Dynamic Systems and Applications 20 (2011) 505-518

A QUENCHING PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE IN AN INFINITE STRIP

C. Y. CHAN AND P. TRAGOONSIRISAK

Department of Mathematics, University of Louisiana at Lafayette
Lafayette, Louisiana 70504, USA
chan@louisiana.edu

Department of Mathematics and Computer Science, Fort Valley State University
Fort Valley, GA 31030, USA
tragoonsirisakp@fvsu.edu

ABSTRACT. This article studies a semilinear parabolic initial-boundary value problem with a concentrated nonlinear source in an infinite strip in the N-dimensional Euclidean space. Existence, uniqueness, and locations where quenching occurs for the solution are investigated.

AMS (MOS) Subject Classification. 35K60, 35B35, 35K55, 35K57

1. INTRODUCTION

Let a point $(x_1, x_2, \ldots, x_{N-1}, x_N)$ in the N-dimensional Euclidean space $\mathbb{R}^N$ be denoted by $(x, \tilde{x})$ with x standing for x_1, L and b be positive numbers such that $b < L$, $S = (-L, L) \times \mathbb{R}^{N-1}$, $s = (-b, b) \times \mathbb{R}^{N-1}$, $\partial S = \left\{(x, \tilde{x}) : x \in \{-L, L\}\text{, and } \tilde{x} \in \mathbb{R}^{N-1}\right\}$, $\partial s = \left\{(x, \tilde{x}) : x \in \{-b, b\}\text{, and } \tilde{x} \in \mathbb{R}^{N-1}\right\}$, $\nu(x, \tilde{x})$ denote the unit outward normal at $(x, \tilde{x}) \in \partial s$, and $\chi_s(x, \tilde{x})$ denote a function which is 1 for $|x| > b$, and 0 for $|x| < b$. Since the Dirac delta function is the derivative of the Heaviside function, it follows that $\partial \chi_s(x, \tilde{x}) / \partial \nu$ gives a Dirac delta function at each point on $x = |b|$, and is zero everywhere else (cf. Chan and Tragoonsirisak [3]), and hence we have a concentrated source on ∂s. We would like to study the following problem with a concentrated nonlinear source on ∂s:

$$(1.1) \qquad \begin{cases} u_t - \triangle u = \alpha \dfrac{\partial \chi_s(x, \tilde{x})}{\partial \nu} f(u) \text{ in } S \times (0, T], \\ u(x, 0) = 0 \text{ on } \bar{S},\ u(x, t) = 0 \text{ on } \partial S \times (0, T], \end{cases}$$

where α and T are positive real numbers, $\bar{S}$ is the closure of S, f is a given function such that $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c, and $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $0 \leq u < c$. We note that a similar problem without a concentrated source was studied by Dai and Gu [6]. For problems involving a

Received August 22, 2011

concentrated nonlinear source on the surface of a ball in $\mathbb{R}^N$, we refer to the papers by Chan and Tragoonsirisak ([3], [4], [5]).

Let $H = \partial/\partial t - \partial^2/\partial x^2$, $D = (0, L)$, $\bar{D} = [0, L]$, and $\Omega = D \times (0, T]$. Due to symmetry, the problem (1.1) is equivalent to the following one-dimensional problem:

$$\left\{ \begin{array}{c} Hu = \alpha\delta\,(x-b)\, f(u) \text{ in } \Omega, \\ u(x,0) = 0 \text{ on } \bar{D},\ u_x\,(0,t) = u\,(L,t) = 0 \text{ for } 0 < t \le T, \end{array} \right. \tag{1.2}$$

where $\delta\,(x-b)$ is the Dirac delta function. Thus, the results obtained in this paper are applicable not only to an infinite strip with $N \ge 2$, but also to $N = 1$ for a one-dimensional problem with mixed boundary conditions. The term $\delta\,(x-b)$ implies that u_x has a jump discontinuity at $x = b$. Therefore, a solution u is at most a continuous function satisfying (1.2).

A solution u is said to quench if there exists an extended real number $t_q \in (0, \infty]$ such that

$$\sup\left\{u(x,t) : x \in \bar{D}\right\} \to c^- \text{ as } t \to t_q.$$

If $t_q < \infty$, then u is said to quench in a finite time. If $t_q = \infty$, then u quenches in infinite time.

In Section 2, we show that the nonlinear integral equation corresponding to the problem (1.2) has a unique nonnegative continuous solution u, which is a strictly increasing function of t for $x \in D$. We then prove that u is the unique solution of the problem (1.2). In Section 3, we show that if t_q is finite, then u quenches at $x = b$ only.

2. EXISTENCE AND UNIQUENESS

Green's function $g\,(x,t;\xi,\tau)$ (cf. Stakgold [8, pp. 197–203]) corresponding to the problem (1.2) with mixed boundary conditions is determined by the following system:

$$Hg = 0 \text{ for } x, \xi \in D \text{ and } 0 < t, \tau < \infty,$$
$$\lim_{t\to\tau^+} g\,(x,t;\xi,\tau) = \delta\,(x-\xi) \text{ for } x, \xi \in D,$$
$$g_x\,(0,t;\xi,\tau) = g\,(L,t;\xi,\tau) = 0 \text{ for } \xi \in D \text{ and } 0 < t, \tau < \infty.$$

By the method of eigenfunction expansions,

$$g\,(x,t;\xi,\tau) = \frac{2}{L}\sum_{n=1}^{\infty} \cos\left(\frac{(2n-1)\,\pi x}{2L}\right) \cos\left(\frac{(2n-1)\,\pi\xi}{2L}\right) \exp\left(-\frac{(2n-1)^2\,\pi^2\,(t-\tau)}{4L^2}\right) \tag{2.1}$$

(cf. Trim [10, pp. 474–478]). By using Green's second identity and the adjoint operator L^*, which is given by $L^*u = -u_t - u_{xx}$, the problem (1.2) is converted into the nonlinear integral equation,

$$u(x,t) = \alpha \int_0^t g\,(x,t;b,\tau)\, f\,(u\,(b,\tau))\, d\tau. \tag{2.2}$$

We modify the techniques in proving Lemma 2.2(a) of Chan and Tian [2] for a blow-up problem to establish the following result.

