Dynamic Systems and Applications 20 (2011) 495-504

GENERALIZED MONOTONE METHOD FOR CAPUTO FRACTIONAL DIFFERENTIAL SYSTEMS VIA COUPLED LOWER AND UPPER SOLUTIONS

DONNA STUTSON AND A. S. VATSALA

Department of Mathematics, Xavier University of New Orleans
New Orleans, LA 70125 USA
Department of Mathematics, University of Louisiana Lafayette, P.O. Box 41010
Lafayette, LA 70504 USA

ABSTRACT. Monotone method combined with the method of upper and lower solutions yields monotone sequences which converge uniformly and monotonically to minimal and maximal solutions of the nonlinear systems, when the forcing function is quasi monotone nondecreasing. In this paper we develop genearalized monotone method for N system of Caputo fractional differential equations when the forcing function is the sum of an increasing and decreasing functions. In generalized monotone method we use coupled upper and lower solutions and the method yields two monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions. This method is applicable to the Lotka-Volterra equation with Caputo fractional derivative of order q when $0 < q \leq 1$. This provides an opportunity to provide better results or improve on the existing results with integer derivatives. Finally, under uniqueness condition we obtain the unique solution of the Caputo fractional differential system.

Keywords and Phrases: Generalized Monotone Method, Caputo Fractional differential System

AMS Subject Classification: 34A08, 34A445, 26A33

1. INTRODUCTION

Nonlinear problems namely nonlinear dynamics systems occur as mathematical models in many branches of science, engineering, finance, economics, etc. For example the well known population model namely the Lotka-Volterra equation is given in the form:

$$u' = a_1 u \left(1 - \frac{b_1 u + c_1 v}{k_1}\right), \quad u(0) = u_0,$$

$$v' = a_2 v \left(1 - \frac{b_2 v + c_2 u}{k_2}\right), \quad v(0) = v_0.$$

This material is based upon work supported by, or in part by, the US Army Research Laboratory and the U.S. Army Research office under contract/grant number W 911 NF-10-1-0396.

Corresponding author: Vatsala

Received August 28, 2011

Here a_i, b_i, c_i, k_i are greater than 0. Also a_i, b_i, c_i, k_i can be continuous functions of t. In general mathematical models or dynamic systems which arise in science and engineering are nonlinear in nature. In the above system of differential equations namely the Volterra-Lotaka model, integer derivatives are used. However, in practice the use of fractional derivatives has been proven to be more useful and economical. See [5] for details. Explicit solutions of nonlinear dynamic systems are rarely possible. It is all the more true of nonlinear fractional dynamic systems. Hence in this work, we develop generalized monotone method for N system of fractional differential equations. Also the forcing functions will have both increasing and decreasing components of the unknown functions. See [3] for monotone method for many types of nonlinear problem. In this work we develop generalized monotone method combined with coupled upper and lower solutions [8, 9] for nonlinear fractional dynamic systems. The method yields monotone sequences which converges uniformly and monotonically to coupled minimal and maximal solutions. In order to prove the uniqueness of the solution for the given nonlinear problem we have developed an auxiliary result which is a comparison result. Using this comparison result we prove that the coupled minimal and maximal solutions reduces to the unique solution of the nonlinear Caputo fractional differential system. Finally our method is applicable to the special case of population model namely the Volterra-Lotka equation with Caputo fractional derivative of order q where $0 < q < 1$, in place of integer derivatives. For stability results on Volterra-Lotka model with fractional derivative see [7].

2. PRELIMINARY AND AUXILIARY RESULTS

In this section we recall some definitions [2, 4, 6] and known results which are needed to develop our main result. We will also prove a comparison result of coupled lower and upper solutions of the related nonlinear Caputo fractional differential system. This will be useful in obtaining the unique solution of the nonlinear problem.

