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An order h^4 numerical technique for solving biharmonic equation

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Abstract

We develop a fourth order finite difference method for the solution of a nonlinear biharmonic problems with third order derivative terms subject to the boundary condition of first kind over a square domain. An important feature of our method is that it uses only 9-grid points and discretize the boundary condition without use of fictitious point. The first order derivative of solution is obtained as a byproduct of the method. Method is successfully applied to solve singular linear equation. The method is used to solve a set of model test problem. Numerical solutions are included to demonstrate computational potency and the order of the method.

Keywords : Difference method, Nonlinear biharmonic problems, Plate problems, Maximum absolute error, Root mean square error.

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1 Introduction

Consider the nonlinear biharmonic equation

$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial x^2 \partial y^2}+\frac{\partial^4 u}{\partial y^4} = f(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy},u_{xxx},u_{xxy},u_{xyy},u_{yyy})$$
$$0<x,\ y<1 \quad (1)$$

subject to boundary conditions

$$u \text{ and } \frac{\partial u}{\partial n} \text{ prescribed.} \quad (2)$$

For the numerical solution of above boundary value problem, difference equations for mesh points near a boundary are obtained using central difference methods requiring minimum 13-grid points (Jain (1987)). Mohanty *et. al.* (1996) have developed difference methods of order two and four using 9-grid points for solving nonlinear biharmonic problems. The idea of inclusion of third order derivative terms in the reference boundary value problem originated while solving third order boundary value problem by method of finite differences [8].

In this paper we discuss 9-point difference methods of order two and four for solving nonlinear biharmonic equation (1) in bounded square region $\Omega = \{(x,y)\,|\,0 < x, y < 1\}$, which may be partitioned into square subregions. We consider square subregion that consist of central point $(\ell,m) = (x_\ell, y_m)$ and eight other points $(\ell \pm 1, m) = (x_\ell \pm h, y_m)$, $(\ell, m \pm 1) = (x_\ell, y_m \pm h)$, $(\ell \pm 1, m \pm 1) = (x_\ell \pm h, y_m \pm h)$, $(\ell \pm 1, m \mp 1) = (x_\ell \pm h, y_m \mp h)$, where $h > 0$ is a grid size and $\ell, m = 0(1)N+1$, N being a positive integer. We denote the grid point (ℓ,m) as $(0,0)$, $(\ell \pm 1, m)$ as $(\pm 1, 0)$ etc. and the values of $f(x,y,u,\ldots), u, u_x$, etc. at the grid point (ℓ,m) by $f_{\ell,m}$, $U_{\ell,m}$, $U_{x\ell,m}, \ldots$ etc. A linear combination of the values of the solution u, u_x, u_y at these grid points are used to derive difference formulas.

Let $U_{\ell,m}$ be the approximate value of solution u. The difference methods which we present here are based on the only eight grid points surrounding each grid point. The systems that are generated by these formulas have a complicated matrix structure and matrices are not positive definite. Since our method is coupled system of equation at each grid point. A system of linear equations can be solved using a variety of iterative methods including excellent results obtained in [1] by using multigrid and preconditioned Krylov methods. But we adopt a direct method for solving a linear system and the Newton-Raphson method otherwise and we have obtained results which are both comparable and competitive.

In Section 2, we describe the finite difference method. Section 3, we have consider singular linear equation. Section 4 we have estimated the order of method and final Section 5 numerical examples are considered to explore the computational potency of the method, to illustrate accuracy and fourth order of the method.

2 The Finite Difference Method

In this paper, we are using the procedure given by Young (1965) etc., Boisvert (1981), Mohanty *et. al.* (1996). We report two sets of difference methods of order two and four. Let us define following expressions,

$$U_1 = U_{\ell+1,m} + U_{\ell-1,m} \tag{3}$$

$$U_2 = U_{\ell,m+1} + U_{\ell,m-1} \tag{4}$$

$$U_3 = U_{\ell+1,m} - U_{\ell-1,m} \tag{5}$$

$$U_4 = U_{\ell,m+1} - U_{\ell,m-1} \tag{6}$$

$$U_5 = U_{\ell+1,m} + U_{\ell-1,m} + U_{\ell,m+1} + U_{\ell,m-1} \tag{7}$$

$$U_6 = U_{\ell+1,m+1} + U_{\ell+1,m-1} + U_{\ell-1,m+1} + U_{\ell-1,m-1} \tag{8}$$

$$U_7 = U_{\ell+1,m+1} - U_{\ell+1,m-1} + U_{\ell-1,m+1} - U_{\ell-1,m-1} \tag{9}$$

$$U_8 = U_{\ell+1,m+1} + U_{\ell+1,m-1} - U_{\ell-1,m+1} - U_{\ell-1,m-1} \tag{10}$$

$$U_9 = U_{\ell+1,m+1} - U_{\ell+1,m-1} - U_{\ell-1,m+1} + U_{\ell-1,m-1} \tag{11}$$

Similarly replacing U by U_x and U_y we may obtain symmetric expressions for U_{xj} and U_{yj} , $j = 1$ (1) 9, respectively.