Lemma 2.1. *For* $(x,t;\xi,\tau) \in \left(\bar{D}\times(\tau,T]\right)\times\left(\bar{D}\times[0,T)\right)$, $g(x,t;\xi,\tau)$ *is continuous.*

Proof. From (2.1),

$$\begin{aligned} |g(x,t;\xi,\tau)| &\le \frac{2}{L}\sum_{n=1}^{\infty}\exp\left(-\frac{(2n-1)^2\pi^2(t-\tau)}{4L^2}\right) \\ &\le \frac{2}{L}\sum_{n=1}^{\infty}\exp\left(-\frac{n^2\pi^2(t-\tau)}{4L^2}\right) \\ &\le \frac{2}{L}\sum_{n=1}^{\infty}\exp\left(-\frac{n\pi^2(t-\tau)}{4L^2}\right), \end{aligned} \tag{2.3}$$

which is a geometric series with the common ratio $\exp\left(-\pi^2(t-\tau)/(4L^2)\right)$. Hence for t in any compact subset of (τ,T),

$$\sum_{n=1}^{\infty}\exp\left(-\frac{n\pi^2(t-\tau)}{4L^2}\right)=\frac{1}{\exp\left(\dfrac{\pi^2(t-\tau)}{4L^2}\right)-1}.$$

By using (2.3) and the Weierstrass M-Test (cf. Stromberg [9, pp. 141–142]), $g(x,t;\xi,\tau)$ converges uniformly on $\bar{D}$ for t in any compact subset of (τ,T). This proves the lemma. □

A physical interpretation of $g(x,t;\xi,\tau)$ is the temperature at the point x on a one-dimensional uniform and homogeneous rod of length L at time t due to a point source situated at the point ξ on the rod at time τ with $\tau<t$; the rod has no heat source, and is subject to an insulated boundary condition at $x=0$ and a zero boundary condition at $x=L$. Thus for $t>\tau$, $g(x,t;\xi,\tau)$ should be positive inside the rod. Our next result proves this positivity property.

Lemma 2.2. *For* $x,\xi\in D$ *and* $0\le\tau<t\le T$, $g(x,t;\xi,\tau)$ *is positive.*

Proof. Let us assume that $g(x,t;\xi,\tau)<0$ somewhere in

$$D_1=\{(x,t;\xi,\tau): x,\xi\in D \text{ and } 0\le\tau<t\le T\}.$$

If g attains its minimum somewhere on the boundary $x=0$, then by the parabolic version of Hopf's lemma (cf. Friedman [7, p. 49]), $g_x>0$ at that point. This contradicts the given boundary condition $g_x=0$ there. Since g is zero on the boundary $x=L$, it follows from $Hg=0$ that g must attain its minimum m somewhere, say $(\bar{x},t_1;\xi_1,\tau_1)$ with $t_1>\tau_1$ in D_1. By the strong maximum principle (cf. Friedman [7, p. 34]), $g(x,t;\xi_1,\tau_1)=m$ for $x\in D$ and $t\in(\tau_1,t_1]$. By Lemma 2.1, g is continuous for $x\in\bar{D}$. Thus, $g=m$ with $t\in(\tau_1,t_1]$ at the boundary $x=L$. This contradiction shows that $g(x,t;\xi,\tau)\ge 0$. Suppose $g(x,t;\xi,\tau)=0$ at some point $(\hat{x},t_2;\xi_2,\tau_2)$ in D_1.

Since $Hg = 0$, it follows from the strong maximum principle that $g(x, t; \xi_2, \tau_2) = 0$ for $x \in D$ and $t \in (\tau_2, t_2]$. On the other hand,

$$g(\xi_2, t_2; \xi_2, \tau_2) = \frac{2}{L} \sum_{n=1}^{\infty} \cos^2 \left(\frac{(2n-1)\pi\xi_2}{2L} \right) \exp \left(-\frac{(2n-1)^2 \pi^2 (t_2 - \tau_2)}{4L^2} \right) > 0,$$

which gives a contradiction. The lemma is then proved. □

Our next result shows that for any continuous function $r(t)$ for $0 \le t \le T$, the function $\int_0^t g(x,t;b,\tau) r(\tau) d\tau$ is continuous.

Lemma 2.3. *If $r(t) \in C([0,T])$, then $\int_0^t g(x,t;b,\tau) r(\tau) d\tau$ is continuous for $x \in \bar{D}$ and $t \in [0,T]$.*

Proof. Let

$$K = \max_{0 \le \tau \le T} r(\tau),$$

and ϵ be any positive number such that $t - \epsilon > 0$. For any $x \in \bar{D}$ *and* $\tau \in [0, t-\epsilon]$,

$$\begin{aligned} &\frac{2}{L} \int_0^{t-\epsilon} \left| \cos\left(\tfrac{(2n-1)\pi x}{2L}\right) \cos\left(\tfrac{(2n-1)\pi b}{2L}\right) \exp\left(-\tfrac{(2n-1)^2\pi^2(t-\tau)}{4L^2}\right) r(\tau) \right| d\tau \\ &\le \frac{2K}{L} \int_0^{t-\epsilon} \exp\left(-\frac{(2n-1)^2 \pi^2 (t-\tau)}{4L^2} \right) d\tau \\ &\le \frac{2K}{L} \int_0^{t-\epsilon} \exp\left(-\frac{n^2\pi^2 (t-\tau)}{4L^2} \right) d\tau \\ &\le \frac{8KL}{n^2\pi^2} \exp\left(-\frac{n^2\pi^2\epsilon}{4L^2} \right). \end{aligned} \tag{2.4}$$