Definition 2.1. The Gamma function $\Gamma(z)$ is defined as

$$\Gamma(z) = \int_0^\infty (t)^{z-1} e^{-z} dt$$

Definition 2.2. The Caputo fractional Derivative of order q of a function, $f(t)$, is defined

$$ {}_a^C D_t^q f(t) = \frac{1}{\Gamma(1-q)} \int\limits_a^t \frac{f'(s)}{(t-s)^q} ds, \quad (0 < q < 1.)$$

Definition 2.3. The Mittag Leffler function is defined

$$E_{\alpha,\beta}(\lambda(t-t_0)^\alpha) = \sum_{k=0}^\infty \frac{(\lambda(t-t_0)^\alpha)^k}{\Gamma(\alpha k + \beta)} \quad (\alpha, \beta > 0).$$

For $t_0 = 0$, $\alpha = q$ and $\beta = 1$, we get,

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)}, \quad (q > 0)$$

which is important in solving linear Caputo fractional differential equations.

Linear Fractional differential equation: Consider the linear Caputo fractional differential equation of the form:

$$^{C}D^q u(t) = \lambda u(t) + f(t), \quad u(t_0) = u_0. \tag{2.1}$$

The solution of (2.1) is given by

$$u(t) = u_0 E_{q,1}(\lambda(t-t_0)^q) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(t-s)^q f(s) ds.$$

For details see [2, 4]. This proves that the solution of a simple linear equation requires the computation of the Mittag Leffler function. However if $\lambda = 0$, the computation of the linear problem is simple. In this paper we use generalized monotone method which requires the computation of linear problems where λ is always zero, which makes our method computationally efficient. If $\lambda = 0$, in equation (2.1), then the solution of (2.1) reduces to

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) ds.$$

Lemma 2.1. *Let $m(t) \in C^1([0,T], \mathbb{R})$. If there exists $t_1 \in [0,T]$ such that $m(t_1) = 0$ and $m(t) \leq 0$ for $0 \leq t \leq T$, then it follows that*

$$^{c}D^q m(t_1) \geq 0.$$

The proof of this has been presented in [4] . However, they have used an additional assumption that $m(t)$ should be Hölder continuous of order $\lambda > q$. This assumption in general does not hold good for the linear iterates which occur naturally in any iterative methods. Thus our Lemma 2.1 is of practical importance. See [1] for proof.

Next we consider the system of N fractional differential equations with initial conditions of the form:

$$^{C}D^q u_i = f_i(t, u_1, u_2, \ldots, u_N) + g_i(t, u_1, u_2, \ldots, u_N), \quad u_i(0) = u_{0i}. \tag{2.2}$$

Here $f_i(t, u_1, u_2, \ldots, u_N)$ and $g_i(t, u_1, u_2, \ldots, u_N) \in C[J \times R^N, R^N]$ where $J = [0,T]$. Further, $f_i(t,u)$ is nondecreasing in u_i and $g_i(t,u)$ is nonincreasing in u_i components for $i = 1, 2, \ldots, N$. This is a more general model than the Lotka-Volterra equation, which is usually two systems with integer derivatives.

Here, and throughout this paper all the inequalities are component wise.

Definition 2.6. Let v_i, w_i for $i = 1, 2, \ldots, N$ be $C[J, R]$. Then v_i and w_i are called coupled lower and upper solutions of (2.2) if they satisfy the following inequalities:

$$^C D^q v_i \le f_i(t, v_1, v_2, \ldots, v_N) + g_i(t, w_1, w_2, \ldots, w_N), \quad v_i(0) \le u_{0i},$$

$$^C D^q w_i \ge f_i(t, w_1, w_2, \ldots, w_N) + g_i(t, v_1, v_2, \ldots, v_N), \quad w_i(0) \ge u_{0i}.$$

Theorem 2.2. *Let $(v_1, v_2, \ldots, v_N)$ and $(w_1, w_2, \ldots, w_N)$ be coupled lower and upper solutions of (2.2). Further let*

(i) *$f_i(t, u)$ is nondecreasing in u_i components and $g_i(t, u)$ is nonincreasing in u_i components for $i = 1, 2, \ldots, N$;*

(ii) *$f_i(t, u)$ and $g_i(t, u)$ satisfy the onesided Lipsicthz condition of the form.*

$$f_i(t, u) - f_i(t, \bar{u}) \le L_i(\sum_{i=1}^{N} u_i - \bar{u}_i), \quad L_i > 0$$

$$g_i(t, u) - g_i(t, \bar{u}) \ge -M_i(\sum_{i=1}^{N} u_i - \bar{u}_i), \quad M_i > 0$$

whenever $u_i \ge \bar{u}_i$, for $i = 1, 2, \ldots, N$. Then $v_i(t) \le w_i(t)$, on J, for $i = 1, 2, \ldots, N$, provided $v_i(0) \le w_i(0)$.