2.1 Second Order Method

Using expressions (3 - 11), let us define these approximation

$$\bar{U}_{xx\ell,m} = (U_{\ell+1,m} + U_{\ell-1,m} - 2U_{\ell,m})/h^2 \tag{2.1.1}$$

$$\bar{U}_{yy\ell,m} = (U_{\ell,m+1} + U_{\ell,m-1} - 2U_{\ell,m})/h^2 \tag{2.1.2}$$

$$\bar{U}_{xy\ell,m} = (U_{\ell+1,m+1} - U_{\ell+1,m-1} - U_{\ell-1,m+1} + U_{\ell-1,m-1})/(4h^2) \tag{2.1.3}$$

$$\bar{U}_{xxx\ell,m} = 3(-U_8 + hU_{x6})/(4h^3) \tag{2.1.4}$$

$$\bar{U}_{xxy\ell,m} = (-2U_4 + U_7)/(2h^3) \tag{2.1.5}$$

$$\bar{U}_{yyy\ell,m} = 3(-U_7 + hU_{y6})/(4h^3) \tag{2.1.6}$$

$$\bar{U}_{xyy\ell,m} = (-2U_3 + U_8)/(2h^3) \tag{2.1.7}$$

Using Taylor's series method it can be easily shown that these are second order approximation.

A second order difference method using (2.1.1)-(2.1.7) for the nonlinear equation (1) may be written as

$$\begin{aligned} \mathrm{LS}[u] &\equiv U_6 - 8U_5 + 3h(U_{x3} + U_{y4}) + 28U_{\ell,m} \\ &= \frac{h^2}{2} f\,(x_\ell, y_m, U_{\ell,m}, U_{x\ell,m}, U_{y\ell,m}, \bar{U}_{xx\ell,m}, \bar{U}_{xy\ell,m}, \bar{U}_{yy\ell,m}, \bar{U}_{xxx\ell,m}, \\ &\qquad \bar{U}_{xxy\ell,m}, \bar{U}_{xyy\ell,m}, \bar{U}_{yyy\ell,m}) \end{aligned} \tag{12}$$

The difference schemes of order two for u_x and u_y at the grid point (ℓ, m) are given by

$$U_{x\ell,m} = (3(U_{\ell+1,m} - U_{\ell-1,m}) - hU_{x1})/(4h) \tag{13}$$

$$U_{y\ell,m} = (3(U_{\ell,m+1} - U_{\ell,m-1}) - hU_{y2})/(4h) \tag{14}$$

(Mohanty *et. al.* (1996))

The proposed second order method has two advantages over the classical 13 point formula. It is based upon a single computational cell and incorporates the boundary conditions in a natural way without the need to introduce fictitious points or special schemes at the boundary.

2.2 Fourth Order Method

Let us define using (3 - 11) and (2.1.1 - 2.1.7) the following sets of approximations.

Set I

$$\bar{\bar{U}}_{xx\ell,m} = 2\bar{U}_{xx\ell,m} - U_{x3}/(2h) \tag{2.2.1}$$

$$\bar{\bar{U}}_{yy\ell,m} = 2\bar{U}_{yy\ell,m} - U_{y4}/(2h) \tag{2.2.2}$$

$$\bar{\bar{U}}_{xy\ell,m} = -\bar{U}_{xy\ell,m} + (U_{x4} + U_{y3})/(2h) \tag{2.2.3}$$

$$\begin{aligned} \bar{\bar{U}}_{xx\ell\pm1,m} = {} & (7U_{\ell\mp1,m} + 16U_{\ell,m} - 23U_{\ell\pm1,m})/(2h^2) \\ & \pm(6U_{x\ell\pm1,m} + 8U_{x\ell,m} + U_{x\ell\mp1,m})/(h) \end{aligned} \tag{2.2.4}$$

$$\begin{aligned} \bar{\bar{U}}_{yy\ell,m\pm1} = {} & (7U_{\ell,m\mp1} + 16U_{\ell,m} - 23U_{\ell,m\pm1})/(2h^2) \\ & \pm(6U_{y\ell,m\pm1} + 8U_{y\ell,m} + U_{y,\ell,m\mp1})/(h) \end{aligned} \tag{2.2.5}$$