It follows from $g(x,t;b,\tau)$ converging uniformly for t in any compact subset of $(\tau, T]$ that we can interchange integration and summation (cf. Wade [11, p. 190]). By (2.4),

$$\int_0^{t-\epsilon} g(x,t;b,\tau) r(\tau) d\tau \le \frac{8KL}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{n^2\pi^2\epsilon}{4L^2} \right) \le \frac{8KL}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ (cf. Stromberg [9, p. 518]), we have

$$\int_0^{t-\epsilon} g(x,t;b,\tau) r(\tau) d\tau \le \frac{4KL}{3},$$

which is independent of ϵ. By the Weierstrass M-test, $\int_0^t g(x,t;b,\tau) r(\tau) d\tau$ converges uniformly with respect to x and t, and hence is continuous for $x \in \bar{D}$ and $t \in [0,T]$.
□

We modify the techniques in proving Theorems 1 and 2 of Chan and Jiang [1] for a first initial-boundary value problem to establish the next two results.

Theorem 2.4. *There exists some t_q such that for $0 \le t < t_q$, the integral equation* (2.2) *has a unique continuous nonnegative solution u, and u is a strictly increasing function of t in D. If t_q is finite, then u quenches at t_q.*

Proof. Let us construct a sequence $\{u_n\}$ in Ω by $u_0(x,t)=0$, and for $n=0,1,2,\ldots$,

$$Hu_{n+1}=\alpha\delta(x-b)f(u_n) \text{ in } \Omega,$$
$$u_{n+1}(x,0)=0 \text{ for } x\in\bar{D},$$
$$\frac{\partial}{\partial x}u_{n+1}(0,t)=0=u_{n+1}(L,t) \text{ for } 0<t\le T.$$

From (2.2),

$$u_{n+1}(x,t)=\alpha\int_0^t g(x,t;b,\tau)f(u_n(b,\tau))d\tau. \tag{2.5}$$

Since $f(0)>0$, and $g(x,t;b,\tau)>0$, it follows from (2.5) that $u_1(x,t)>u_0(x,t)$ in Ω. Using the principle of mathematical induction, we have $0<u_1<u_2<\cdots<u_{n-1}<u_n$ in Ω for any positive integer n.

To show that each u_n is an increasing function of t in D, we construct a sequence $\{w_n\}$ such that for $n=0,1,2,\ldots$, $w_n(x,t)=u_n(x,t+h)-u_n(x,t)$, where h is any positive number less than T. Then, $w_0(x,t)=0$. By (2.5), we have

$$w_1(x,t)=\alpha f(0)\left(\int_0^{t+h}g(x,t+h;b,\tau)d\tau-\int_0^t g(x,t;b,\tau)d\tau\right).$$

Let $\sigma=\tau-h$. Then,

$$\begin{aligned}\int_0^{t+h}g(x,t+h;b,\tau)d\tau&=\int_0^h g(x,t+h;b,\tau)d\tau+\int_0^t g(x,t+h;b,\sigma+h)d\sigma\\&=\int_0^h g(x,t+h;b,\tau)d\tau+\int_0^t g(x,t;b,\sigma)d\sigma\end{aligned}$$

since $g(x,t+h;b,\sigma+h)=g(x,t;b,\sigma)$. Thus in D, we have for $0<t\le T-h$,

$$w_1(x,t)=\alpha f(0)\int_0^h g(x,t+h;b,\tau)d\tau>0.$$

In D, let us assume that for some positive integer j, $w_j>0$ for $0<t\le T-h$. Then,

$$w_{j+1}(x,t)=\alpha\left(\int_0^{t+h}g(x,t+h;b,\tau)f(u_j(b,\tau))d\tau-\int_0^t g(x,t;b,\tau)f(u_j(b,\tau))d\tau\right).$$

Let $\sigma=\tau-h$. We have

$$\begin{aligned}&\int_0^{t+h}g(x,t+h;b,\tau)f(u_j(b,\tau))d\tau\\&=\int_0^h g(x,t+h;b,\tau)f(u_j(b,\tau))d\tau+\int_0^t g(x,t+h;b,\sigma+h)f(u_j(b,\sigma+h))d\sigma\\&=\int_0^h g(x,t+h;b,\tau)f(u_j(b,\tau))d\tau+\int_0^t g(x,t;b,\sigma)f(u_j(b,\sigma+h))d\sigma\\&>\int_0^h g(x,t+h;b,\tau)f(u_j(b,\tau))d\tau+\int_0^t g(x,t;b,\sigma)f(u_j(b,\sigma))d\sigma. \qquad (2.6)\end{aligned}$$

Thus in D,

$$w_{j+1}(x,t)>\alpha\int_0^h g(x,t+h;b,\tau)f(u_j(b,\tau))d\tau>0.$$

By the principle of mathematical induction, $w_n > 0$ in D for $0 < t \leq T - h$ and all positive integers n. Thus, each u_n is an increasing function of t in D.

