t) - x(t)| \leq \left(\int_0^\infty |G(t,s)| \, [Q(s) + R(s)] \, ds \right) |y|_\infty \,,$$

we can see that, if $\Theta(t)$ converges to 0 as $t \to \infty$, then (1.4) is true. The proof of a) is complete. To prove b), we take a bounded solution y of (1.1) and we observe that

$$x(t) = y(t) - \int_0^\infty G(t,s)F(s, y(s), \tilde{y}(s))ds$$

satisfies equation (1.3) and it verifies condition (1.4). Finally, the correspondence $x \leftrightarrow y$ given by equation (2.5) is bicontinuous since the vectorial inequalities

$$(I + \Omega)^{-1} \, |x_1 - x_2|_\infty \leq |y_1 - y_2|_\infty \leq (I - \Omega)^{-1} \, |x_1 - x_2|_\infty$$

hold and c) follows. □

We have the following result for the “linear” differential system with maxima

$$y'(t) = A(t)x(t) + Q(t)y(t) + R(t)\tilde{y}(t) + g(t). \tag{2.8}$$

Corollary 2.2. *Let $Q(t)$ and $R(t)$ be locally integrable $n \times n$ matrices, and denote by $\Theta(t)$, Ω the $n \times n$ matrices given by*

$$\Theta(t) = \int_0^\infty \{ |G(t,s)Q(s)| + |G(t,s)R(s)| \} \, ds, \quad \Omega = |\Theta|_\infty \,,$$

where $G(t,s)$ is a Green matrix of the system (1.3)*. If $\rho(\Omega) < 1$, where $\rho(\Omega)$ is the spectral radious of the matrix Ω, then conclutions a), b) and c) of Theorem* 2.1 *are verified, where* (1.1) *is changed by* (2.8) *and* (1.3) *is changed by* (2.3)*.*

Remark 2.3. When we have an ordinary dichotomy, i.e., when there exists a constant c such that $\|G(t,s)\| \le c$, for all (t,s), then we have $\Theta(t) \to 0$ as $t \to \infty$ if $\Phi P_1 \to 0$ as $t \to \infty$ and Q, $R \in L^1$. If now, the dichotomy is exponential, i.e., there exist the constants $\alpha > 0$ and c such that $\|G(t,s)\| \le ce^{-\alpha|t-s|}$, $t \neq s$, then $\Theta(t) \to 0$ as $t \to \infty$ in much cases, such as $Q(t)$, $R(t) \to 0$ as $t \to \infty$ or Q, $R \in L^p$, $p \ge 1$.

Example 2.4. Consider the differential system with maxima

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 & q_{12} \\ q_{21} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 & r_{12} \\ r_{21} & 0 \end{pmatrix}\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}$$

where q_{12}, q_{21}, r_{12} and r_{21} are locally integrable real functions, $\operatorname{Re}\alpha < 0$ and $\operatorname{Re}\beta > 0$.

For projections $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, the matrix Θ in Corollary 2.2 is given by $\theta_{ii} = \omega_{ii} = 0$, $i = 1, 2$, and $\omega_{12} = \sup_{t\in I} \theta_{12}(t)$, $\omega_{21} = \sup_{t\in I} \theta_{21}(t)$, where

$$\theta_{12}(t) = \int_0^t (|q_{12}| + |r_{12}|)(s)e^{\operatorname{Re}\alpha(t-s)}ds, \quad \theta_{21}(t) = \int_t^\infty (|q_{21}| + |r_{21}|)(s)e^{\operatorname{Re}\beta(t-s)}ds.$$

If $\omega_{12}\omega_{21} < 1$, then the spectral radius $\rho(\Omega) < 1$, where $\Omega = |\Theta|_\infty = (\omega_{ij})$ and we can apply Corollary 2.2. Therefore, for each bounded solution $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ of (1.3), there exists a solution $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ of (2.8) such that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + O\begin{pmatrix} \theta_{21}(t) \sup\limits_{t\in I} |y_2(t)| \\ \theta_{12}(t) \sup\limits_{t\in I} |y_1(t)| \end{pmatrix}$$

and, for all $t \in I$, we have

$$\begin{pmatrix} |y_1(t)| \\ |y_2(t)| \end{pmatrix} \le \frac{1}{1 - \omega_{12}\omega_{21}} \begin{pmatrix} 1 & \omega_{12} \\ \omega_{21} & 1 \end{pmatrix}\begin{pmatrix} \sup\limits_{t\in I} |x_1(t)| \\ \sup\limits_{t\in I} |x_2(t)| \end{pmatrix}.$$