$$\begin{aligned} \bar{\bar{U}}_{xx\ell,m\pm1} = {} & 2(U_{\ell+1,m\pm1} - 2U_{\ell,m\pm1} + U_{\ell-1,m\pm1})/(h^2) \\ & -(U_{x\ell+1,m\pm1} - U_{x\ell-1,m\pm1})/(2h) \end{aligned} \tag{2.2.6}$$

$$\begin{aligned}
\bar{\bar{U}}_{yy\ell\pm1,m} &= 2(U_{\ell\pm1,m+1} - 2U_{\ell\pm1,m} + U_{\ell\pm1,m-1})/(h^2) \\
&\quad -(U_{y\ell\pm1,m+1} - U_{y\ell\pm1,m-1})/(2h) && (2.2.7) \\
\bar{\bar{U}}_{xy\ell\pm1,m} &= -\bar{U}_{xy\ell,m} + U_{y3}/(2h) + (U_{x7} \pm U_{x9})/(4h) \\
&\quad \mp(U_{y6} - 2U_{y5} + 4U_{y\ell,m})/(6h) && (2.2.8) \\
\bar{\bar{U}}_{xy\ell,m\pm1} &= -\bar{U}_{xy\ell,m} + U_{x4}/(2h) + (U_{y8} \pm U_{y9})/(4h) \\
&\quad \mp(U_{x6} - 2U_{x5} + 4U_{x\ell,m})/(6h) && (2.2.9) \\
\bar{\bar{U}}_{xxx\ell\pm1,m} &= (\mp 99U_{\ell\pm1,m} \pm 48U_{\ell,m} \pm 51U_{\ell\mp1}, m)/(2h^3) \\
&\quad +(39U_{x\ell\pm1,m} + 96U_{x\ell,m} + 15U_{x\ell\mp1,m})/(2h^2) && (2.2.10) \\
\bar{\bar{U}}_{yyy\ell,m\pm1} &= (\mp 99U_{\ell,m\pm1} \pm 48U_{\ell,m} \pm 51U_{\ell,m\mp1})/(2h^3) \\
&\quad +(39_{y\ell,m\pm1} + 96U_{y\ell,m} + 15U_{y\ell,m\mp1})/(2h^2) && (2.2.11) \\
\bar{\bar{U}}_{xxx\ell,m\pm1} &= 15(U_{\ell+1,m\pm1} - U_{\ell-1,m\pm1})/(2h^3) - 3(U_{x\ell+1,m\pm1} \\
&\quad +8U_{x\ell,m\pm1} + U_{x\ell-1,m\pm1})/(2h^2) && (2.2.12) \\
\bar{\bar{U}}_{yyy\ell\pm1,m} &= 15(U_{\ell\pm1,m+1} - U_{\ell\pm1,m-1})/(2h^3) - 3(U_{y\ell\pm1,m+1} \\
&\quad +8U_{y\ell\pm1,m} + U_{y\ell\pm1,m-1})/(2h^2) && (2.2.13) \\
\bar{\bar{U}}_{xxx\ell,m} &= 3[3U_8 - U_3 + h(U_{x1} - 4U_{x2} - U_{x6})]/(2h^3) && (2.2.14) \\
\bar{\bar{U}}_{yyy\ell,m} &= 3[3U_7 - U_4 + h(U_{y2} - 4U_{y1} - U_{y6})]/(2h^3) && (2.2.15) \\
\bar{\bar{U}}_{xxy\ell,m} &= 5\bar{U}_{xxy\ell,m}/2 + (2U_{y2} - U_{y6} - U_{x9})/(4h^2) && (2.2.16) \\
\bar{\bar{U}}_{xyy\ell,m} &= 5\bar{U}_{xyy\ell,m}/2 + (2U_{x1} - U_{x6} - U_{y9})/(4h^2)\,. && (2.2.17)
\end{aligned}$$

Set II

$$\begin{aligned}
\bar{\bar{U}}_{xxy\ell\pm1,m} &= 7[U_{\ell\pm1,m+1} - U_{\ell\pm1,m-1} + U_4 - 2(U_{\ell\mp1,m+1} - U_{\ell\mp1,m-1})]/(2h^3) \\
&\quad +[2U_{y2} \mp 10(U_{x\ell\mp1,m+1} - U_{x\ell\mp1,m-1}) \mp 32U_{x4} - U_{y6}]/(4h^2) \\
&&& (2.2.18)
\end{aligned}$$

$$\bar{\bar{U}}_{xxy\ell,m\pm1} = 4\bar{U}_{xxy\ell,m} - (U_{x9} \mp 2U_{y7} + 8U_{y1} - 16U_{y\ell,m} \pm 4U_{y4})/(4h^2) \tag{2.2.19}$$