For any given positive constant M $(< c)$, it follows from (2.5) and u_n being an increasing function of t in D that there exists some t_1 such that $u_{n+1} \leq M$ for $0 \leq t \leq t_1$ and $n = 0, 1, 2, \ldots$. In fact, t_1 satisfies

$$u_{n+1}(x, t_1) \leq \alpha f(M) \int_0^{t_1} g(x, t_1; b, \tau) d\tau \leq M.$$

Let u denote $\lim_{n\to\infty} u_n$. From (2.5) and the Monotone Convergence Theorem (cf. Stromberg [9, pp. 266–268]), we have (2.2) for $0 \leq t \leq t_1$.

To prove that u is unique, we assume that the integral equation (2.2) has two distinct solutions u and $\tilde{u}$ on the interval $[0, t_1]$. From (2.2),

$$u(x, t) - \tilde{u}(x, t) = \alpha \int_0^t g(x, t; b, \tau) \left(f(u(b, \tau)) - f(\tilde{u}(b, \tau))\right) d\tau. \tag{2.7}$$

Since $f''(u) > 0$ for $u \in [0, c)$, it follows from the Mean Value Theorem that $|f(u) - f(\tilde{u})| \leq f'(M) |u - \tilde{u}|$. From (2.7),

$$|u(x, t) - \tilde{u}(x, t)| \leq \alpha f'(M) \int_0^t g(x, t; b, \tau) \left|u(b, \tau) - \tilde{u}(b, \tau)\right| d\tau.$$

By Lemma 2.3, there exists some t_2 $(\leq t_1)$ such that

$$\alpha f'(M) \max_{\bar{D}\times[0,t_2]} \left(\int_0^t g(x, t; b, \tau) d\tau\right) < 1. \tag{2.8}$$

Let $\Theta = \max_{\bar{D}\times[0,t_2]} |u - \tilde{u}|$. Then,

$$\Theta \leq \alpha f'(M) \max_{\bar{D}\times[0,t_2]} \left(\int_0^t g(x, t; b, \tau) d\tau\right) \Theta.$$

By (2.8), this gives a contradiction. Thus, we have uniqueness of a solution for $0 \leq t \leq t_2$.

If $t_2 < t_1$, then for $t_2 \leq t \leq t_1$,

$$u(x, t) = \int_0^L g(x, t; \xi, t_2) u(\xi, t_2) d\xi + \alpha \int_{t_2}^t g(x, t; b, \tau) f(u(b, \tau)) d\tau \tag{2.9}$$

(cf. Chan and Tian [2]). Thus for $t_2 \leq t \leq t_1$,

$$u(x, t) - \tilde{u}(x, t) = \alpha \int_{t_2}^t g(x, t; b, \tau) \left(f(u(b, \tau)) - f(\tilde{u}(b, \tau))\right) d\tau.$$

Let $\tilde{\Theta} = \max_{\bar{D}\times[t_2,\min\{2t_2,t_1\}]} |u - \tilde{u}|$. Then,

$$\tilde{\Theta} \leq \alpha f'(M) \max_{\bar{D}\times[t_2,\min\{2t_2,t_1\}]} \left(\int_{t_2}^t g(x, t; b, \tau) d\tau\right) \tilde{\Theta}.$$

Let $\sigma = \tau - t_2$. Then for $t \in [t_2, \min\{2t_2, t_1\}]$,

$$\alpha f'(M) \max_{\bar{D}\times[t_2,\min\{2t_2,t_1\}]} \left(\int_{t_2}^t g(x, t; b, \tau) d\tau\right)$$

$$= \alpha f'(M) \max_{\bar{D}\times[t_2,\min\{2t_2,t_1\}]} \left(\int_0^{t-t_2} g(x,t;b,\sigma+t_2)d\sigma \right)$$

$$= \alpha f'(M) \max_{\bar{D}\times[t_2,\min\{2t_2,t_1\}]} \left(\int_0^{t-t_2} g(x,t-t_2;b,\sigma)d\sigma \right) < 1 \tag{2.10}$$

by (2.8). This gives a contradiction. Hence, we have uniqueness of a solution for $0 \le t \le \min\{2t_2, t_1\}$. By proceeding in this way, the integral equation (2.2) has a unique solution u for $0 \le t \le t_1$.

To prove that u is continuous on $\bar{D} \times [0, t_1]$, we note that $f(u_n(\xi,\tau))$ is bounded by $f(M)$. It follows from (2.5), Lemma 2.3 and f being continuous that for $n = 0, 1, 2, \ldots$, $u_{n+1}(x,t)$ is continuous on $\bar{D} \times [0, t_1]$. From (2.5),

$$u_{n+1}(x,t) - u_n(x,t) = \alpha \int_0^t g(x,t;b,\tau) \left(f(u_n(b,\tau)) - f(u_{n-1}(b,\tau)) \right) d\tau. \tag{2.11}$$

Using the Mean Value Theorem, we have

$$f(u_n) - f(u_{n-1}) \le f'(M) \left(u_n - u_{n-1} \right).$$

Let $\Lambda_n = \max_{\bar{D}\times[0,t_2]} (u_n - u_{n-1})$. From (2.11),

$$\Lambda_{n+1} \le \alpha f'(M) \max_{\bar{D}\times[0,t_2]} \left(\int_0^t g(x,t;b,\tau)d\tau \right) \Lambda_n.$$

By (2.8), the sequence $\{u_n(x,t)\}$ converges uniformly to $u(x,t)$ on $\bar{D} \times [0, t_2]$, and hence, u is continuous there.