In particular, if $q_{ij} = 0$, r_{12} is a bounded real function in the interval I, such that $|r_{12}(t)| \le M$ for all $t \in I$ and $r_{21}(t) = \frac{1}{t+1}$ then, if $-\frac{M}{\operatorname{Re}\alpha \operatorname{Re}\beta} < 1$, we can apply Corollary 2.2. Note that $r_{21} \notin L^1$ and general cases where $r_{21} \notin L^p$ for any $p > 1$, may be studied.

The cases $\operatorname{Re}\beta \ge 0, \operatorname{Re}\alpha < 0$ or $\operatorname{Re}\beta = \operatorname{Re}\alpha = 0$ can be similarly studied

Remark 2.5. If $\operatorname{Re}\alpha$, $\operatorname{Re}\beta \neq 0$, then condition $\theta_{21}\theta_{12} < 1$ given in Example 2.4, is verified for $t \ge t_0$, and t_0 large enough, for example in the following cases:

1) If $q_{21} \in L^{p_1}$, $r_{21} \in L^{p_2}$ and $|q_{12}| \le \epsilon$, $|r_{12}| \le \epsilon$, where ϵ is small enough, $\epsilon < \min\{|\operatorname{Re}\alpha|, |\operatorname{Re}\beta|\}$.

2) If $q_{12} \in L^{p_1}$, $r_{12} \in L^{p_2}$ and $|q_{12}| \leq \epsilon$, $|r_{12}| \leq \epsilon$, where ϵ is small enough, $\epsilon < \min\{|\mathrm{Re}\,\alpha|, |\mathrm{Re}\,\beta|\}$.
3) If q_{21}, $r_{21} \in L^{p_{21}}$ and q_{12}, r_{12} are bounded, or, q_{21}, $r_{21} \to 0$ and q_{12}, r_{12} are bounded, then, we have the asymptotic equivalence.
4) If q_{12}, $r_{12} \in L^{p_{12}}$ and q_{21}, r_{21} are bounded, or, q_{12}, $r_{12} \to 0$ and q_{21}, r_{21} are bounded.

In every case, we have the asymptotic equivalence.

3. LEVINSON'S TYPE THEOREMS

A typical situation for the matrix A is $A = diag\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}$. For $\lambda \in \{\lambda_i(t)\}_{i=1}^n$ the function $x_\lambda(t) = \exp\left(\int_0^t \lambda(s)ds\right) e_\lambda$ is a solution of (1.3), where $e_\lambda = e_k$ if $\lambda = \lambda_k$ and $e_k = (0, \ldots, 1, \ldots, 0)$ is the vector of the canonical bases of R^n. Let $P = P_\lambda$ the constant diagonal projection such that $|x_\lambda(t)|^{-1} \Phi(t) P_\lambda \to 0$ as $t \to \infty$ and denotes G_λ its corresponding Green matrix. We have the following result:

Theorem 3.1. *Suppose that $x_\lambda(t) = \exp\left(\int_0^t \lambda(s)ds\right) v_\lambda$ satisfies equation* (1.3), *where v_λ is a constant vector, F verifies conditions of Theorem* 2.1, *and the spectral radious $\rho(\Omega_\lambda) < 1$, where $\Omega_\lambda = |\Theta_\lambda|_\infty$ with*

$$\Theta_\lambda(t) = \int_0^\infty |G_\lambda(t,s)| \exp\left(-\mathrm{Re}\int_s^t \lambda(\xi)d\xi\right)\{(Q(s) + R(s)m_\lambda(s))\}\,ds \tag{3.1}$$

and

$$m_\lambda(s) = \exp\left(-\mathrm{Re}\int_0^s \lambda(\xi)d\xi\right) \max_{\sigma\in I_s}\left\{\exp\left(\mathrm{Re}\int_0^\sigma \lambda(\xi)d\xi\right)\right\}. \tag{3.2}$$