Proof. We will initially prove the result when one of the inequalities is strict. If the conclusion is not true the set

$$[Z : t \in J : w_i(t) \le v_i(t)]$$

is not empty. Let $t_1 = \inf t \in Z$. Then there exist an i such that $v_i(t_1) = w_i(t_1)$, $v_i(t) < w_i(t)$ for $t \in [0, t_1)$ and $v_j(t) \le w_j(t)$ on $t \in [0, t_1]$, $i \ne j$. From Lemma 2.1 we get $^C D^q m_i(t_1) \ge 0$ by setting $m_i(t) = v_i(t) - w_i(t)$. This implies that $^C D^q v_i(t_1) \ge ^C D^q w_i(t_1)$ for $t_1 \in (0, T]$. Hence, from the strict inequality of the coupled upper and lower solutions we get

$$\begin{aligned}
&f_i(t_1, v_1(t_1), v_2(t_1), \ldots, w_i(t_1)., v_N(t_1)) + g_i(t_1, w_1(t_1), w_2(t_1), \ldots, v_i(t_1), w_N t_1))\\
&>^C D^q v_i(t_1)\\
&\ge^C D^q w_i(t_1)\\
&\ge f_i(t_1, w_1(t_1), w_2(t_1), \ldots, v_i(t_1), w_i(t_1)) + g_i(t_1, v_1(t_1), v_2(t_1), v_i(t_1), \ldots, v_i(t_1))\\
&\ge f_i(t_1, v_1(t_1), v_2(t_1), \ldots, w_i(t_1)., v_N(t_1)) + g_i(t_1, w_1(t_1), w_2(t_1), dots, v_i(t_1), v_N t_1))
\end{aligned}$$

using the fact that f_i and g_i are nondecreasing and nonincreasing in their u_i components. This is a contradiction. Hence, $v_i(t) < w_i(t)$ on J.

To obtain the result for non-strict inequality, we consider

$$v_{i,\epsilon}(t) = v_i(t) - \epsilon E_q \left(N \left(\sum_{i=1}^{i=N} (L_i + M_i) \right) + 1 \right) t^q)$$

$$w_{i,\epsilon}(t) = w_i(t) + \epsilon E_q\left(N\left(\sum_{i=1}^{i=N}(L_i + M_i)\right) + 1\right)t^q)$$

for each $i = 1, 2, \ldots, N$.

Then using Lipschitz condition of f_i and g_i in u , we have

$$\begin{aligned}
{}^C D^q v_{i,\epsilon} &\leq f_i(t_1, v_1(t_1), v_2(t_1), \ldots, v_i(t_1), v_N(t_1)) + g_i(t_1, w_1(t_1), w_2(t_1), \ldots, w_i(t_1), w_N t_1)) \\
&\quad - (N(\sum_{i=1}^{i=N}(L_i + M_i)) + 1)\epsilon E_q((\sum_{i=1}^{i=N}(L_i + M_i) + 1)t^q) \\
&\leq f_i(t, v_{1,\epsilon}, v_{2,\epsilon}, \ldots, v_{N,\epsilon}) + g_i(t, w_{1,\epsilon}, w_{2,\epsilon}, .w_{iN\epsilon}) \\
&\quad - \epsilon(N(\sum_{(j=1, j\neq i)}^{j=N}(L_i + M_i))E_q(N(\sum_{i=1}^{i=N}(L_i + M_i)) + 1)t^q) \\
&< f_i(t, v_{1,\epsilon}, v_{2,\epsilon}, \ldots, v_{N,\epsilon}) + g_i(t, w_{1,\epsilon}, w_{2,\epsilon}, .w_{iN\epsilon}).
\end{aligned}$$