If $t_2 < t_1$, then from (2.9),

$$u_{n+1}(x,t) = \int_0^L g(x,t;\xi,t_2)u(\xi,t_2)\,d\xi + \alpha \int_{t_2}^t g(x,t;b,\tau)f(u_n(b,\tau))d\tau.$$

Let $\tilde{\Lambda}_n = \max_{\bar{D}\times[t_2,\min\{2t_2,t_1\}]} (u_n - u_{n-1})$. Then for $t_2 \le t \le t_1$,

$$\begin{aligned} \tilde{\Lambda}_{n+1} &\le \alpha f'(M) \max_{\bar{D}\times[t_2,\min\{2t_2,t_1\}]} \left(\int_{t_2}^t g(x,t;b,\tau)d\tau \right) \tilde{\Lambda}_n \\ &= \alpha f'(M) \max_{\bar{D}\times[t_2,\min\{2t_2,t_1\}]} \left(\int_0^{t-t_2} g(x,t;b,t_2+\sigma)d\sigma \right) \tilde{\Lambda}_n \\ &= \alpha f'(M) \max_{\bar{D}\times[t_2,\min\{2t_2,t_1\}]} \left(\int_0^{t-t_2} g(x,t-t_2;b,\sigma)d\sigma \right) \tilde{\Lambda}_n. \end{aligned}$$

It follows from (2.10) that for $t \in [t_2, \min\{2t_2, t_1\}]$, u is continuous. By proceeding in this way, the integral equation (2.2) has a unique continuous solution u on $\bar{D} \times [0, t_1]$.

Let t_q be the supremum of the intervals for which the integral equation (2.2) has a unique continuous solution u. If t_q is finite, and u does not reach c^- at t_q, then for any positive constant between $\max_{\bar{D}} u(x, t_q)$ and c, a proof similar to the above shows that there exists some $t_3\,(> t_q)$ such that the integral equation (2.2) has a unique continuous solution u for $0 \le t \le t_3$. This contradicts the definition of t_q. Thus, if t_q is finite, then u reaches c^- somewhere at t_q.

Since u_n is an increasing function of t, we have for $x \in D$ and any positive number h such that $t+h < t_q$,

$$\begin{aligned} &u(x,t+h) - u(x,t) \\ &\quad = \alpha \left(\int_0^{t+h} g(x,t+h;b,\tau) f(u(b,\tau)) d\tau - \int_0^t g(x,t;b,\tau) f(u(b,\tau)) d\tau \right). \end{aligned}$$

As in the derivation of (2.6), we have

$$\begin{aligned} &\int_0^{t+h} g(x,t+h;b,\tau) f(u(b,\tau)) d\tau \\ &\quad > \int_0^h g(x,t+h;b,\tau) f(u(b,\tau)) d\tau + \int_0^t g(x,t;b,\sigma) f(u(b,\sigma)) d\sigma. \end{aligned}$$

Hence,

$$u(x,t+h) - u(x,t) > \alpha \int_0^h g(x,t+h;b,\tau) f(u(b,\tau)) d\tau > 0,$$

which shows that u is a strictly increasing function of t in D. □

Theorem 2.5. *The problem* (1.2) *has a unique solution* u *for* $0 \le t < t_q$.

Proof. For any $t_4 \in (0,t)$,

$$\begin{aligned} &\int_0^t g(x,t;b,\tau) f(u(b,\tau)) d\tau \\ &= \lim_{m\to\infty} \int_0^{t-1/m} g(x,t;b,\tau) f(u(b,\tau)) d\tau \\ &= \lim_{m\to\infty} \int_{t_4}^t \frac{\partial}{\partial\zeta} \left(\int_0^{\zeta-1/m} g(x,\zeta;b,\tau) f(u(b,\tau)) d\tau \right) d\zeta \\ &+ \lim_{m\to\infty} \int_0^{t_4-1/m} g(x,t_4;b,\tau) f(u(b,\tau)) d\tau. \end{aligned} \tag{2.12}$$

Differentiating $g(x,\zeta;b,\tau)$ with respect to ζ term by term for any $\tau \in [0,\zeta-1/m]$, we have

$$\begin{aligned} &\frac{2}{L} \left| \sum_{n=1}^{\infty} \left(-\tfrac{(2n-1)^2\pi^2}{4L^2} \right) \cos\left(\tfrac{(2n-1)\pi x}{2L} \right) \cos\left(\tfrac{(2n-1)\pi b}{2L} \right) \exp\left(-\tfrac{(2n-1)^2\pi^2(\zeta-\tau)}{4L^2} \right) \right| \\ &\le \frac{\pi^2}{2L^3} \sum_{n=1}^{\infty} (2n-1)^2 \exp\left(-\frac{(2n-1)^2\pi^2}{4mL^2} \right) \\ &\le \frac{\pi^2}{2L^3} \sum_{n=1}^{\infty} n^2 \exp\left(-\frac{n^2\pi^2}{4mL^2} \right), \end{aligned}$$

which converges by the Ratio Test, and by the Weierstrass M-Test, converges uniformly for t in any compact subset of (τ, t_q) (cf. Wade [11, pp. 190–191]). Thus, we can differentiate $g(x,t;\xi,\tau)$ with respect to t term by term, and $g_t(x,t;\xi,\tau)$ is bounded for $\tau \in [0,t)$.