Then, there exists a solution $y_\lambda(t)$ of (1.1) *such that*

$$y_\lambda(t) = \exp\left(\int_0^t \lambda(s)ds\right)(v_\lambda + O(\Omega_\lambda |v_\lambda|)) \tag{3.3}$$

and

$$y_\lambda(t) = \exp\left(\int_0^t \lambda(s)ds\right)(v_\lambda + O(\Theta_\lambda(t) |v_\lambda|)) \tag{3.4}$$

for some constant vector $v_\lambda \in R^n$.

Proof. Consider the integral equation

$$y(t) = x_\lambda(t) + \int_0^\infty G_\lambda(t,s)F(s, y(s), \tilde{y}(s))ds,$$

where $y(t) = \chi(t)$, for $t \in [-h, 0]$ and G is the Green Matrix associated to linear system (1.3). If we denote $e(t) = \exp \int_0^t \lambda(\sigma)d\sigma$. We define the operator $\mathcal{A} : B_e \to B_e$ by the formula

$$\mathcal{A}(y)(t) = x_\lambda(t) + \int_0^\infty G_\lambda(t,s)F(s, y(s), \tilde{y}(s))ds.$$

We have

$$\left|\max_{\sigma\in I_s} \varphi_1(\sigma) - \max_{\sigma\in I_s} \varphi_2(\sigma)\right| \le \max_{\sigma\in I_s} |e(\sigma)| \left|\max_{\sigma\in I_s} \hat{\varphi}_1(\sigma) - \max_{\sigma\in I_s} \hat{\varphi}_2(\sigma)\right|,$$

where $\hat{\varphi} = \varphi e^{-1}$. Then

$$\begin{aligned}
&|(\mathcal{A}\varphi_1 - \mathcal{A}\varphi_2)(t)| \\
&\le \int_0^\infty |G_\lambda(t,s)| \left\{ Q(s) \left|\varphi_1(s) - \varphi_2(s)\right| + R(s) \left|\max_{\sigma\in I_s} \varphi_1(\sigma) - \max_{\sigma\in I_s} \varphi_2(\sigma)\right| \right\} ds \\
&\le \left(\int_0^\infty |G_\lambda(t,s)e(s)| \,(Q(s) + R(s) \left|e^{-1}(s)\right| \max_{\sigma\in I_s} |e(\sigma)|)ds \right) \|\varphi_1 - \varphi_2\|_e \,.
\end{aligned}$$

Therefore, we have $|e^{-1}(t)(\mathcal{A}\varphi_1 - \mathcal{A}\varphi_2)(t)| \le \Theta_{\lambda(t)} \|\varphi_1 - \varphi_2\|_e$ and $\|(\mathcal{A}\varphi_1 - \mathcal{A}\varphi_2)\|_e \le (\sup_{t\in J} \Theta_\lambda(t)) \|\varphi_1 - \varphi_2\|_e$.

By the Banach fixed point theorem, there exists a solution $y = y(t)$ of (1.1) in B_e and conditions (3.3) and (3.4) are verified. □

Remark 3.2. The function x_λ is solution of system (1.3) if and only if $A(t)v_\lambda = \lambda(t)v_\lambda$. The function m_λ in (3.1) is important because it can be unbounded. For example, in the case $\lambda(s) = -2s$ we have $m_\lambda(t) = e^{2th-h^2}$.

For the "linear" system with maxima

$$y'(t) = A(t)y(t) + Q(t)y(t) + R(t)\tilde{y}(t), \tag{3.5}$$

where matrices A, Q and R are defined in the interval $[0, \infty)$, we have:

Corollary 3.3. *Let $Q(t)$, $R(t)$, and $m_\lambda(t)$ be locally integrable in the interval I, and we denote by $\Theta(t)$ the $n \times n$ matrix given by*

$$\Theta(t) = \int_0^\infty \left\{|G(t,s)Q(s)| + |G(t,s)R(s)| \, m_\lambda(s)\right\} \exp\left(-\operatorname{Re} \int_s^t \lambda(\xi)d\xi \right) ds,$$

where $G(t,s)$ is the Green matrix of system (1.3) and $m_\lambda(s)$ is defined in Theorem 3.1. If for a constant vector v, $x_\lambda = \exp\left(\int_0^t \lambda(s)ds\right) v$ satisfies equation (1.3), and $\rho(\Omega) < 1$, where $\Omega = \|\Theta\|_\infty$, then there exists a solution y of (3.5) such that (3.4) is verified.