This proves that

$${}^C D^q v_{i,\epsilon} < f_i(t, v_{1,\epsilon}, v_{2,\epsilon}, \ldots, v_{N,\epsilon}) + g_i(t, w_{1,\epsilon}, w_{2,\epsilon}, .w_{N\epsilon}).$$

From the strict inequality result we have

$$v_{i,\epsilon}(t) < w_{i,\epsilon}(t)$$

for $i = 1, 2, \ldots, N$. Letting $\epsilon \longrightarrow 0$, we can conclude $v_{01}(t) \leq w_{01}(t)$. □

We note that the conclusion of the above result is valid under the weaker assumption that f_i, and g_i are quasi-monotone nondecreasing and quasi-monotone non increasing in u respectively. We also note that to obtain the result for non-strict inequality result in the proof we can assume that

$$v_{i,\epsilon}(t) = v_i(t) - \epsilon E_q(N((L + M)) + 1)t^q)$$

$$w_{i,\epsilon}(t) = w_i(t) + \epsilon E_q(N((L + M)) + 1)t^q)$$

where $L = \max_{i=1,2,\ldots,N} L_i$, $M = \max_{i=1,2,\ldots,N} M_i$.

In particular if $f(t, u)$ and $g(t, u)$ are linear then we can obtain the following corollary as a special case of Theorem 2.2. For that purpose let

$$f_i(t, u) = \sum_{j=1}^{N} L_{ij} u_j, \quad \text{for } i = 1, 2, \ldots, N$$

and

$$g_i(t, u) = \sum_{j=1}^{N} -M_{ij} u_j, \quad \text{for } i = 1, 2, \ldots, N.$$

Also, let $p_i = v_i(t) - w_i(t)$. Then we have the following result as a special case of Theorem 2.2.

Corollary 2.3. *Let*

$$^{C}D^{q}p_i(t) \leq \sum_{j=1}^{N}(L_{ij}+M_{ij})p_j, \quad \textit{for } i=1,2,\ldots,N.$$

Then we have $p_i(t) \leq 0$, for $i=1,2,\ldots,N$, on $J=[0,T]$, whenever $p_i \leq 0$, for $i=1,2,\ldots,N$.

3. MAIN RESULTS

In this section we develop generalized monotone method for the nonlinear fractional differential system (2.2) combined with the method of coupled upper and lower solutions relative to 2.2. We obtain monotone sequences which converges uniformly and monotonically to coupled minimal and maximal solutions of (2.2). Finally, using the special case of the comparison Theorem 2.2, namely its corollary and the one sided lipschitz condition of f_i and g_i for $i=1,2,\ldots,N$ in its u components, we prove that there exists a unique solution of the system (2.2). The next result proves the existence of coupled minimal and maximal solution of (2.2).

Theorem 3.1. *Let $f_i, g_i \in C[J\times R^N, R^N]$ such that $f_i(t,u)$ is nondecreasing in u and $g_i(t,u)$ is nonincreasing in u for $t\in J$, and for each $i=1,2,\ldots,N$. Let $v_0, w_0 \in C^1[J,R^N]$ be coupled lower and upper solutions of (2.2), such that $v_{0i}(t) \leq w_{0i}(t)$ for $i=1,2,N$ on J. Then there exists monotone sequences $\{v_n\}$ and $\{w_n\}$ which converges uniformly and monotonically to coupled minimal and maximal solutions of (2.2) such that $v_n \to v$ and $w_n \to w$ as $n\to\infty$, provided $v_{0i}(0) \leq u_i(0) \leq w_{0i}(0)$ for $i=1,2,\ldots,N$. Further if u is any solution of (2.1) such that $v_{0i} \leq u_i \leq w_{0i}$, then $v \leq u \leq w$ on J.*