$$\bar{\bar{U}}_{xyy\ell\pm1,m} = 4\bar{U}_{xyy\ell,m} - (U_{y9} \mp 2U_{x8} + 8U_{x2} - 16U_{x\ell,m} \pm 4U_{x3})/(4h^2) \tag{2.2.20}$$

$$\begin{aligned}\bar{\bar{U}}_{xyy\ell,m\pm1} &= 7[U_{\ell+1,m\pm1} - U_{\ell-1,m\pm1} + U_3 - 2(U_{\ell+1,m\mp1} - U_{\ell-1,m\mp1})]/(2h^3) \\ &\quad + [2U_{x1} \mp 10(U_{y\ell+1,m\mp1} - U_{y\ell-1,m\mp1}) \mp 32U_{y3} - U_{x6}]/(4h^2)\end{aligned} \tag{2.2.21}$$

It is easy to verify by Taylor's series method that the approximation in set I are of order four whereas in set II are of order three.

Further, let

$$\begin{aligned}\bar{\bar{F}}_{\ell,m} = f(&x_\ell, y_m, U_{\ell,m}, U_{x\ell,m}, U_{y\ell,m}, \bar{\bar{U}}_{xx\ell,m}, \bar{\bar{U}}_{xy\ell,m}, \bar{\bar{U}}_{yy\ell,m}, \bar{\bar{U}}_{xxx\ell,m}, \bar{\bar{U}}_{xxy\ell,m}, \\ &\bar{\bar{U}}_{xyy\ell,m}, \bar{\bar{U}}_{yyy\ell,m}),\end{aligned}$$

similarly we can define

$$\bar{\bar{F}}_{\ell\pm1,m} \quad \text{and} \quad \bar{\bar{F}}_{\ell,m\pm1}$$

The nonlinear equation (1), using (2.2.1)-(2.2.21) at the gird point (ℓ, m) may be discretized as

$$\begin{aligned}\text{LF}[u] &\equiv 264U_{\ell,m} + 6U_6 - 75U_5 + h[U_{x8} + U_{y7} + 28U_{x3} + 28U_{y4}] \\ &= \frac{h^4}{3}[\bar{\bar{F}}_{\ell+1,m} + \bar{\bar{F}}_{\ell-1,m} + \bar{\bar{F}}_{\ell,m+1} + \bar{\bar{F}}_{\ell,m-1} + 11\bar{\bar{F}}_{\ell,m}]\end{aligned} \tag{15}$$

and the difference methods for u_x and u_y are given by

$$\begin{aligned}\text{LF}[u_x] &\equiv h[40U_{x\ell,m} + 8U_{x1} + U_{x6} + U_{y9}] - 3[U_8 + 8U_3] \\ &= \frac{h^4}{6}[\bar{\bar{F}}_{\ell+1,m} - \bar{\bar{F}}_{\ell-1,m}]\end{aligned} \tag{16}$$

$$\begin{aligned}\mathrm{LF}[u_y] &\equiv h[40U_{y\ell,m}+8U_{y2}+U_{y6}+U_{x9}]-3[U_7+8U_4]\\ &= \frac{h^4}{6}[\bar{\bar{F}}_{\ell,m+1}-\bar{\bar{F}}_{\ell,m-1}] \qquad (17)\end{aligned}$$

(Mohanty *et. al.* (1996))

Here we see that at interior mesh points we have three unknowns namely u, u_x and u_y. This means that the number of bands with nonzero entries is increased and so is the size of the matrix for the same mesh size. However, the values of the derivatives of the solution, which are often of interest, are also computed.

3 The Singular Linear Equation

Let us consider the singular linear equation of the form

$$\nabla^4 u = u_{xx}+\frac{1}{x^2}u_{yy}+\frac{\alpha}{x}u_x+f(x,y), \quad 0<x,\ y<1 \qquad (18)$$

where α is constant, then

$$\begin{aligned}\bar{\bar{F}}_{\ell+a,m} &= \bar{\bar{U}}_{xx\ell+a,m}+\frac{1}{(x_{\ell+a})^2}\bar{\bar{U}}_{yy\ell+a,m}+\frac{\alpha}{x_{\ell+a}}U_{x\ell+a,m}+f_{\ell+a,m},\ a=0,\pm1\\ \bar{\bar{F}}_{\ell,m\pm1} &= \bar{\bar{U}}_{xx\ell,m\pm1}+\frac{1}{(x_\ell)^2}\bar{\bar{U}}_{yy\ell,m\pm1}+\frac{\alpha}{x_\ell}U_{x\ell,m\pm1}+f_{\ell,m\pm1}\end{aligned}$$

By the help of the approximations in Set I and using the methods (15, 16, 17), we may obtain a difference scheme of $O(h^4)$ for equation (18). However scheme fails when solution is to be determined at $\ell=1$, the vicinity of $x=0$. We overcome this difficulty by modifying the method in such a way that the solutions retain the order and accuracy even in the vicinity of the singularity.