Remark 3.4. The functions in system (1.1) can be defined on the interval $[a, \infty)$, instead of $I = [0, \infty)$. In this case all the results remain valid in $[a, \infty)$, eventually, for a big enough.

Example 3.5. Consider the differential system with maxima

$$y' = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} y + \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} y + \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \tilde{y} \tag{3.6}$$

where q_{ij}, r_{ij} , $1 \leq i,\ j \leq 2$ are locally integrable real functions and α, β are real constants such that $\alpha < \beta$.

For $\lambda = \alpha$ and projections $P_1 = 0$ and $P_2 = I_2$, we see that $m_\lambda(s) = 1$ if $\alpha > 0$ and $m_\lambda(s) = e^{-\alpha h}$ if $\alpha \leq 0$. We have:

$$\Theta_\alpha(t) = \int_t^\infty (|Q(s)| + |R(s)|\, m_\alpha(s)) e^{(\beta-\alpha)(t-s)} ds$$

If a is big enough then in the following cases we have $\rho\,(\Omega) < 1$:

1) $q_{ij} \in L^{c_{ij}}$, $r_{ij} \in L^{d_{ij}}$, $1 \leq c_{ij}, d_{ij} < \infty$
2) $|q_{ij}| \leq \epsilon$, $|r_{ij}| \leq \epsilon$, $1 \leq i, j \leq 2$, for ϵ small enough

As $x = e^{\alpha t} e_1$ is a solution of (1.3), where e_1 is the vector of the canonical bases of R^2, in any of these cases, there exists a solution φ_1 of the system (3.6), defined in $[a, \infty)$ such that

$$\varphi_1(t) = e^{\alpha t} \left\{ I_2 + O(\Theta_\alpha(t)) \right\} e_1.$$

For $\lambda = \beta$ and projections $P_1 = I_2$ and $P_2 = 0$ we see that $m_\lambda(s) = 1$ if $\beta > 0$ and $m_\lambda(s) = e^{-\beta h}$ if $\beta \leq 0$. In this case, we have:

$$\Theta_\beta(t) = \int_a^t (|Q(s)| + |R(s)|\, m_\beta(s)) e^{(\alpha-\beta)(t-s)} ds.$$

Similarly, there exists a solution φ_2 of the system (3.6), defined in $[a, \infty)$ such that

$$\varphi_2(t) = e^{\beta t} \left\{ I_2 + O(\Theta_\beta(t)) \right\} e_2.$$

So, we have

$$\varphi_2(t) = e^{\beta t} \left\{ I_2 + O\left(\int_a^t (|Q(s)| + |R(s)|\, m_\beta(s)) e^{(\alpha-\beta)(t-s)} ds \right) \right\} e_2.$$

We have considered α and β are constants, but we can take α and β non constants. For example, if $\alpha(t) = -2t$ and $\beta(t) = \varepsilon - 2t$, for $t \geq 0$ and $\varepsilon > 0$, we observe that $m_\alpha(t) = e^{2th-h^2}$ is not a bounded function and $\Theta_\alpha(t)$ is given by

$$\Theta_\alpha(t) = \int_t^\infty \left(|Q(s)| + |R(s)|\, e^{2sh-h^2} \right) e^{\varepsilon(t-s)} ds.$$

Then, Corollary 3.3 and Theorem 3.1 are not valid asking only $r_{ij} \in L^1$, it is necessary $r_{ij}(s)e^{2sh-h^2} \in L^1$.