By the Leibnitz Rule (cf. Stromberg [9, p. 380]),

$$\begin{aligned}&\frac{\partial}{\partial\zeta}\left(\int_0^{\zeta-1/m} g(x,\zeta;b,\tau)f(u(b,\tau))d\tau\right)\\&=g(x,\zeta;b,\zeta-1/m)f(u(b,\zeta-1/m))+\int_0^{\zeta-1/m}\frac{\partial}{\partial\zeta}g(x,\zeta;b,\tau)f(u(b,\tau))d\tau\\&=g(x,1/m;b,0)f(u(b,\zeta-1/m))+\int_0^{\zeta-1/m} g_\zeta(x,\zeta;b,\tau)f(u(b,\tau))d\tau.\end{aligned}$$

We have

$$\begin{aligned}&\lim_{m\to\infty}\int_{t_4}^t\frac{\partial}{\partial\zeta}\left(\int_0^{\zeta-1/m} g(x,\zeta;b,\tau)f(u(b,\tau))d\tau\right)d\zeta\\&=\lim_{m\to\infty}\int_{t_4}^t g(x,1/m;b,0)f(u(b,\zeta-1/m))d\zeta\\&\quad+\lim_{m\to\infty}\int_{t_4}^t\int_0^{\zeta-1/m} g_\zeta(x,\zeta;b,\tau)f(u(b,\tau))d\tau d\zeta.\end{aligned}$$

Since $g(x,1/m;b,0)$ is independent of ζ, and $f(u(b,\zeta-1/m))$ increases as m increases, it follows from the Monotone Convergence Theorem that

$$\begin{aligned}&\lim_{m\to\infty}\int_{t_4}^t g(x,1/m;b,0)f(u(b,\zeta-1/m))d\zeta\\&=\int_{t_4}^t\lim_{m\to\infty}g(x,1/m;b,0)f(u(b,\zeta-1/m))d\zeta\\&=\int_{t_4}^t\delta(x-b)f(u(b,\zeta))d\zeta\end{aligned}$$

since $\lim_{t\to\tau^+} g\left(x,t;\xi,\tau\right)=\delta\left(x-\xi\right)$. To show that

$$\begin{aligned}&\lim_{m\to\infty}\int_{t_4}^t\int_0^{\zeta-1/m} g_\zeta(x,\zeta;b,\tau)f(u(b,\tau))d\tau d\zeta\\&=\int_{t_4}^t\lim_{m\to\infty}\int_0^{\zeta-1/m} g_\zeta(x,\zeta;b,\tau)f(u(b,\tau))d\tau d\zeta,\end{aligned}\tag{2.13}$$

let

$$h_m(x,\zeta)=\int_0^{\zeta-1/m} g_\zeta(x,\zeta;b,\tau)f(u(b,\tau))d\tau.$$

Without loss of generality, let $m>l$. We have

$$h_m(x,\zeta)-h_l(x,\zeta)=\int_{\zeta-1/l}^{\zeta-1/m} g_\zeta(x,\zeta;b,\tau)f(u(b,\tau))d\tau.$$

Since $f(u(b,\tau))$ is an increasing function of τ, it follows from the Second Mean Value Theorem (cf. Stromberg [9, p. 328]) that for ζ in any compact subset of $(0,t_q)$, there exists some real number γ with $\zeta-\gamma\in[\zeta-1/l,\zeta-1/m]$ such that

$$\int_{\zeta-1/l}^{\zeta-1/m} g_\zeta(x,\zeta;b,\tau)f(u(b,\tau))d\tau$$

$$= f\left(u\left(b, \zeta - \frac{1}{l}\right)\right) \int_{\zeta-1/l}^{\zeta-\gamma} g_\zeta(x, \zeta; b, \tau) d\tau$$
$$+ f\left(u\left(b, \zeta - \frac{1}{m}\right)\right) \int_{\zeta-\gamma}^{\zeta-1/m} g_\zeta(x, \zeta; b, \tau) d\tau.$$

From (2.1), $g_\zeta(x, \zeta; b, \tau) = -g_\tau(x, \zeta; b, \tau)$. Thus, we have

$$\begin{aligned}
&h_m(x, \zeta) - h_l(x, \zeta) \\
&= -f\left(u\left(b, \zeta - \frac{1}{l}\right)\right) \int_{\zeta-1/l}^{\zeta-\gamma} g_\tau(x, \zeta; b, \tau) d\tau \\
&\quad - f\left(u\left(b, \zeta - \frac{1}{m}\right)\right) \int_{\zeta-\gamma}^{\zeta-1/m} g_\tau(x, \zeta; b, \tau) d\tau \\
&= f\left(u\left(b, \zeta - \frac{1}{l}\right)\right) \left(g\left(x, \zeta; b, \zeta - \frac{1}{l}\right) - g(x, \zeta; b, \zeta - \gamma)\right) \\
&\quad - f\left(u\left(b, \zeta - \frac{1}{m}\right)\right) \left(g\left(x, \zeta; b, \zeta - \frac{1}{m}\right) - g(x, \zeta; b, \zeta - \gamma)\right) \\
&= f\left(u\left(b, \zeta - \frac{1}{l}\right)\right) \left(g\left(x, \frac{1}{l}; b, 0\right) - g(x, \gamma; b, 0)\right) \\
&\quad - f\left(u\left(b, \zeta - \frac{1}{m}\right)\right) \left(g\left(x, \frac{1}{m}; b, 0\right) - g(x, \gamma; b, 0)\right)
\end{aligned}$$

which can be made as small as we wish by choosing l and m sufficiently large since $f(u)$ is bounded, and $g(x, \epsilon; b, 0)$ is continuous for any $\epsilon > 0$. Thus, $\{h_m\}$ is a Cauchy sequence converging uniformly with respect to ζ in any compact subset of $(0, t_q)$, and hence, we have (2.13) (cf. Wade [11, pp. 186–187]).