Further we may write

$$\frac{1}{x_{\ell\pm1}} = \frac{1}{x_\ell}\mp\frac{h}{(x_\ell)^2}+\frac{h^2}{(x_\ell)^3}+O(\mp h^3+h^4)$$

$$
\begin{aligned}
\frac{1}{(x_{\ell\pm1})^2} &= \frac{1}{(x_\ell)^2} \mp \frac{2h}{(x_\ell)^3} + \frac{3h^2}{(x_\ell)^4} + O(\mp h^3 + h^4) \\
f_{\ell\pm1,m} &= f_{\ell,m} \pm h f_{x\ell,m} + \frac{h^2}{2} f_{xx\ell,m} + O(\pm h^3 + h^2) \\
f_{\ell,m\pm1} &= f_{\ell,m} \pm h f_{y\ell,m} + \frac{h^2}{2} f_{yy\ell,m} + O(\pm h^3 + h^2)\,.
\end{aligned}
$$

By the help of these approximation and neglecting high order terms, we obtain a new difference scheme of $O(h^4)$ for solving the equation (18) as

$$
\mathrm{LF}[u] = \frac{h^4}{3}[R_0 + R_1 + R_2 + R_3 + R_4 + R_5] \tag{19}
$$

$$
\mathrm{LF}[u_x] = \frac{h^4}{6}[2hf_{x\ell,m} + S_1 + S_2 + S_3 + S_4] \tag{20}
$$

$$
\mathrm{LF}[u_y] = \frac{h^4}{6}[2hf_{y\ell,m} + T_1 + T_2 + T_3 + T_4]\,. \tag{21}
$$

where

$$
\begin{aligned}
R_0 &= 15 f_{\ell,m} + h^2(f_{xx\ell,m} + f_{yy\ell,m}) \\
R_1 &= \frac{\alpha}{\ell h}\left[(15 + \delta_x^2 + \delta_y^2) - \frac{1}{\ell}(2\mu_x\delta_x) + \frac{2}{\ell^2}\right] U_{x\ell,m} \\
R_2 &= \frac{2}{h^2}(9 + \delta_y^2)\delta_x^2 U_{\ell,m} - \frac{1}{2h}(3 + \delta_y^2)(2\mu_x\delta_x)U_{x\ell,m} \\
R_3 &= \frac{1}{\ell^2 h^2}\left[\frac{2}{h^2}(9 + \delta_x^2)\delta_y^2 U_{\ell,m} - \frac{1}{2h}(3 + \delta_x^2)(2\mu_y\delta_y)U_{y\ell,m}\right] \\
R_4 &= \frac{-2}{\ell^3 h^3}\left[\frac{2}{h}(\delta_y^2 2\mu_x\delta_x)U_{\ell,m} - \frac{1}{2}(4\mu_x\delta_x\mu_y\delta_y)U_{y\ell,m}\right] \\
R_5 &= \frac{6}{\ell^4 h^4}\left[2\delta_y^2 U_{\ell,m} - \frac{h}{2}(2\mu_y\delta_y)U_{y\ell,m}\right] \\
S_1 &= \frac{1}{h^2}\left[\left(-15 + \frac{2}{\ell^2 h^2}\delta_y^2\right)(2\mu_x\delta_x)U_{\ell,m} - \frac{8}{\ell^3 h^2}\delta_y^2 U_{\ell,m}\right] \\
S_2 &= \frac{1}{h}[30 + 7\delta_x^2]U_{x\ell,m} \\
S_3 &= -\frac{1}{2\ell^2 h^3}\left[(2\mu_x\delta_x) - \frac{4}{\ell}\right](2\mu_y\delta_y)U_{y\ell,m}
\end{aligned}
$$

$$\begin{aligned}
S_4 &= \frac{\alpha}{\ell h}\left[(2\mu_x\delta_x) - \frac{2}{\ell}\right] U_{x\ell,m} \\
T_1 &= \frac{1}{h^2}\left[\frac{-15}{\ell^2 h^2} + 2\delta_x^2\right][2\mu_y\delta_y]U_{\ell,m} \\
T_2 &= \frac{1}{\ell^2 h^3}[30 + 7\delta_y^2]U_{y\ell,m} \\
T_3 &= -\frac{1}{2h}(4\mu_x\delta_x\mu_y\delta_y)U_{x\ell,m} \\
T_4 &= \frac{\alpha}{\ell h}(2\mu_y\delta_y)U_{x\ell,m}
\end{aligned}$$

where μ, δ are averaging and central difference operators respectively and defined as

$$\begin{aligned}
\mu_x U_{\ell,m} &= \frac{1}{2}(U_{\ell+1/2,m} + U_{\ell-1/2,m}) \\
\delta_x U_{\ell,m} &= (U_{\ell+1/2,m} - I_{\ell-1/2,m})\,.
\end{aligned}$$