4. APPLICATIONS TO SECOND ORDER DIFFERENTIAL EQUATIONS

In the particular case when the matrix A in Theorem 2.1 is null, that is

$$y' = F\left(t, y(t), \max_{s \in t-h,t} y(s)\right)$$

we have the following vectorial result, extending the scalar ones in [12]:

Corollary 4.1. *Assume that there exist integrable matrices $Q(t)$ and $R(t)$ with non-negative entries, such that (1.2) holds, for all $(t, x_i, y_i) \in I \times R^n \times R^n$, $i = 1, 2$. If $\rho(\Omega) < 1$, where*

$$\Theta(t) = \int_t^\infty \{Q(s) + R(s)\}\, ds, \quad \Omega = |\Theta|_\infty$$

then we have:

a) *For each $v \in R^n$ there exists a unique solution y of (1.2) such that we have the componentwise formulas*

$$y = v + O(\Omega\, |v|) \tag{4.1}$$

$$y = v + O(\Theta(t)\, |v|), \tag{4.2}$$

as t approaches infinity, and we have the vectorial estimate

$$|y(t)| \le (I - \Omega)^{-1} (|v|)\,. \tag{4.3}$$

b) *For each bounded solution y of (1.2) there exists $v \in R^n$ such that (4.1), (4.2) and (4.3) are verified.*

c) *The correspondance $v \longrightarrow y$ is bicontinuous and its represents an asymptotic equivalence between the bounded solutions of (1.1) and R^n, when $\Theta(t) \to 0$ as $t \to \infty$.*

Proof. It follows at once from Theorem 2.1. $\square$

Consider the differential system

$$z' = A(t)z + Q(t)z + R(t)\tilde{z}. \tag{4.4}$$

The substitution $z = \Phi(t)y$ changes (4.4) into the differential system

$$y' = \Phi^{-1}(t)Q(t)\Phi(t)y + \Phi^{-1}(t)R(t)(\Phi(\tilde{t})y).$$

Then, by Corollary 4.1 we obtain the following result:

Corollary 4.2. *Let $Q(t)$ and $R(t)$ be locally integrable $n \times n$ matrices, and denote by $\Theta(t), \Omega$ the $n \times n$ matrices given by*

$$\Theta(t) = \int_t^\infty \left\{ \left|\Phi^{-1}(s)Q(s)\Phi(s)\right| + \left|\Phi^{-1}(s)R(s)\right| \left|\tilde{\Phi}\right|(s) \right\} ds, \quad \Omega = |\Theta|_\infty . \tag{4.5}$$

If $\rho(\Omega) < 1$, where $\rho(\Omega)$ is the spectral radious of the matrix Ω, then conclusions a), b) and c) of Theorem 2.1 *are verified, where* (1.1) *is changed by* (4.4) *and* (1.4), (1.5) *and* (1.6) *are respectivelly changed by:*

$$z(t) = \Phi(t)(v + O(\Theta(t)\,|v|), \tag{4.6}$$

$$z(t) = \Phi(t)(v + O(\Omega\,|v|), \tag{4.7}$$

$$|z(t)| \le |\Phi(t)|\,(I - \Omega)^{-1}\,|v| . \tag{4.8}$$

In Corollary 4.2, the correspondence $z \longrightarrow \Phi(t)v$ is an equivalence between solutions not necessarilly bounded. Moreover, every solution has a representation as (4.6), (4.7) and (4.8). For that, the stronger integrability condition (4.5) has been necessary, see next example. In Theorem 3.1 and Corollary 3.3, only some special solutions have an asymptotic formula as (3.3) or (3.4).

Example 4.3. Consider the second order differential equation with maxima

$$u''(t) = u(t) + \delta r(t)\tilde{u}(t), \tag{4.9}$$

where r is an integrable function in $I = [0, \infty)$. Corollaries 4.2 and 3.3 will be applied.