From (2.12),

$$\begin{aligned}
&\int_0^t g(x, t; b, \tau) f(u(b, \tau)) d\tau \\
&= \int_{t_4}^t \delta(x - b) f(u(b, \zeta)) d\zeta + \int_{t_4}^t \int_0^\zeta g_\zeta(x, \zeta; b, \tau) f(u(b, \tau)) d\tau d\zeta \\
&\quad + \int_0^{t_4} g(x, t_4; b, \tau) f(u(b, \tau)) d\tau.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\partial}{\partial t} \int_0^t g(x, t; b, \tau) f(u(b, \tau)) d\tau \\
&\quad = \delta(x - b) f(u(b, t)) + \int_0^t g_t(x, t; b, \tau) f(u(b, \tau)) d\tau. \qquad (2.14)
\end{aligned}$$

By Lemma 2.3, $\int_0^t g(x, t; b, \tau) f(u(b, \tau)) d\tau$ is continuous for $x \in \bar{D}$. Differentiating $g(x, t; b, \tau)$ with respect to x term by term for any $\tau \in [0, t - \epsilon]$ with ϵ being any positive number such that $t-\epsilon>0$, we have

$$\begin{aligned}
&\frac{2}{L}\left|\sum_{n=1}^{\infty}\left(-\tfrac{(2n-1)\pi}{2L}\right)\sin\left(\tfrac{(2n-1)\pi x}{2L}\right)\cos\left(\tfrac{(2n-1)\pi b}{2L}\right)\exp\left(-\tfrac{(2n-1)^2\pi^2(t-\tau)}{4L^2}\right)\right| \\
&\leq \frac{\pi}{L^2}\sum_{n=1}^{\infty}(2n-1)\exp\left(-\frac{(2n-1)^2\pi^2(t-\tau)}{4L^2}\right) \\
&\leq \frac{\pi}{L^2}\sum_{n=1}^{\infty}n\exp\left(-\frac{n^2\pi^2\epsilon}{4L^2}\right),
\end{aligned}$$

which converges by the Ratio Test, and by the Weierstrass M-Test, converges uniformly on $\bar{D}$. Thus, $g_x(x,t;b,\tau)$ exists, and is continuous on $\bar{D}\times(0,t)$. Let ϵ be any positive number such that $t-\epsilon>0$. For any $y\in D$ and $\tau\in[0,t-\epsilon]$,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^{t-\epsilon}g(x,t;b,\tau)f(u(b,\tau))d\tau \\
&=\lim_{\epsilon\to 0}\int_y^x\left(\frac{\partial}{\partial\rho}\int_0^{t-\epsilon}g(\rho,t;b,\tau)f(u(b,\tau))d\tau\right)d\rho+\lim_{\epsilon\to 0}\int_0^{t-\epsilon}g(y,t;b,\tau)f(u(b,\tau))d\tau \\
&=\lim_{\epsilon\to 0}\int_y^x\int_0^{t-\epsilon}g_\rho(\rho,t;b,\tau)f(u(b,\tau))d\tau d\rho+\int_0^t g(y,t;b,\tau)f(u(b,\tau))d\tau
\end{aligned}$$

(cf. Wade [11, pp. 319–320]). By the Fubini Theorem (cf. Stromberg [9, pp. 352–353]),

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_y^x\int_0^{t-\epsilon}g_\rho(\rho,t;b,\tau)f(u(b,\tau))d\tau d\rho \\
&=\lim_{\epsilon\to 0}\int_0^{t-\epsilon}f(u(b,\tau))\int_y^x g_\rho(\rho,t;b,\tau)d\rho d\tau \\
&=\lim_{\epsilon\to 0}\int_0^{t-\epsilon}f(u(b,\tau))\left(g(x,t;b,\tau)-g(y,t;b,\tau)\right)d\tau \\
&=\int_0^t f(u(b,\tau))\left(g(x,t;b,\tau)-g(y,t;b,\tau)\right)d\tau,
\end{aligned}$$

which exists by Lemma 2.3. Thus,

$$\begin{aligned}
&\int_0^t f(u(b,\tau))\left(g(x,t;b,\tau)-g(y,t;b,\tau)\right)d\tau \\
&=\int_0^t f(u(b,\tau))\int_y^x g_\rho(\rho,t;b,\tau)d\rho d\tau \\
&=\int_y^x\int_0^t g_\rho(\rho,t;b,\tau)f(u(b,\tau))d\tau d\rho.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^{t-\epsilon}g(x,t;b,\tau)f(u(b,\tau))d\tau \\
&=\int_y^x\int_0^t g_\rho(\rho,t;b,\tau)f(u(b,\tau))d\tau d\rho+\int_0^t g(y,t;b,\tau)f(u(b,\tau))d\tau.
\end{aligned}$$

Differentiating this with respect to x, we obtain

$$\frac{\partial}{\partial x}\int_0^t g(x,t;b,\tau)f(u(b,\tau))d\tau = \int_0^t g_x(x,t;b,\tau)f(u(b,\tau))d\tau.$$

A similar argument shows that

$$\frac{\partial}{\partial x}\int_0^t g_x(x,t;b,\tau)f(u(b,\tau))d\tau = \int_0^t g_{xx}(x,t;b,\tau)f(u(b,\tau))d\tau,$$

which gives

$$\frac{\partial^2}{\partial x^2}\int_0^t g(x,t;b,\tau)f(u(b,\tau))d\tau = \int_0^t g_{xx}(x,t;b,\tau)f(u(b,\tau))d\tau. \tag{2.15}$$

From (2.2), (2.14) and (2.15), we have for $x \in D$ and $0 < t < t_q$,

$$\begin{aligned} Hu &= \alpha\delta(x-b)f(u(b,t)) + \alpha\int_0^t Hg(x,t;b,\tau)f(u(b,\tau))d\tau \\ &= \alpha\delta(x-b)f(u(x,t)) \end{aligned}$$

since $Hg(x,t;b,\tau)=0$. From (2.2), $\lim_{t\to 0^+} u(x,t) = 0$ on $\bar{D}$. Since $g_x(0,t;b,\tau)=0$, and $g_x(x,t;b,\tau)$ is continuous on $\bar{D}\times(0,t_q)$, we have

$$u_x(0,t) = \lim_{x\to 0^+} u_x(x,t) = \alpha \lim_{x\to 0^+}\int_0^t g_x(x,t;b,\tau)f(u(b,\tau))d\tau = 0.$$