Proof. We define the sequences $\{v_n\}$ and $\{w_n\}$ as follows:

$$^{C}D^{q}v_{ni} = f_i(t, v_{n-1,1}, v_{n-1,2}, \ldots, v_{n-1,N}) + g_i(t, w_{n-1,1}, w_{n-1,2}, \ldots, w_{n-1,N}), \tag{3.1}$$

$$v_{n,i}(0) = u_{0i} \tag{3.2}$$

$$^{C}D^{q}w_{ni} = f_i(t, w_{n-1,1}, w_{n-1,2}, \ldots, w_{n-1,N}) + g_i(t, v_{n-1,1}, v_{n-1,2}, \ldots, v_{n-1,N}), \tag{3.3}$$

$$w_{n,i}(0) = u_{0i}. \tag{3.4}$$

It is easy to observe that each $v_{1i}(t)$ and $w_{1i}(t)$ for each $i=1,2,\ldots,N$ are solutions of scalar linear equation of the form (2.1) with $\lambda=0$. They can be computed without involving Mittag-Lefleur function. The solution is unique as well. Now continuing the process, the solution $v_{ni}(t)$ and $w_{ni}(t)$ for each $i=1,2,\ldots,N$, exists and is unique for each $n\in N$.

Initially, we prove that $v_{0i} \leq v_{1i} \leq w_{1i} \leq w_{0i}$ for each $i=1,2,\ldots,N$. For that purpose, set $p_i(t) = v_{0i}(t) - v_{1i}(t)$ which implies that $p_i(0) = v_{0i}(0) - v_{1i}(0) \leq$

$u_{0i} - u_{0i} = 0$. From the construction of v_{1i}, the definition of coupled lower solution and the hypotheses we get,

$$\begin{aligned} {}^{C}D^{q}p_i(t) \leq\ & f_i(t, v_{01}, v_{02}, \ldots, v_{0N}) + g_i(t, w_{01}, w_{02}, \ldots, w_{0N}) \\ & - f_i(t, v_{01}, v_{02}, \ldots, v_{0N}) + g_i(t, w_{01}, w_{02}, \ldots, w_{0N}) = 0. \end{aligned}$$

Thus, $p_i \leq 0$ on J from scalar version of Corollary 2.3. This proves $v_{0,i} \leq v_{1,i}$ on J for $i = 1, 2, \ldots, N$. Similarly we can prove $w_{1,i} \leq w_{0,i}$ on J.

Now we will prove $v_{1,i} \leq w_{1,i}$ on J for each $i = 1, 2, \ldots, N$. For that purpose, let $p_i(t) = v_{1,i} - w_{1,i}$. Then $p_i(0) = v_{1,i}(0) - w_{1,i}(0) = 0$. Since $v_{0i} \leq w_{0i}$, and by the nondecreasing and nonincreasing nature of f_i and g_i, it follows that

$$\begin{aligned} {}^{C}D^{q}p_i(t) \leq\ & f_i(t, v_{01}, v_{02}, \ldots, , v_{0N}) + g_i(t, w_{01}, w_{02}, \ldots, , w_{0N}) \\ & - f_i(t, w_{01}, w_{02}, \ldots, w_{0N}) + g_i(t, v_{01}, v_{02}, \ldots, v_{0N}) = 0. \end{aligned}$$

Then from scalar version of Corollary 2.3, we have $v_{1,i} \leq w_{1,i}$ on J, for $i = 1, 2, \ldots, N$.

Let u_i for $i = 1, 2, \ldots, N$ be any solution of (2.1) such that $v_{0i} \leq u_i \leq w_{0i}$ on J, then we can prove $v_{1i} \leq u_i \leq w_{1i}$. In fact we can prove $v_{0i} \leq v_{1i} \leq u_i \leq w_{1i} \leq w_{0i}$. We will prove $v_{1i} \leq u_i$, for $i = 1, 2, \ldots, N$ and the proof for $u_i \leq w_{1,i}$ follows on the same lines.