Equation (4.9) is equivalent to the differential linear system

$$z' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z + \delta r(t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tilde{z}, \quad z = \begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{4.10}$$

If $\int_0^\infty |r(t)|\, e^{2t} dt < \infty$ then for parameter $|\delta|$ small enough we get $\rho(\Omega) < 1$, where $\Omega = |\Theta|_\infty$ and the matrix $\Theta(t)$ defined in (4.5) is given by

$$\Theta(t) = \int_t^\infty \delta\,|r(s)| \begin{pmatrix} \frac{1}{2}e^h + \frac{1}{2} & e^{2s} \\ \frac{1}{2}e^{-2s} + \frac{1}{2}e^{h-2s} & 1 \end{pmatrix} ds.$$

Then, applying Corollary 4.2 we have:

For all $\nu = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in R^2$ there exists a unique solution z of (4.10) such that (4.6), (4.7) and (4.8) are verified. Then a consequence of the Corollary 4.2 is that, for all solution u of equation (4.9), there exists $\nu \in R^2$ such that

$$u(t) = e^{-t}\nu_1 + e^t\nu_2 + |\delta|\, O\left\{ (|\nu_1| \int_t^\infty |r(s)|\, ds + \nu_2 \int_t^\infty |r(s)|\, e^{2s} ds) \right.$$
$$\left. + |\delta|\, e^t(|\nu_1| \int_t^\infty |r(s)|\, e^{-2s} ds + |\nu_2| \int_t^\infty |r(s)|\, ds) \right\},$$

and with an analogous formula for u'.

For every solution u of (4.9), equation (4.7) gives the formulae

$$u(t) = e^{-t}\nu_1 + e^t\nu_2 + O(\omega_{11}\,|\nu_1| + \omega_{12}\,|\nu_2|)$$

$$u'(t) = -e^{-t}\nu_1 + e^t\nu_2 + O(\omega_{21}\,|\nu_1| + \omega_{22}\,|\nu_2|),$$

where $\omega_{ij} = \sup_{t\in I}\theta_{ij}(t)$, and inequality (4.8) implies

$$u(t) \le \frac{1}{\Delta}\left\{ e^{-t}((1-\omega_{22})\,|\nu_1| + \omega_{21}\,|\nu_2|) + e^t(\omega_{12}\,|\nu_1| + (1-\omega_{11})\,|\nu_2| \right\}$$

$$u'(t) \le \frac{1}{\Delta}\left\{ -e^{-t}((1-\omega_{22})\,|\nu_1| + \omega_{21}\,|\nu_2|) + e^t(\omega_{12}\,|\nu_1| + (1-\omega_{11})\,|\nu_2| \right\},$$

where $\Delta = (1-\omega_{11})(1-\omega_{22}) - \omega_{12}\omega_{21}$.

Then,under the strong condition $r(t)e^{2t} \in L^1(I)$, applying Corollary 4.2 we obtain asymptotic formulae for all the solutions of the equation (4.9).

Only with $r \in L^1(I)$ in equation (4.9), we can also apply Corollary 3.3. First, we transform (4.10) into the differential system

$$y' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} y + \frac{1}{2}\delta r(t) \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} (\tilde{Ty}). \tag{4.11}$$

Since $\left|\tilde{Ty}\right| \le |T|\,|\tilde{y}|$ we can apply Corollary 3.3 to equation (4.11). For the solution $z_{-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$ of the homogeneous system associated to (4.11), we have

$$\Theta(t) = \int_t^\infty \frac{1}{2}\,|\delta|\,|r(s)|\, e^h \begin{pmatrix} 1 & 1 \\ e^{2(t-s)} & e^{2(t-s)} \end{pmatrix} ds.$$

If r is integrable in the interval I and $|\delta|$ is small enough, we have $\rho(\Omega) < 1$,then there exists $\nu_{-1} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in R^2$ and a solution $y_{-1}(t) = \begin{pmatrix} y_{-11} \\ y_{-12} \end{pmatrix}$ of (4.11) such that $y_{-1}(t) = e^{-t}(v_{-1} + O(\Theta(t)\,|v_{-1}|)$. Then, there is a solution y_{-11} of (4.11) such that

$$y_{-11} = e^{-t}\left(c_1 + O\left(|\delta| \int_t^\infty |r(s)|\, ds \right)\right).$$

$$y_{-12} = e^{-t}\left(c_2 + O\left(|\delta| \int_t^\infty |r(s)|\, e^{2(t-s)} ds\right)\right).$$