Using $g(L,t;b,\tau)=0$ and u being continuous, we have

$$u(L,t) = \alpha\int_0^t g(L,t;b,\tau)f(u(b,\tau))d\tau = 0.$$

Thus, the nonnegative continuous solution of the integral equation (2.2) is a solution of the problem (1.2). Since a solution of the problem (1.2) is a solution of the integral equation (2.2), which has a unique solution before quenching occurs, it follows that u is the solution of the problem (1.2), and the theorem is proved. □

3. LOCATIONS FOR QUENCHING

We modify the proof of Theorem 3 of Chan and Jiang [1] for a first initial-boundary value problem to prove the following result.

Theorem 3.1. *For any $t\in(0,t_q)$, $u(x,t)$ attains its absolute maximum at (b,t) on the region $\bar{D}\times[0,t]$. If t_q is finite, then at t_q, u quenches at $x=b$ only.*

Proof. By Theorems 2.4 and 2.5, there exists some t_q such that for $0\le t<t_q$, the problem (1.2) has a unique nonnegative (continuous) solution u, which is a strictly increasing function of t in D. Since $u(b,t)$ for $t\in(0,t_q)$ is known, let us denote it by $\eta(t)$, which is positive and increasing for $t>0$. The problem (1.2) is equivalent to the following two initial-boundary value problems:

$$\begin{cases} Hu = 0 \text{ in } (0,b)\times(0,t_q), \\ u(x,0)=0 \text{ on } [0,b],\ u_x(0,t)=0 \text{ and } u(b,t)=\eta(t) \text{ for } t\in(0,t_q). \end{cases} \tag{3.1}$$

$$(3.2)\qquad \begin{cases} Hu = 0 \text{ in } (b,L)\times(0,t_q), \\ u(x,0)=0 \text{ on } [b,L]\,,\ u\,(b,t)=\eta\,(t) \text{ and } u\,(L,t)=0 \text{ for } t\in(0,t_q). \end{cases}$$

For the problem (3.1), if u attains its maximum or minimum somewhere on the boundary $x = 0$ (with $t \in (0,t_q)$), then by the parabolic version of Hopf's lemma, $u_x \neq 0$ there. This contradicts the boundary condition. Thus by the weak maximum principle, we have for each $t \in (0,t_q)$, u attains its absolute maximum at (b,t) on $[0,b]\times[0,t]$. For the problem (3.2), it follows from $\eta\,(t)$ being a strictly increasing function of t that u attains its absolute maximum at (b,t) on $[b,L]\times[0,t]$. Thus, if u quenches, then it quenches at $x=b$.

Since u is a strictly increasing function of t in D, $u_t \geq 0$ there. For the problem (3.1), it follows from the parabolic version of Hopf's Lemma that for any fixed $t \in (0,t_q)$, $u_x\,(b,t) > 0$. For any $x \in (0,b)$, $u_{xx} = u_t \geq 0$, and hence u is concave up. Similarly, for the problem (3.2), we have that for any arbitrarily fixed $t \in (0,t_q)$, $u_x\,(b,t) < 0$. For any $x \in (b,L)$, $u_{xx} = u_t \geq 0$, and hence u is concave up. Therefore, if u quenches, then it quenches at $x = b$ only. If t_q is finite, then by Theorems 2.4 and 2.5, u quenches at t_q. The theorem is then proved. □

REFERENCES

[1] C. Y. Chan and X. O. Jiang, Quenching for a degenerate parabolic problem due to a concentrated nonlinear source, *Quart. Appl. Math.*, 62:553–568, 2004.

[2] C. Y. Chan and H. Y. Tian, Single-point blow-up for a degenerate parabolic problem due to a concentrated nonlinear source, *Quart. Appl. Math.*, 61:363–385, 2003.

[3] C. Y. Chan and P. Tragoonsirisak, A multi-dimensional quenching problem due to a concentrated nonlinear source in $\mathbb{R}^N$, *Nonlinear Anal.*, 69:1494–1514, 2008.

[4] C. Y. Chan and P. Tragoonsirisak, Effects of a concentrated nonlinear source on quenching in $\mathbb{R}^N$, *Dynam. Systems Appl.*, 18:47–54, 2009.

[5] C. Y. Chan and P. Tragoonsirisak, A multi-dimensional blow-up problem due to a concentrated nonlinear source in $\mathbb{R}^N$, *Quart. Appl. Math.*, 69:317–330, 2011.

[6] Q. Y. Dai and Y. G. Gu, A short note on quenching phenomena for semilinear parabolic equations, *J. Differential Equations*, 137:240–250, 1997.

[7] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964, pp. 34, and 49.

[8] I. Stakgold, *Boundary Value Problems of Mathematical Physics, vol. II*, Macmillan, New York, NY, 1968, pp. 197–203.

[9] K. R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth, Belmont, CA, 1981, pp. 141–142, 266–268, 328, 352–353, 380, and 518.

[10] D. W. Trim, *Applied Partial Differential Equations*, PWS-KENT, Boston, MA, 1990, pp. 474–478.

[11] W. R. Wade, *An Introduction to Analysis*, 2nd ed., Prentice-Hall, Upper Saddle River, NJ, 2000, pp. 186–187, 190–191, and 319–320.