Note that, the scheme $(19, 20, 21)$ is of $O(h^4)$ and free from the term $1/(\ell \pm 1)$, hence very easily solved for $\ell = 1\ (1)\ N$ in the region Ω, when the value of u is known explicitly on the boundary $\partial\Omega$ and the boundary is parallel to the coordinate axes, then we also know the tangential derivatives and formulas $(12, 13, 14)$, $(15, 16, 17)$ and $(19, 20, 21)$ can be used.

4 The Estimation of the Order of Method

We illustrate a method of determining the orders of the formulas by considering the formula (15). At the grid point (ℓ, m), we may write differential equation (1) as

$$\begin{aligned}
&\left(\frac{\partial^4 u}{\partial x^4}\right)_{\ell,m} + 2\left(\frac{\partial^4 u}{\partial x^2 y^2}\right)_{\ell,m} + \left(\frac{\partial^4 u}{\partial y^4}\right)_{\ell,m} \\
&\quad = f(x_\ell, y_m, U_{\ell,m}, U_{x\ell,m}, U_{y\ell,m}, U_{xx\ell,m}, U_{xy\ell,m}, U_{yy\ell,m}, U_{xxx\ell,m}, U_{xxy\ell,m}, \\
&\qquad U_{xyy\ell,m}, U_{yyy\ell,m}) \\
&\quad = F_{\ell,m}
\end{aligned}$$

similarly we can define $F_{\ell\pm1,m}$, $F_{\ell,m\pm1}$.

With help of approximation in set I, II expand u, u_x, u_y in Taylor series about central mesh point (ℓ, m) and let us denote $H_{\ell,m} = (\partial f/\partial U_{xxy})_{\ell,m}$, $G_{\ell,m} = (\partial f/\partial U_{xyy})_{\ell,m}$. Similarly we define $H_{\ell\pm1,m}$ and $G_{\ell,m\pm1}$. We obtain

$$\bar{\bar{F}}_{\ell,m} = F_{\ell,m} + O(h^4) \tag{4.1}$$

$$\bar{\bar{F}}_{\ell+1,m} = F_{\ell+1,m} + \mathrm{RT}_1\ H_{\ell+1,m} + \mathrm{ST}_1\ G_{\ell+1,m} + O(h^4) \tag{4.2}$$

$$\bar{\bar{F}}_{\ell-1,m} = F_{\ell-1,m} + \mathrm{RT}_2\ H_{\ell-1,m} + \mathrm{ST}_2\ G_{\ell-1,m} + O(h^4) \tag{4.3}$$

$$\bar{\bar{F}}_{\ell,m+1} = F_{\ell,m+1} + \mathrm{RT}_3\ H_{\ell,m+1} + \mathrm{ST}_3\ G_{\ell,m+1} + O(h^4) \tag{4.4}$$

$$\bar{\bar{F}}_{\ell,m-1} = F_{\ell,m-1} + \mathrm{RT}_4\ H_{\ell,m-1} + \mathrm{ST}_4\ G_{\ell,m-1} + O(h^4) \tag{4.5}$$

where we denote $U_{rs} = \partial^{r+s}U/\partial x^r \partial y^s$ and

$$\mathrm{RT}_1 = h^3(-144\, U_{51\ell+1,m} + 120\, U_{33\ell+1,m})/6\,! \tag{4.6}$$

$$\mathrm{RT}_2 = h^3(144\, U_{51\ell-1,m} - 120\, U_{33\ell-1,m})/6\,! \tag{4.7}$$

$$\mathrm{RT}_3 = 10h^3 U_{42\ell,m+1}/5\,! \tag{4.8}$$

$$\mathrm{RT}_4 = -10h^3 U_{42\ell,m-1}/5\,! \tag{4.9}$$

$$\mathrm{ST}_1 = 10h^3 U_{24\ell+1,m}/5\,! \tag{4.10}$$

$$\mathrm{ST}_2 = -10h^3 U_{24\ell-1,m}/5\,! \tag{4.11}$$

$$\mathrm{ST}_3 = h^3(120\, U_{33\ell,m+1} - 144\, U_{15\ell,m+1})/6\,! \tag{4.12}$$

$$\mathrm{ST}_4 = h^3(-120\, U_{33\ell,m-1} + 144\, U_{15\ell,m-1})/6\,! \tag{4.13}$$