Now let $p_i = v_{1,i} - u_i$. Then, we have $p_i(0) = 0$. Also

$$\begin{aligned} {}^{C}D^{q}p_i(t) &=^{C} D^{q}v_{1i} -^{C} D^{q}u_i \\ &= f_i(t, v_{0i}, v_{02}, \ldots, v_{0N}) + g_i(t, w_{0i}, w_{02}, \ldots, w_{0N}) \\ &\quad - [f_i(t, u_1, u_2, \ldots, u_{0N}) + g_i(t, u_1, u_2, \ldots, u_{0N})] \leq 0, \end{aligned}$$

since $v_{0i} \leq u_{0i} \leq w_{0i}$. Thus we have ${}^{C}D^{q}p_i(t) \leq 0$ and $p_i(0) = 0$ which implies $p_i(t) \leq 0$, on J. This proves that $v_{1i} \leq u_i$ for $i = 1, 2, \ldots, N$ on J. Similarly we can prove $u_i \leq w_{1i}$ on J. Now it follows that $v_{1i} \leq u_i \leq w_{0i}$, on J, for $i = 1, 2, \ldots, N$. From the result we have $v_{0i} \leq v_{1i} \leq u_i \leq w_{1i} \leq w_{0i}$ on J. Now we claim that sequences defined by (3.1—3.2) and (3.3—3.4) will satisfy.

$$v_{0i} \leq v_{1i} \leq v_{2i} \leq v_{ni} \leq u_i \leq w_{ni} \leq w_{2i} \leq w_{1i} \leq w_{0i}$$

on J, for $i = 1, 2, \ldots, N$, for all n. We will prove this by method of mathematical induction. We assume that the inequalities above hold true for some $n = k$. Then we will prove that is true for $n = k + 1$. Setting $p_i(t) = v_{ki} - v_{k+1,i}$, we have $p_i(0) = 0$ Then

$$\begin{aligned} {}^{C}D^{q}p_i &= {}^{C}D^{q}v_{k,i} - {}^{C}D^{q}v_{k+1,i} \\ &= f_i(t, v_{k-1,1}, v_{k-1,2}, \ldots, v_{k-1,N}) + g_i(t, w_{k-1,1}, w_{k-1,2}, \ldots, w_{k-1,N}) \\ &\quad - [f_i(t, v_{k,1}, v_{k,2}, \ldots, v_{k,N}) + g_i(t, w_{k,1}, w_{k,2}, \ldots, w_{k,N})] \leq 0 \end{aligned}$$

on J, since $v_{k-1,i} \leq v_{k,i}$ and $w_{k+1,i} \leq w_{k,i}$. Similarly we can prove $w_{k+1,i} \leq w_{k,i}$. Next we prove $v_{k+1,,i} \leq u_i$. Setting $p_i(t) = v_{k+1,i}(t) - u_i$, we have $p_i(0) = 0$. Then,

$$\begin{aligned} {}^C D^q p_i(t) &= {}^C D^q v_{k+1,i} - {}^C D^q u_i \\ &= f_i(t, v_{k,1}, v_{k,2}, \ldots, v_{k,N}) + g_i(t, w_{k,1}, w_{k,2}, \ldots, v_{k,N}) \\ &\quad - [f_i(t, u_1, u_2, \ldots, u_N) + g_i(t, u_1, u_2, \ldots, u_N)] \leq 0 \end{aligned}$$

on J, since $v_{k,i} \leq u_i$ and $u_i \leq w_{k,i}$ for $i = 1, 2$. This implies $p_i(t) \leq 0$ on J, which proves $v_{k+1,i} \leq u_i$. On the same lines we can prove $u_i \leq w_{k+1,i}$. Since $v_{n,i}(t)$ and $w_{n,i}(t)$ are continuous functions on a closed bounded set on

$$\Omega = (t, u)|t \in J, v_0 \leq u \leq w_0,$$

one can easily prove that sequences $\{v_{n,i}\}$ and $\{w_{n,i}\}$ are uniformly bounded. Using the integral representation of $v_{n,i}(t)$ and $w_{n,i}(t)$ we can show that the sequences $\{v_{n,i}\}$ and $\{w_{n,i}\}$ for $i = 1, 2, \ldots, N$ are equi-continuous on J.