Since $u_{-1} = -y_{-11} + y_{-12}$ is a solution of (4.9) and $u'_{-1} = y_{-11} + y_{-12}$:

$$u_{-1}(t) = e^{-t}\left(c_1 + O\left(|\delta| \int_t^\infty |r(s)|\, ds\right)\right) + e^{-t}\left(c_2 + O\left(|\delta| \int_t^\infty |r(s)|\, e^{2(t-s)} ds\right)\right)$$

$$u'_{-1}(t) = e^{-t}\left(-c_1 + O\left(|\delta| \int_t^\infty |r(s)|\, ds\right)\right) + e^{-t}\left(c_2 + O\left(|\delta| \int_t^\infty |r(s)|\, e^{2(t-s)} ds\right)\right)$$

for $|\delta|$ small enough.

Also for $|\delta|$ small enough, there exists a solution u of (4.9) such that

(4.12)
$$u = -e^t\left(d_1 + O\left(|\delta| \int_t^\infty |r(s)| e^{-2(t-s)}\, ds\right)\right) + e^t\left(d_2 + O\left(|\delta| \int_t^\infty |r(s)\, ds|\right)\right),$$

and

(4.13)
$$u' = e^t\left(d_1 + O\left(|\delta| \int_t^\infty |r(s)| e^{-2(t-s)}\, ds\right)\right) + e^t\left(d_2 + O\left(|\delta| \int_t^\infty |r(s)\, ds|\right)\right).$$

We remark that although the integrability $r(t)e^{2t} \in L^1$ asked for r in Corollary 4.2 to obtain the asymptotic formulae (4.12)–(4.13) is stronger, this conclusion is valid for every solution u of equation (4.9). Using Corollary 3.3, r needs to be only integrable, but the conclusion is weaker.

Example 4.4. Consider the ordinary differential equation with maxima

$$u'' = \frac{2}{t^2} u + q(t)u + r(t)\tilde{u}, \quad t \geq 1. \tag{4.14}$$

The substitution $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} u \\ u' \end{pmatrix}$ changes equation (4.14) into the ordinary differential system

$$y' = \begin{pmatrix} 0 & 1 \\ \frac{2}{t^2} & 0 \end{pmatrix} y + q(t)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y + r(t)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tilde{y}.$$

Since, the ordinary differential equation $u'' = \frac{2}{t^2}u$ has the fundamental set of solutions $\{u_j\}_{j=1}^2$, where $u_1(t) = t^2$ and $u_2(t) = t^{-1}$, then the matrix Θ defined by (4.5) is given by

$$\Theta(t) = \int_t^\infty \left\{ |q(s)| \begin{pmatrix} \frac{1}{3}s & \frac{1}{3s^2} \\ \frac{1}{3}s^4 & \frac{1}{3}s \end{pmatrix} + |r(s)| \begin{pmatrix} \frac{1}{3}s & \frac{1}{3}s\,|-h+s| \\ \frac{1}{3}s^4 & \frac{s^2}{3|-h+s|} \end{pmatrix} \right\} dt.$$

If $q(s)s^4$ and $r(s)s^4$ are integrables in the interval I, then there exists a nonnegative number a big enough such that $\rho(\Omega) < 1$ and then, we can apply Corollary 4.2.

Proceeding as in example 4.3, we have that for every solution u of equation (4.14), there exists $\nu_i \in R$, $i = 1, 2$, such that

$$u(t) = t^2 \left(\nu_1 + |\delta|\, O\left(\int_t^\infty (|r| + |q)| \right)(s)ds \right) |\nu_1|)$$
$$+ t^{-1}\left(\nu_2 + O\left(\int_t^\infty (|r| + |q|)(s)s^4 ds \right) |\nu_1| \right),$$

$$u'(t) = 2t\left(\nu_1 + O\left(\int_t^\infty (|r| + |q|)(s)s^4 ds \right) |\nu_1| \right)$$
$$- t^{-2}(\nu_2 + O\left(\int_t^\infty \left(\frac{|r(s)|}{es - h} s^2 ds \right) \right) |\nu_2| .$$

Again the result is valid for every solution u of equation (4.14). The integrability conditions have been very strong.

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