Expanding the $U_{rs\ell\pm1,m}$ in Taylor's series about central grid point (ℓ, m), we find, $\mathrm{RT}_1 = -\mathrm{RT}_2 = O(h^3)$, so from (4.6) and (4.7)

$$\mathrm{RT}_1\, H_{\ell+1,m} - \mathrm{RT}_1\, H_{\ell-1,m} = \mathrm{RT}_1[H_{\ell+1,m} - H_{\ell-1,m}] \tag{4.14}$$

again expanding $H_{\ell+1,m}$ and $H_{\ell-1,m}$ in Taylor's series about central grid point (ℓ, m), from (4.14) we get

$$\mathrm{RT}_1[H_{\ell+1,m} - H_{\ell-1,m}] = O(h^4).$$

We can obtain similar result for other expression from (4.8) to (4.13).

So from (4.1) to (4.5) and (4.14) we obtained

$$\begin{aligned}&\bar{\bar{F}}_{\ell-1,m} + \bar{\bar{F}}_{\ell-1,m} + \bar{\bar{F}}_{\ell,m+1} + \bar{\bar{F}}_{\ell,m-1} + 11\bar{\bar{F}}_{\ell,m}\\ &\qquad = F_{\ell+1,m} + F_{\ell-1,m} + F_{\ell,m+1} + F_{\ell,m-1} + 11F_{\ell,m} + O(h^4).\end{aligned} \tag{4.15}$$

The local truncation error associated with (15) is $O(h^4)$.

Thus the finite difference formula (15) is fourth order descretization of nonlinear biharmonic equation (1). Using this method, we can estimate the order of other formulas.

5 Numerical Illustrations

In this section we have solved the coupled nonlinear plate problem, the singular problem and two other problems with third derivative terms whose exact solutions are known to us. In each case we have taken the unit square $0 < x,\ y < 1$ as the region of integration and covered it with a uniform grid of width h. The right hand side functions and boundary conditions may be obtained using the exact solution. We have used Gauss-Seidel iterative method for solving linear equations and Newton-Raphson method for solving nonlinear equations. However both iterative methods were very slow. In order to avoid a large number of iterations and because of the machine storage limitations, we have restricted to $h = 1/4$, $h = 1/8$ and $h = 1/16$ in computations work. The same initial vector is used for both the difference methods and iterations were stopped when the error to tolerance 10^{-14} is achieved. All computations were carried out using double precision arithmetic on an IBM AIX VERSION 4 at the computer center, University of Delhi, Delhi - 110007.

Example 1. Coupled nonlinear plate problem

$$\nabla^4 u \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 2\,\varepsilon(u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}) + f(x,y)$$

$$\nabla^4 v \equiv \frac{\partial^4 v}{\partial x^4} + 2\frac{\partial^4 v}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial y^4} = (u_{xy})^2 - u_{xx}u_{yy} + g(x,y)$$

where $\varepsilon > 0$ is the thickness of the plate. Its exact solution are given by $u = \sin x \cos y$ and $v = \cos x \sin y$. The maximum absolute errors are calculated in Table 1 for $\varepsilon = 0.5$, 1.0 and 5.0.

Table 1

		Scheme (15)			Scheme (12)		
h	ε	0.5	1.0	5.0	0.5	1.0	5.0
$\frac{1}{4}$	u	.2540 (−6)	.5047 (−6)	.2665 (−5)	.1280 (−4)	.1314 (−4)	.1609 (−4)
	v	.5485 (−7)	.5380 (−7)	.4467 (−7)	.1228 (−4)	.1227 (−4)	.1224 (−4)
$\frac{1}{8}$	u	.1283 (−7)	.3107 (−7)	.1937 (−6)	.2853 (−5)	.2906 (−5)	.3369 (−5)
	v	.3303 (−8)	.3773 (−8)	.3159 (−8)	.2772 (−5)	.2772 (−5)	.2769 (−5)

Example 2. The problem is to solve (18), whose exact solution is given by $u = x^4 \cos y$. The root mean square errors are calculated in Table 2 for $\alpha = 1$ and 2.

Table 2

		Scheme (19, 20, 21)		Scheme (12, 13, 14)	
h	α	1	2	1	2
$\frac{1}{4}$	u	0.1144 (−4)	0.2601 (−5)	0.6518 (−3)	0.9291 (−3)
	u_x	0.3874 (−4)	0.3159 (−4)	0.1957 (−2)	0.2787 (−2)
	u_y	0.1144 (−4)	0.1528 (−4)	0.1985 (−2)	0.2830 (−2)
$\frac{1}{8}$	u	0.9762 (−6)	0.3005 (−6)	0.1552 (−3)	0.2213 (−3)
	u_x	0.4404 (−5)	0.1667 (−5)	0.4523 (−3)	0.6440 (−3)
	u_y	0.3048 (−5)	0.9779 (−6)	0.4720 (−3)	0.6727 (−3)

Example 3. The model problem with nonlinear third derivative term is

$$\nabla^4 u \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = \alpha u(u_{xxx} + u_{yyy}) + f(x, y)$$

whose exact solution is given by $u = x^4 \cos y/120$. The maximum absolute errors are calculated in Table 3 for $\alpha = 1.0$, 5.0 and 10.0.