Hence by Ascoli-Arzela theorem there exist subsequences $\{v_{n_k,i}\}$ and $\{w_{n_k,i}\}$ for $i = 1, 2, \ldots, N$ which converge uniformly and monotonically to ρ_i and r_i for $i = 1, 2, \ldots, N$, respectively. Since the sequences are monotone the entire sequence converges. This proves that

$$(3.5) \qquad {}^C D^q \rho i = f_i(t, \rho_1, \rho_2, \ldots, \rho_N) + g_i(t, r_1, r_2, \ldots, r_N) \quad \rho_i(0) = u_{0i}$$

$$(3.6) \qquad {}^C D^q r_i = f_i(t, r_1, r_2, \ldots, r_N) + g_i(t, \rho_1, \rho_2, \ldots, \rho_N) \quad r_i(0) = u_{0i}$$

for $i = 1, 2, \ldots, N$. This also proves that $\rho_i(t) \leq r_i(t)$ for $i = 1, 2, \ldots, N$, on J. □

In the next result we prove the uniqueness of the solution of fractional dynamic systems (2.2).

Theorem 3.2 (Uniqueness)**.** *Let the hypothesis of Theorem* 3.1 *hold. Further let f_i and g_i satisfy the one sided Lipschitz condition as in Theorem* 2.2*. Then $\rho_i = u_i = r_i$ for $i = 1, 2, \ldots, N$, are the unique solutions of the system* (2.2)*.*

Proof. From Theorem 3.1 we have $\rho_i(t) \leq r_i(t)$ on J, for $i = 1.2, \ldots, N$. It is enough to prove that $r_i(t) \leq \rho_i(t)$ for $i = 1, 2, \ldots, N$, on J. This can be achieved by setting $p_i(t) = r_i(t) - \rho_i(t)$, using the one sided Lipschitz condition on f_i, g_i and Corollary 2.3. This enable us to prove that $r_i(t) \leq \rho_i(t)$ on J, for $i = 1.2, \ldots, N$. This concludes the proof. □

Conclusion: In this paper we have developed generalized monotone method for nonlinear fractional differential systems via coupled lower and upper solutions of type 1. The advantage of the generalized monotone method is that each component of the iterates are scalar fractional differential equation whose solution is easy to compute compared to computing the solution corresponding linear system. In addition, when we use generalized monotone method the computation of the solution of these scalar fractional differential equation is considerably easier, since we do not require the computation of Mittag-Lefleur function.

REFERENCES

[1] Z. Denton and A.S. Vatsala. Monotone Iterative Technique for Finite Systems of Nonlinear Riemann-Lioville Fractional Differential Equations, Opuscula Mathematica, Vol. 31, No. 3, 2011 .

[2] A.A. Kilbas, H.M. Srivastava, and J.J Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier, North Holland, 2006.

[3] G.S. Ladde, V. Lakshmikantham and A.S. Vatsala. *Monotone Iterative Techniques for Nonlinear Differential Equations*. Pitman, (1985).

[4] V. Lakshmikantham, S. Leela, and D.J. Vasundhara Devi. *Theory of fractional dynamic systems*. Cambridge Scientific Publishers, 2009.

[5] B. Oldham and J. Spanier. *The Fractional Calculus*. Academic Press, New York, London, 1974.

[6] I. Podlubny. *Fractional Differential Equations*. Academics Press, San Diego, 1999.

[7] M. Rivero,Juan J. Trujillo, and M.P. Velasco. On Deterministic Fractional Models, *New Trends in Nanotechnology and Fractional Caculus Applications*, Edited by D. Baleanu et al.pp 123–150 (2010) Springer.

[8] M. Sokol and A.S. Vatsala, A unified exhaustive study of monotone iterative method for initial value problems, *Nonlinear Studies*, 8 (4), 429–438 (2001).

[9] A.S. Vatsala and I. H. West. Generalized Monotone Iterative Method for Initial Value Problems, *Applied Mathematics Letters*, 17 (2004) 1231–1237.