Table 3

		Scheme (15, 16, 17)			Scheme (12, 13, 14)		
h	α	1	5	10	1	5	10
$\frac{1}{4}$	u	.7682 (−8)	.7695 (−8)	.7712 (−8)	.2079 (−5)	.2067 (−5)	.2052 (−5)
	u_x	.5860 (−7)	.5870 (−7)	.5882 (−7)	.6339 (−5)	.6310 (−5)	.6273 (−5)
	u_y	.6445 (−7)	.6449 (−7)	.6455 (−7)	.6333 (−5)	.6296 (−5)	.6250 (−5)
$\frac{1}{8}$	u	.6796 (−9)	.6812 (−9)	.6833 (−9)	.4982 (−6)	.4952 (−6)	.4914 (−6)
	u_x	.2526 (−8)	.2531 (−8)	.2537 (−8)	.1504 (−5)	.1500 (−5)	.1494 (−5)
	u_y	.2609 (−8)	.2614 (−8)	.2621 (−8)	.1499 (−5)	.1490 (−5)	.1477 (−5)
$\frac{1}{16}$	u	.3997 (−10)	.4008 (−10)	.4022 (−10)	.1225 (−6)	.1217 (−6)	.1208 (−6)
	u_x	.1332 (−09)	.1337 (−09)	.1343 (−09)	.3953 (−6)	.3938 (−6)	.3919 (−6)
	u_y	.1292 (−09)	.1295 (−09)	.1300 (−09)	.3943 (−6)	.3917 (−6)	.3885 (−6)

Example 4. The model problem with linear third derivative term is

$$\nabla^4 u \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = \alpha(u_{xxx} + u_{xxy} + u_{xyy} + u_{yyy}) + f(x, y)$$

whose exact solution is given by $u = e^x \cos y$.

The maximum absolute errors are calculated in Table 4 for $\alpha = 1.0$, 5.0 and 10.0.

Table 4

		Scheme (15, 16, 17)			Scheme (12, 13, 14)		
h	α	1	5	10	1	5	10
$\frac{1}{4}$	u	.3671 (−6)	.1784 (−5)	.5135 (−5)	.2756 (−4)	.1338 (−3)	.2350 (−3)
	u_x	.1124 (−5)	.5736 (−5)	.1086 (−4)	.1048 (−3)	.3942 (−3)	.7525 (−3)
	u_y	.1199 (−5)	.5932 (−5)	.1274 (−4)	.1064 (−3)	.4159 (−3)	.7377 (−3)
$\frac{1}{8}$	u	.2010 (−7)	.3536 (−7)	.2953 (−7)	.6720 (−5)	.3206 (−4)	.5535 (−4)
	u_x	.6489 (−7)	.1155 (−6)	.2550 (−6)	.2098 (−4)	.1241 (−3)	.2483 (−3)
	u_y	.6061 (−7)	.1223 (−6)	.2698 (−6)	.2413 (−4)	.1006 (−3)	.1683 (−3)
$\frac{1}{16}$	u	.1223 (−8)	.1653 (−8)	.3899 (−8)	.1716 (−5)	.7893 (−5)	.1312 (−4)
	u_x	.4102 (−8)	.6035 (−8)	.1943 (−7)	.5653 (−5)	.3061 (−4)	.5705 (−4)
	u_y	.3780 (−8)	.5832 (−8)	.2231 (−7)	.6108 (−5)	.2632 (−4)	.4079 (−4)

Conclusions

In this paper, we have outlined a procedure for obtaining difference methods of $O(h^2)$ and $O(h^4)$ for the nonlinear biharmonic problems with third order derivative terms and method applied to solve coupled nonlinear plate problem. The procedure described in this article offers significant advantages for generalization of nonlinear biharmonic problem (Mohanty *et. al.* (1996)). A drawback of these methods is that solutions to the resulting systems of equations cannot be obtained quickly because of the lack of the simple structure and positive-definiteness. From the numerical results, we conclude that the higher order methods may be attractive for those problem where solution is sufficiently smooth. It is hoped that the ideas presented here may lead to development of new techniques for solving even more general problems with accurate/computational efficiency.

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